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Estimate of *K*-functionals and modulus of smoothness constructed by generalized spherical mean operator

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Abstract. Using a generalized spherical mean operator, we define generalized modulus of smoothness in the space $L_k^2(\mathbb{R}^d)$. Based on the Dunkl operator we define Sobolev-type space and K-functionals. The main result of the paper is the proof of the equivalence theorem for a K-functional and a modulus of smoothness for the Dunkl transform on $\mathbb{R}^d$.

Keywords. Dunkl operator; generalized spherical mean operator; K-functional; modulus of smoothness.

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1. Introduction and preliminaries

In [2], Belkina and Platonov proved the equivalence theorem for a K-functional and a modulus of smoothness for the Dunkl transform in the Hilbert space $L_2(\mathbb{R}, |x|^{2\alpha+1})$, $\alpha > -1/2$, using a Dunkl translation operator.

In this paper, we prove the analog of this result (see [2]) in the Hilbert space $L^2(\mathbb{R}^d, w_k)$. For this purpose, we use a generalized spherical mean operator in the place of the Dunkl translation operator.

Dunkl [4] defined a family of first-order differential-difference operators related to some reflection groups. These operators generalize in a certain manner the usual differentiation and have gained considerable interest in various fields of mathematics and also in physical applications. The theory of Dunkl operators provides generalizations of various multivariable analytic structures. Among others, we cite the exponential function, the Fourier transform and the translation operator. For more details about these operators, see [3, 4, 6, 7, 9, 10, 12] and the references therein.

Let R be a root system in $\mathbb{R}^d$, W the corresponding reflection group, R_+ a positive subsystem of R and k a non-negative and W-invariant function defined on R. The Dunkl operator is defined for $f \in C^1(\mathbb{R}^d)$ by

$$\mathrm{D}_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in \mathrm{R}_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \quad x \in \mathbb{R}^d.$$

Here $\langle , \rangle$ is the usual Euclidean scalar product on $\mathbb{R}^d$ with the associated norm $|.|$ and σ_α the reflection with respect to the hyperplane H_α orthogonal to α. We consider the weight function

$$w_k(x) = \prod_{\alpha \in \mathrm{R}_+} |\langle \alpha, x\rangle|^{2k(\alpha)},$$

where w_k is W-invariant and homogeneous of degree 2γ where

$$\gamma = \sum_{\alpha \in \mathrm{R}_+} k(\alpha).$$

We let η be the normalized surface measure on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ and set

$$\mathrm{d}\eta_k(y) = w_k(y)\mathrm{d}\eta(y).$$

Then η_k is a W-invariant measure on $\mathbb{S}^{d-1}$, and we let $d_k = \eta_k(\mathbb{S}^{d-1})$.

The Dunkl kernel E_k on $\mathbb{R}^d \times \mathbb{R}^d$ has been introduced by Dunkl in [5]. For $y \in \mathbb{R}^d$, the function $x \longmapsto E_k(x, y)$ can be viewed as the solution on $\mathbb{R}^d$ of the following initial problem:

$$\mathrm{D}_j u(x, y) = y_i u(x, y), \quad 1 \le j \le d,$$

$$u(0, y) = 1.$$

This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

Rösler has proved in [12] the following integral representation for the Dunkl kernel,

$$E_k(x, z) = \int_{\mathbb{R}^d} \mathrm{e}^{\langle y, z\rangle} \mathrm{d}\mu_x(y), \ x \in \mathbb{R}^d, \quad z \in \mathbb{C}^d,$$

where μ_x is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B(0, |x|)$ of center 0 and radius $|x|$.

PROPOSITION 1.1

Let $z, w \in \mathbb{C}^d$ and $\lambda \in \mathbb{C}$. Then

(1) $E_k(z, 0) = 1$,
(2) $E_k(z, w) = E_k(w, z)$,
(3) $E_k(\lambda z, w) = E_k(z, \lambda w)$,
(4) *For all* $\nu = (\nu_1, ..., \nu_d) \in \mathbb{N}^d, \ x \in \mathbb{R}^d, \ z \in \mathbb{C}^d$, *we have*

$$|D_z^\nu E_k(x, z)| \le |x|^{|\nu|} \exp(|x||\mathrm{Re}(z)|),$$

where

$$D_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} \ldots \partial z_d^{\nu_d}}, \quad |\nu| = \nu_1 + \cdots + \nu_d.$$

In particular,

$$|D_z^\nu E_k(ix, z)| \le |x|^\nu$$

for all $x, z \in \mathbb{R}^d$.

Proof. See [3].

The Dunkl transform is defined for $f \in L^1_k(\mathbb{R}^d) = L^1(\mathbb{R}^d, w_k(x)\mathrm{d}x)$ by

$$\hat{f}(\xi) = c_k^{-1} \int_{\mathbb{R}^d} f(x) E_k(-i\xi, x) w_k(x)\mathrm{d}x,$$

where the constant c_k is given by

$$c_k = \int_{\mathbb{R}^d} \mathrm{e}^{\frac{-|z|^2}{2}} w_k(z)\mathrm{d}z.$$

The inverse Dunkl transform is defined by the formula

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) E_k(ix, \xi) w_k(\xi)\mathrm{d}\xi, \quad x \in \mathbb{R}^d.$$

The Dunkl Laplacian D_k is defined by

$$D_k = \sum_{i=1}^{d} D_i^2.$$

From [11], we have that if $f \in L^2_k(\mathbb{R}^d)$,

$$\widehat{\mathrm{D}_k f}(\xi) = -|\xi|^2 \hat{f}(\xi). \tag{1}$$

The Dunkl transform shares several properties with its counterpart in the classical case. We mention here, in particular that Parseval theorem holds in $L^2_k(\mathbb{R}^d)$. As in the classical case, a generalized translation operator is defined in the Dunkl (see [13, 14]). Namely, for $f \in L^2_k(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we define $\tau_x(f)$ to be the unique function in $L^2_k(\mathbb{R}^d)$ satisfying

$$\widehat{\tau_x f}(y) = E_k(ix, y)\hat{f}(y) \quad \text{a.e. } y \in \mathbb{R}^d.$$

Form to Parseval theorem and Proposition 1.1, we see that

$$\|\tau_x f\|_{L^2_k(\mathbb{R}^d)} \le \|f\|_{L^2_k(\mathbb{R}^d)} \quad \text{for all } x \in \mathbb{R}^d.$$

The generalized spherical mean value of $f \in L^2_k(\mathbb{R}^d)$ is defined by

$$M_h f(x) = \frac{1}{d_k} \int_{\mathbb{S}^{d-1}} \tau_x(f)(hy)\mathrm{d}\eta_k(y), \quad (x \in \mathbb{R}^d, h > 0).$$

We have

$$\|M_h f\|_{L^2_k(\mathbb{R}^d)} \le \|f\|_{L^2_k(\mathbb{R}^d)}. \tag{2}$$

PROPOSITION 1.2

Let $f \in L^2_k(\mathbb{R}^d)$ and fix $h > 0$. Then $M_h f \in L^2_k(\mathbb{R}^d)$ and

$$\widehat{\mathrm{M}_h f}(\xi) = j_{\gamma+\frac{d}{2}-1}(h|\xi|)\hat{f}(\xi), \quad \xi \in \mathbb{R}^d. \tag{3}$$

Proof. See [8].

Let the function $f(x) \in L_k^2(\mathbb{R}^d)$. We define differences of the order m ($m \in 1, 2, \ldots$) with a step $h > 0$.

$$\Delta_h^m f(x) = (I - M_h)^m f(x),$$

where I is the unit operator.

For any positive integer m, we define the generalized module of smoothness of the mth order by the formula

$$w_m(f, \delta)_{2,k} = \sup_{0<h\leq\delta} \|\Delta_h^m f\|_{L_k^2(\mathbb{R}^d)}, \quad \delta > 0.$$

Let $W_{2,k}^m$ be the Sobolev space constructed by the operator D_k, i.e.,

$$\mathrm{W}_{2,k}^m = \{f \in L_k^2(\mathbb{R}^d) : \ D_k^j f \in L_k^2(\mathbb{R}^d); \ j = 1, 2, \ldots, m\}.$$

Let us define the K-functional constructed by the spaces $L_k^2(\mathbb{R}^d)$ and $W_{2,k}^m$,

$$\begin{aligned} K_m(f, t)_{2,k} &= K(f, t; L_k^2(\mathbb{R}^d); W_{2,k}^m) \\ &= \inf\{\|f - g\|_{L_k^2(\mathbb{R}^d)} + t\|D_k^m g\|_{L_k^2(\mathbb{R}^d)}; \ g \in W_{2,k}^m\}, \end{aligned}$$

where $f \in L_k^2(\mathbb{R}^d), \ t > 0$.

For $\alpha > \frac{-1}{2}$, let $j_\alpha(x)$ be a normalized Bessel function of the first kind, i.e.,

$$j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha + 1) J_\alpha(x)}{x^\alpha},$$

where $J_\alpha(x)$ is a Bessel function of the first kind (Chap. 7 of [1]).

The function $j_\alpha(x)$ is infinitely differentiable, $j_\alpha(0) = 1$.
We understand a generalized exponential function as the function [2]

$$e_\alpha(x) = j_\alpha(x) + i c_\alpha x j_{\alpha+1}(x), \tag{4}$$

where $c_\alpha = (2\alpha + 2)^{-1}, i = \sqrt{-1}$.

From (4), we have $|1 - j_\alpha(x)| \leq |1 - e_\alpha(x)|$

2. Main results

Lemma 2.1. *Let $f(x) \in L_k^2(\mathbb{R}^d)$. Then*

$$\|\Delta_h^m f\|_{\mathrm{L}_k^2(\mathbb{R}^d)} \leq 2^m \|f\|_{\mathrm{L}_k^2(\mathbb{R}^d)}.$$

Proof. We use the proof of recurrence for m and the formula (2).

Lemma 2.2. *For $x \in \mathbb{R}$, the following inequalities are fulfilled*:

(1) $|e_\alpha(x)| \leq 1$,
(2) $|1 - e_\alpha(x)| \leq 2|x|$,
(3) $|1 - e_\alpha(x)| \geq c$ *with* $|x| \geq 1$, *where* $c > 0$ *is a certain constant which depends only on* α.

Proof. See [2].

Lemma 2.3. *For $x \in \mathbb{R}$, the following inequalities are fulfilled*:

(1) $|j_\alpha(x)| \le 1$,
(2) $|1 - j_\alpha(x)| \ge c_1$ *with* $|x| \ge 1$, *where* $c_1 > 0$ *is a certain constant which depends only on* α.

Proof. Analog of proof of Lemma 2.2.

In what follows, $f(x)$ is an arbitrary function of the space $L_k^2(\mathbb{R}^d)$; $c, c_1, c_2, c_3, \ldots$ are positive constants.

Lemma 2.4. *Let* $f \in W_{2,k}^m$, $t > 0$. *Then*

$$w_m(f,t)_{2,k} \le c_2 t^{2m} \|D_k^m f\|_{L_k^2(\mathbb{R}^d)}.$$

Proof. Assume that $h \in (0,t]$, $\Delta_h^m f = (I - M_h)^m f$ is the difference with the step h. From Proposition 1.2, formula (1) and the Parseval equality,

$$\begin{aligned} \|\Delta_h^m f\|_{L_k^2(\mathbb{R}^d)} &= \|(1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|))^m \hat{f}(\xi)\|_{L_k^2(\mathbb{R}^d)}; \\ \|D_k^m f\|_{L_k^2(\mathbb{R}^d)} &= |\xi|^{2m} \|\hat{f}(\xi)\|_{L_k^2(\mathbb{R}^d)}. \end{aligned} \tag{5}$$

Formula (5) implies the equality

$$\|\Delta_h^m f\|_{L_k^2(\mathbb{R}^d)} = h^{2m} \left\| \frac{(1 - j_{\gamma+\frac{d}{2}-1}(h|\xi|))^m}{h^{2m}|\xi|^{2m}} |\xi|^{2m} \hat{f}(\xi) \right\|_{L_k^2(\mathbb{R}^d)}.$$

Then

$$\|\Delta_h^m f\|_{L_k^2(\mathbb{R}^d)} \le h^{2m} \left\| \frac{(1 - e_{\gamma+\frac{d}{2}-1}(h|\xi|))^{2m}}{(h|\xi|)^{2m}} |\xi|^{2m} \hat{f}(\xi) \right\|_{L_k^2(\mathbb{R}^d)}. \tag{6}$$

According to Lemma 2.2, for all $s \in \mathbb{R}$ we have the inequality $|(1 - e_\alpha(x))^{2m} s^{-2m}| \le c_2$, where $c_2 = 2^{2m}$. We have

$$\begin{aligned} \|\Delta_h^m f\|_{L_k^2(\mathbb{R}^d)} &\le c_2 h^{2m} \||\xi|^{2m} \hat{f}(\xi)\|_{L_k^2(\mathbb{R}^d)} \\ &= c_2 h^{2m} \|D_k^m f\|_{L_k^2(\mathbb{R}^d)} \le c_2 t^{2m} \|D_k^m f\|_{L_k^2(\mathbb{R}^d)}. \end{aligned}$$

Calculating the supremum with respect to all $h \in (0,t]$, we obtain

$$w_m(f,t)_{2,k} \le c_2 t^{2m} \|D_k^m f\|_{\mathrm{L}_k^2(\mathbb{R}^d)}$$

For any $f \in L_k^2(\mathbb{R}^d)$ and any number $\nu > 0$, let us define the function

$$P_\nu(f)(x) = \digamma^{-1}(\hat{f}(\xi)\chi_\nu(\xi)),$$

where $\chi_\nu(\xi)$ is the function defined by $\chi_\nu(\xi) = 1$, for $|\xi| \le \nu$ and $\chi(\xi) = 0$, for $|\xi| > \nu$, $\digamma^{-1}$ is the inverse Dunkl transform. One can easily prove that the function $P_\nu(f)$ is infinitely differentiable and belongs to all classes $W_{2,k}^m$.

Lemma 2.5. *For any function* $f \in L_k^2(\mathbb{R}^d)$. *Then*

$$\|f - P_\nu(f)\|_{L_k^2(\mathbb{R}^d)} \leq c_4 \|\Delta_{1/\nu}^m f\|_{L_k^2(\mathbb{R}^d)}, \quad \nu > 0$$

Proof. Let $|1 - j_{\gamma+\frac{d}{2}-1}(t)| \geq c_1$ with $|t| \geq 1$ (see Lemma 2.3). Using the Parseval equality, we have

$$\begin{aligned}\|f - P_\nu(f)\|_{L_k^2(\mathbb{R}^d)} &= \|(1-\chi_\nu(\xi))\hat{f}(\xi)\|_{L_k^2(\mathbb{R}^d)} \\ &= \left\| \frac{1-\chi_\nu(\xi)}{\left(1-j_{\gamma+\frac{d}{2}-1}\left(\frac{|\xi|}{\nu}\right)\right)^m} \left(1 - j_{\gamma+\frac{d}{2}-1}\left(\frac{|\xi|}{\nu}\right)\right)^m \right. \\ &\quad \left. \times \hat{f}(\xi) \right\|_{L_k^2(\mathbb{R}^d)}.\end{aligned}$$

Note that

$$\sup_{|\xi|\in\mathbb{R}} \frac{1-\chi_\nu(\xi)}{\left|\left(1 - j_{\gamma+\frac{d}{2}-1}\left(\frac{|\xi|}{\nu}\right)\right)\right|} \leq \frac{1}{c_1^m}$$

Then $\|f - P_\nu(f)\|_{L_k^2(\mathbb{R}^d)} \leq c_1^{-m} \|\left(1-j_{\gamma+\frac{d}{2}-1}\left(\frac{|\xi|}{\nu}\right)\right)^m \hat{f}(\xi)\|_{L_k^2(\mathbb{R}^d)} = c_4 \|\Delta_{1/\nu}^m f\|_{L_k^2(\mathbb{R}^d)}$.

COROLLARY 2.6

$\|f - P_\nu(f)\|_{L_k^2(\mathbb{R}^d)} \leq c_4 w_m(f, 1/\nu)_{2,k}$.

Lemma 2.7. *The following inequality is true*:

$$\|D_k^m(P_\nu(f))\|_{L_k^2(\mathbb{R}^d)} \leq c_5 \nu^{2m} \|\Delta_{1/\nu}^m f\|_{L_k^2(\mathbb{R}^d)}, \quad \nu > 0,\ m \in \{1, 2, \ldots\}.$$

Proof. Using the Parseval equality, we have

$$\begin{aligned}\|D_k^m(P_\nu(f))\|_{L_k^2(\mathbb{R}^d)} &= \|\widehat{D_k^m(P_\nu}(f))\|_{L_k^2(\mathbb{R}^d)} = \||\xi|^{2m}\chi_\nu(\xi)\hat{f}(\xi)\|_{L_k^2(\mathbb{R}^d)} \\ &= \left\| \frac{|\xi|^{2m}\chi_\nu(\xi)}{\left(1 - j_{\gamma+\frac{d}{2}-1}\left(\frac{|\xi|}{\nu}\right)\right)^m} \left(1 - j_{\gamma+\frac{d}{2}-1}\left(\frac{|\xi|}{\nu}\right)\right)^m \right. \\ &\quad \left. \times \hat{f}(\xi) \right\|_{L_k^2(\mathbb{R}^d)}.\end{aligned}$$

Note that

$$\begin{aligned}\sup_{|\xi|\in\mathbb{R}} \frac{|\xi|^{2m}\chi_\nu(\xi)}{|1 - j_{\gamma+\frac{d}{2}-1}\left(\frac{|\xi|}{\nu}\right)|^m} &= \nu^{2m} \sup_{|\xi|\leq\nu} \frac{\left(\frac{|\xi|}{\nu}\right)^{2m}}{|1 - j_{\gamma+\frac{d}{2}-1}\left(\frac{|\xi|}{\nu}\right)|^m} \\ &= \nu^{2m} \sup_{|t|\leq 1} \frac{t^{2m}}{|1 - j_{\gamma+\frac{d}{2}-1}(t)|^m}.\end{aligned}$$

Let

$$c_5 = \sup_{|t|\le 1} \frac{t^{2m}}{|1 - j_{\gamma+\frac{d}{2}-1}(t)|^m}.$$

Then, we have

$$\|D_k^m(P_\nu(f))\|_{L_k^2(\mathbb{R}^d)} \le c_5 \nu^{2m} \|\Delta_{1/\nu}^m f\|_{L_k^2}.$$

COROLLARY 2.8

$\|D_k^m(P_\nu(f))\|_{L_k^2(\mathbb{R}^d)} \le c_5 \nu^{2m} w_m(f, 1/\nu)_{2,k}.$

Theorem 2.9. *One can find positive numbers c_6 and c_7 which the inequality*

$$c_6 w_m(f,\delta)_{2,k} \le K_m(f, \delta^{2m})_{2,k} \le c_7 w_m(f,\delta)_{2,k},$$

$f \in L_k^2(\mathbb{R}^d),\ \delta > 0.$

Proof. Firstly prove of the inequality

$$c_6 w_m(f,\delta)_{2,k} \le K_m(f,\delta^{2m})_{2,k}.$$

Let $h \in (0,\delta],\ g \in W_{2,k}^m$. Using Lemmas 2.1 and 2.4, we have

$$\begin{aligned}\|\Delta_h^m f\|_{L_k^2(\mathbb{R}^d)} &\le \|\Delta_h^m (f-g)\|_{L_k^2(\mathbb{R}^d)} + \|\Delta_h^m g\|_{L_k^2(\mathbb{R}^d)} \\ &\le 2^m \|f-g\|_{L_k^2(\mathbb{R}^d)} + c_2 h^{2m} \|D_k^m g\|_{L_k^2(\mathbb{R}^d)} \\ &\le c_8(\|f-g\|_{L_k^2(\mathbb{R}^d)} + \delta^{2m}\|D_k^m g\|_{L_k^2(\mathbb{R}^d)}),\end{aligned}$$

where $c_8 = \max(2^m, c_2)$. Calculating the supremum with respect to $h \in (0,\delta]$ and the infimum with respect to all possible functions $g \in \mathrm{W}_{2,k}^m$, we obtain

$$w_m(f,\delta)_{2,k} \le c_8 K_m(f,\delta^{2m})_{2,k},$$

whence we get the inequality.

Now, we prove the inequality

$$K_m(f,\delta^{2m})_{2,k} \le c_7 w_m(f,\delta)_{2,k}.$$

Since $P_\nu(f) \in W_{2,k}^m$, by the definition of a K-functional we have

$$K_m(f,\delta^{2m})_{2,k} \le \|f - P_\nu(f)\|_{L_k^2(\mathbb{R}^d)} + \delta^{2m}\|D_k^m P_\nu(f)\|_{L_k^2(\mathbb{R}^d)}.$$

Using Corollaries 2.6 and 2.8, we obtain

$$K_m(f,\delta^{2m})_{2,k} \le c_4 w_m(f,1/\nu)_{2,k} + c_5 \nu^{2m}\delta^{2m} w_m(f,1/\nu)_{2,k},$$

$$K_m(f,\delta^{2m})_{2,k} \le c_4 w_m(f,1/\nu)_{2,k} + c_5 (\nu\delta)^{2m} w_m(f,1/\nu)_{2,k}.$$

Since ν is an arbitrary positive value, choosing $\nu = 1/\delta$, we obtain the inequality.

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