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Perturbation of operators and approximation of spectrum

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Abstract. Let $A(x)$ be a norm continuous family of bounded self-adjoint operators on a separable Hilbert space $\mathbb{H}$ and let $A(x)_n$ be the orthogonal compressions of $A(x)$ to the span of first n elements of an orthonormal basis of $\mathbb{H}$. The problem considered here is to approximate the spectrum of $A(x)$ using the sequence of eigenvalues of $A(x)_n$. We show that the bounds of the essential spectrum and the discrete spectral values outside the bounds of essential spectrum of $A(x)$ can be approximated uniformly on all compact subsets by the sequence of eigenvalue functions of $A(x)_n$. The known results, for a bounded self-adjoint operator, are translated into the case of a norm continuous family of operators. Also an attempt is made to predict the existence of spectral gaps that may occur between the bounds of essential spectrum of $A(0) = A$ and study the effect of norm continuous perturbation of operators in the prediction of spectral gaps. As an example, gap issues of some block Toeplitz–Laurent operators are discussed. The pure linear algebraic approach is the main advantage of the results here.

Keywords. Operator; perturbation; essential spectrum; spectral gap; Toeplitz operators and matrices.

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1. Introduction

Perturbation theory of operators incorporates a good deal of spectral theory. There are many instances in quantum mechanics, where the perturbation of operators arises. For example, the Schrödinger operator

$$\tilde{A}(u) = -\ddot{u} + V \cdot u \tag{1.1}$$

defined on a suitable subspace of $L^2(\mathbb{R})$ can be viewed as a perturbation of differential operator. If we consider the discretized version of this operator, we obtain a bounded operator on $l^2(\mathbb{Z})$, which can be seen as a perturbation of the difference operator, up to some scaling and translation by the identity as defined below:

$$A(\{x_j\}) = \{x_{j+1} + x_{j-1} + v(j)x_j\},\ \ \{x_j\} \in l^2(\mathbb{Z}).$$

Here we discuss the linear algebraic techniques used in [5] and [16], under a norm continuous perturbation of the operator.

Let $\mathbb{H}$ be a separable Hilbert space and A be a bounded self-adjoint operator defined on $\mathbb{H}$. The spectrum of A is denoted by $\sigma(A)$ with m, M as its lower and upper bounds. Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for $\mathbb{H}$. Consider the finite dimensional truncations of A, that is $A_n = P_n A P_n$, where P_n is the projection of $\mathbb{H}$ onto the span of first n elements $\{e_1, e_2, \ldots, e_n\}$ of the basis.

Various mathematicians have done extensive research to use the spectrum of A_n, for computing spectrum $\sigma(A)$ and essential spectrum $\sigma_e(A)$ of A [1, 5, 11, 12, 17]. However prediction of spectral gaps and related problems using truncation method is yet to be investigated in detail, though a brief attempt in this direction has been done in [21]. In this paper, the approximation results in [5] are translated into the case of a norm continuous family of operators $A(x)$. We prove that the bounds of essential spectrum and the discrete spectral values outside the bounds of essential spectrum of $A(x)$ can be approximated uniformly on all compact subsets by a sequence of eigenvalue functions.

Also, some spectral gap prediction results are proved using the finite dimensional truncations. We should mention that gap related problems were studied using analytical and variational techniques, especially for Schrödinger operators with different kinds of potentials. This refers to classical Borg-type theorems which characterized the periodic potentials depending on the nature of spectral gaps (see [8, 14, 24] and references therein and refer to [6, 16] for new perspectives). Here we try for such results in the case of some perturbed discrete Schrödinger operators treating them as block Toeplitz–Laurent operators.

The following is a brief account of some developments in the linear algebraic techniques to the spectral approximation problem, which will play a key role throughout this paper.

1.1 *Linear algebraic approach*

Let ν, μ be the lower and upper bounds of $\sigma_e(A)$ respectively, with A being self-adjoint. Let $\lambda_R^+(A) \le \cdots \le \lambda_2^+(A) \le \lambda_1^+(A)$ be the discrete eigenvalues of A lying above μ and $\lambda_1^-(A) \le \lambda_2^-(A) \le \cdots \le \lambda_S^-(A)$ be the eigenvalues of A lying below ν. Here R and S can be infinity. Denote by $\lambda_1(A_n) \ge \lambda_2(A_n) \ge \cdots \ge \lambda_n(A_n)$ the eigenvalues of A_n. The following result from [5] is of interest in our context:

Theorem 1.1. *For every fixed integer k we have*

$$\lim_{n\to\infty} \lambda_k(A_n) = \begin{cases} \lambda_k^+(A), & \text{if } R = \infty \text{ or } 1 \le k \le R, \\ \mu, & \text{if } R < \infty \text{ and } k \ge R+1, \end{cases}$$

$$\lim_{n\to\infty} \lambda_{n+1-k}(A_n) = \begin{cases} \lambda_k^-(A), & \text{if } S = \infty \text{ or } 1 \le k \le S, \\ \nu, & \text{if } S < \infty \text{ and } k \ge S+1. \end{cases}$$

In particular,

$$\lim_{k\to\infty} \lim_{n\to\infty} \lambda_k(A_n) = \mu \quad \text{and} \quad \lim_{k\to\infty} \lim_{n\to\infty} \lambda_{n+1-k}(A_n) = \nu.$$

Remark 1.2. The above results are also true if we replace A_n by some other sequence A_{1n} with the property that $\|A_n - A_{1n}\| \to 0$ as $n \to \infty$, with $\|\cdot\|$ being the spectral norm. In order to justify this, we need only to recall an important inequality concerning the eigenvalues of self-adjoint matrices A, B (refer e.g. to [2]),

$$|\lambda_k(A) - \lambda_k(B)| \leq \|A - B\|\,. \tag{1.2}$$

The notion of essential points and transient points and their relation with spectrum are quoted below from [1].

DEFINITION 1.3 [1]

Essential points: A real number λ is an essential point if for every open set U containing λ, $\lim_{n\to\infty} N_n(U) = \infty$, where $N_n(U)$ is the number of eigenvalues of A_n in U.

DEFINITION 1.4 [1]

Transient points: A real number λ is transient if there is an open set U containing λ, such that $\sup_{n\geq 1} N_n(U) < \infty$.

DEFINITION 1.5 [1]

The degree of an operator A is defined by the relation

$$\deg(A) = \sup_{n\geq 1} \text{ rank } (P_n A - A P_n).$$

DEFINITION 1.6 [1]

A is an operator in the Arveson's class if $A = \Sigma_n A_n, \deg(A_n) < \infty$ for every n and $\Sigma_n\big(1 + \deg\,(A_n)^{\frac{1}{2}}\big)\|A_n\| < \infty$.

Theorem 1.7 [1]. *If A is a bounded self-adjoint operator, and if we denote*

$$\Lambda = \{\lambda \in R;\, \lambda = \lim \lambda_n,\, \lambda_n \in \sigma(A_n)\}$$

and Λ_e, the set of all essential points, then

$$\sigma(A) \subseteq \Lambda \subseteq [m, M] \quad \text{and} \quad \sigma_e(A) \subseteq \Lambda_e.$$

Theorem 1.8 [1]. *If A is a bounded self-adjoint operator in the Arveson's class, then $\sigma_e(A) = \Lambda_e$ and every point in Λ is either transient or essential.*

The subsequent theorem taken from [5] denies the existence of spurious eigenvalues (points in Λ which are not spectral values), under the assumption that the essential spectrum is connected.

Theorem 1.9. *If A is a self-adjoint operator and if $\sigma_e(A)$ is connected, then $\sigma(A) = \Lambda$.*

Remark 1.10. It is worthwhile to notice that the connectedness of essential spectrum enables us to compute the spectrum using finite dimensional truncations.

The paper is organized as follows. In §2, the approximation results are extended to the case of a one-parameter norm continuous family of operators. In §3, the spectral gap prediction results are proved with some examples. Also, we make observations of what happens to the spectral gaps under a norm continuous perturbation. In the fourth section, results on the spectral gaps of some block Toeplitz–Laurent operators are reported. We present the modified version of discrete Borg's theorem with the techniques used in [16]. This section is not directly linked to the previous sections, however it deals with examples of perturbed operators and their spectral gap issues. To be more precise, the absence of spectral gaps ensures the triviality of the potential sequence that appears in the discretized Schrödinger operator. The proofs are obtained by looking at the block Toeplitz structure of the matrix representation of the operators. A concluding section ends the paper.

2. Spectrum under perturbation

Let $A(x)$ be a norm continuous family of operators with domain D_0 in the complex plane. Our aim is to study the changes in the behavior of spectrum, under these perturbations. We generalize the approximation techniques used in the case of a single operator, to a norm continuous family of operators.

First we define the approximation number functions as follows.

DEFINITION 2.1

Consider the singular number s_k, k natural number,

$$s_k(A(x)) = \inf\{\|A(x) - F\|\,,\ \operatorname{rank}(F) \leq k-1\}, \quad x \in D_0$$

which is the k-th *approximation number function* of $A(x)$.

Clearly we have for each $x \in D_0$,

$$\|A(x)\| = s_1(A(x)) \geq s_2(A(x)) \geq \cdots \geq s_k(A(x)) \geq \cdots \geq 0. \tag{2.1}$$

Recall the definition of essential norm.

DEFINITION 2.2

$\|A(x)\|_{\text{ess}} = \inf\{\|A(x) - K\|, K \text{ compact}\}, x \in D_0$.

The following lemmas are easy consequences of the continuity of $A(x)$.

Lemma 2.3. *$s_k(A(.)) \to \|A(.)\|_{\text{ess}}$ as $k \to \infty$ uniformly on all compact subsets of D_0.*

Proof. Consider the sequence of functions $f_k(x) = s_k(A(x))$. From [15], we have for each x,

$$f_k(x) = s_k(A(x)) \to \|A(x)\|_{\text{ess}}\,.$$

Also since

$$|f_k(x) - f_k(y)| = |s_k(A(x)) - s_k(A(y))| \leq \|A(x) - A(y)\|\,, \tag{2.2}$$

and $A(x)$ is norm continuous, we observe that each function in the sequence are continuous. Hence using the monotonicity of the sequence of functions in (2.1), we conclude that the convergence is uniform in each compact subsets, by Dini's theorem (see p. 150 of [23]). Hence the proof. □

Now we consider the truncations $A(x)_n = P_n A(x) P_n$ and singular numbers $s_k(A(x)_n) = \inf\{\|A(x)_n - F_n\|,\ \text{rank}\,(F_n) \leq k-1\}$.

Lemma 2.4. *$s_k(A(x)_n) \to s_k(A(x))$ as $n \to \infty$, for each k, and the convergence is uniform on all compact subsets of D_0.*

Proof. Our first observation is that the sequence of functions $f_{n,k}(x) = s_k(A(x)_n)$ form an equicontinuous family of functions. This follows from the following inequality:

$$\begin{aligned}|f_{n,k}(x) - f_{n,k}(y)| = |s_k(A(x)_n) - s_k(A(y)_n)| &\leq \|A(x)_n - A(y)_n\| \\ &\leq \|A(x) - A(y)\|.\end{aligned}$$

Also from the interlacing theorem for singular values (see [2] for the proof), we have

$$f_{n,k}(x) = s_k(A(x)_n) \geq s_k(A(x)_{n-1}) = f_{n-1,k}(x),$$

for each k and for every $x \in D_0$. Hence the sequence of singular value functions form a monotone sequence of functions. Also by Theorem (1.1) of [5],

$$f_{n,k}(x) = s_k(A(x)_n) \to s_k(A(x)) \text{ as } n \to \infty,$$

for each k and for all $x \in D_0$. Now by Dini's theorem, the convergence is uniform on all compact subsets of D_0 and the proof is complete. □

For the rest of this paper, we assume that $A(x)$ is self-adjoint for each x. Let $\nu(x)$, $\mu(x)$ be the lower and upper bounds of $\sigma_e(A(x))$ respectively, and also let the numbers $\lambda_R^+(A(x)) \leq \cdots \leq \lambda_2^+(A(x)) \leq \lambda_1^+(A(x))$ be the discrete eigenvalues of $A(x)$ lying above $\mu(x)$, and $\lambda_1^-(A(x)) \leq \lambda_2^-(A(x)) \leq \cdots \leq \lambda_S^-(A(x))$ be the eigenvalues lying below $\nu(x)$. Here R and S can be infinity. The quantities $\lambda_{1,n}(x) \geq \lambda_{2,n}(x) \geq \cdots \geq \lambda_{n,n}(x)$ denote the eigenvalues of $A(x)_n$ in non increasing order.

Theorem 2.5.

$$\lim_{n\to\infty} \lambda_{k,n}(x) = \begin{cases} \lambda_k^+(x), & \text{if } R = \infty \text{ or } 1 \leq k \leq R, \\ \mu(x), & \text{if } R < \infty \text{ and } k \geq R+1, \end{cases}$$

$$\lim_{n\to\infty} \lambda_{n+1-k,n}(x) = \begin{cases} \lambda_k^-(x), & \text{if } S = \infty \text{ or } 1 \leq k \leq S, \\ \nu(x), & \text{if } S < \infty \text{ and } k \geq S+1. \end{cases}$$

In particular,

$$\lim_{k\to\infty}\lim_{n\to\infty} \lambda_{k,n}(x) = \mu(x) \quad \textit{and} \quad \lim_{k\to\infty}\lim_{n\to\infty} \lambda_{n+1-k,n}(x) = \nu(x).$$

Furthermore, *in each of the cases given above*, *the convergence is uniform on all compact subsets of D_0.*

Proof. For each fixed $x \in D_0$, these limits exist by Theorem 1.1. We observe the fact that the sequence of eigenvalue functions, $f_{n,k}(x) = \lambda_{k,n}(x)$ form an equicontinuous family of functions, from the following inequalities:

$$\begin{aligned}|f_{n,k}(x) - f_{n,k}(y)| &= |\lambda_{k,n}(x) - \lambda_{k,n}(y)| \leq \|A(x)_n - A(y)_n\| \\ &\leq \|A(x) - A(y)\|.\end{aligned}$$

Also by Cauchy's interlacing theorem for eigenvalues,

$$\lambda_{1,n+1}(x) \geq \lambda_{1,n}(x) \geq \lambda_{2,n+1}(x) \geq \cdots \lambda_{n,n+1}(x) \geq \lambda_{n,n}(x) \geq \lambda_{n+1,n+1}(x),$$

for each $x \in D_0$. In particular, for each k and for every $x \in D_0$,

$$f_{n+1,k}(x) = \lambda_{k,n+1}(x) \geq \lambda_{k,n}(x) = f_{n,k}(x).$$

Hence $f_{n,k}(.)$ forms a monotone sequence of continuous functions that converges point wise. Therefore by Dini's theorem, the convergence is uniform on all compact subsets of D_0. Hence the proof is complete. □

Remark 2.6. Using Theorem 2.5, we can approximate the discrete spectrum of a norm continuous family of operators, lying outside the bounds of essential spectrum by the eigenvalue functions of truncations uniformly on all compact subsets.

It was observed in [5] that norm of ${A_n}^{-1}$ is uniformly bounded if A is invertible and the essential spectrum is connected. The perturbed version of this result is proved below.

COROLLARY 2.7

Let $A(x)$ be a norm continuous family of bounded self-adjoint operators such that $\sigma_e(A(x))$ is connected for all x in the domain D_0. Then

$$\lim_{n\to\infty} \|(A(x)_n - \lambda I_n)^{-1}\| = \|(A(x) - \lambda I)^{-1}\| \quad \textit{for every } \lambda \in \mathbb{C} - \mathbb{R}.$$

Also the convergence is uniform on all compact subsets of D_0.

Proof. By Theorem 1.9, $\sigma(A(x)) = \Lambda(A(x))$. Hence we can easily observe the following:

$$\begin{aligned} d(z, \sigma(A(x)_n)) &\to d(z, \Lambda(A(x)) \\ &= d(z, \sigma(A(x))) \text{ for every complex number } z. \end{aligned}$$

Therefore, for every non real z,

$$\begin{aligned} \|(A(x)_n - \lambda I_n)^{-1}\| &= \frac{1}{d(z, \sigma(A(x)_n))} \to \frac{1}{d(z, \sigma(A(x)))} \\ &= \|(A(x) - \lambda I)^{-1}\|. \end{aligned}$$

Also the convergence is uniform on all compact subsets of D_0 as observed in the previous theorems. □

3. Gaps in the essential spectrum

Now we consider the problem to predict the gaps in the essential spectrum, if any. An interval I is called spectral gap if there exist real sets J_1, J_2 containing the spectrum of A such that $\sup J_1 \leq \inf I < \sup I \leq \inf J_2$. We are interested in the gaps that lie between the bounds of essential spectrum of A. Also the intervals between these bounds, containing only discrete eigenvalues, are treated as spectral gaps. The following theorem is an attempt to predict the existence of spectral gaps, using the finite dimensional truncations. The perturbed versions are attempted at the end of this section.

We use the notation $\#S$ to denote the number of elements in the set S. From here onwards the convex combination of eigenvalues of truncations will be considered at many places. The notation w_{nk} is used to denote an averaging sequence. That is, $0 \leq w_{nk} \leq 1$ and $\sum_{k=1}^{n} w_{nk} = 1$.

Theorem 3.1. *Let A be a bounded self-adjoint operator, and $\lambda_{n1}(A_n) \geq \lambda_{n2}(A_n) \geq \cdots \geq \lambda_{nn}(A_n)$ be the eigenvalues of A_n arranged in decreasing order. For each positive integer n, let $a_n = \sum_{k=1}^{n} w_{nk}\lambda_{nk}$ be the convex combination of eigenvalues of A_n. If there exists a $\delta > 0$ and $K > 0$ such that*

$$\#\{\lambda_{nj};\ |a_n - \lambda_{nj}| < \delta\} < K \tag{3.1}$$

and in addition if $\sigma_e(A)$ and $\sigma(A)$ has the same upper and lower bounds, then $\sigma_e(A)$ has a gap.

Proof. Consider the set $S = \{a_n, n = 1, 2, 3, \ldots\}$ and observe that $\lambda_{nn} \leq a_n \leq \lambda_{n1}$. Also since each λ_{nj}'s lie in the interval $[m, M]$, we see that the set S is contained in the interval $[m, M] = [\nu, \mu]$.

First we consider the case when S is a finite set, say $S = \{a_{s1}, a_{s2}, a_{s3}, \ldots, a_{sm}\}$. In this case, there exists finitely many numbers, say $a_{n1}, a_{n2}, a_{n3}, \ldots, a_{np}$ such that the value of a_n equals some of the numbers a_{ni}'s, $i = 1, 2, \ldots, p$, for infinitely many n. That is

$$a_n = a_{ni} \quad \text{for infinitely many } n \text{ where } i = 1, 2, \ldots, p.$$

From this and by condition (3.1), for each $i = 1, 2, \ldots, p$, we have

$$N_n(a_{ni} - \delta, a_{ni} + \delta) = \#\{\lambda_{nj};\ |a_{ni} - \lambda_{nj}| < \delta\} < K \ \text{ for infinitely many } n.$$

Hence $N_n(a_{ni} - \delta, a_{ni} + \delta)$ will not go to infinity as n goes to infinity. Therefore no number in the interval $(a_{ni} - \delta, a_{ni} + \delta)$ is an essential point. Since the essential spectrum is contained in the set of all essential points, by Theorem 1.7, there is no essential spectral values in this interval. Also since each a_{ni} lies between the bounds of essential spectrum, we can choose an appropriate $\epsilon > 0$ such that $(a_{ni} - \epsilon, a_{ni} + \epsilon)$ lies between the bounds and contained in the interval $(a_{ni} - \delta, a_{ni} + \delta)$. Then the interval $(a_{ni} - \epsilon, a_{ni} + \epsilon)$ is a spectral gap.

Now we consider the case when S is an infinite set. Hence S has a limit point in $\mathbb{R}$. Now if w_0 is a limit point of the set S, then we have $\nu \leq w_0 \leq \mu$.

Now the interval $(w_0 - \delta/2, w_0 + \delta/2)$ will contain infinitely many points from the set S. Corresponding to these points, there are infinitely many A_n's for which the number of eigenvalues in $(w_0 - \delta/2, w_0 + \delta/2)$ is bounded by K due to (3.1). Hence the sequence $N_n\left(w_0 - \frac{\delta}{2}, w_0 + \frac{\delta}{2}\right)$ will not go to infinity, since a subsequence of it, is bounded by K. Hence no point in the interval $(w_0 - \delta/2, w_0 + \delta/2)$ is an essential point. Since the essential spectrum is contained the set of all essential points, by Theorem 1.7, $(w_0 - \delta/2, w_0 + \delta/2)$ contains no essential spectral values. Hence, as in Case 1, we can choose an $\epsilon > 0$, such that the interval $(w_0 - \epsilon, w_0 + \epsilon)$ is a spectral gap between the bounds of the essential spectrum and the proof is complete. □

Remark 3.2. There is possibility for the presence of discrete eigenvalues inside the gaps in the above case.

Remark 3.3. The special case which is more interesting is when $w_{nk} = \frac{1}{n}$, for all n. In that case, we are actually looking at the averages of eigenvalues of truncations and these averages can be computed using the trace at each level.

Remark 3.4. It is to be noted that all the points of the form $a_n = \sum_{k=1}^{n} w_{nk}\lambda_{nk}$ are in the numerical range of A_n. Therefore the result can be made simpler in the language of numerical range. However it is not easy to compute the numbers in the expression (3.1). Here we treated it as a deviation from the mean value. Hence the condition (3.1) may be interpreted as a restriction to the deviation of the eigenvalues of truncations from their central tendency. Nevertheless the computations still remains difficult.

Special choice I

Let us consider an instance where these weights w_{nk} arises naturally associated to a self-adjoint operator on a Hilbert space. Let $A_n = \sum_{k=1}^{n} \lambda_{n,k} Q_{n,k}$ be the spectral resolution of A_n. Define $w_{nk} = \langle Q_{n,k}e_1, e_1\rangle$. Then $0 \leq w_{nk} \leq 1$ and $\sum_{k=1}^{n} w_{nk} = 1$. Now

$$\sum_{k=1}^{n} w_{nk}\lambda_{nk} = \sum_{k=1}^{n} \lambda_{nk}\langle Q_{n,k}e_1, e_1\rangle = \langle A_n e_1, e_1\rangle = \langle Ae_1, e_1\rangle = a_{11}.$$

Therefore by Theorem 3.1, if there exists a $\delta > 0$ and a $K > 0$, such that

$$\#\{\lambda_{nj};\, |a_{11} - \lambda_{nj}| < \delta\} < K$$

then there exists a gap in the essential spectrum of A. Hence if the first entry in the matrix representation of A, is not an essential point, then there exists a gap in the essential spectrum.

Remark 3.5. All points of the form $\langle Ae_i, e_i\rangle = a_{ii}$ are in the numerical range which lies between the bounds of the essential spectrum, in the case that the bounds coincide with the bounds of the spectrum. Hence in that case, if a_{ii} is not an essential point for some i, then that will lead to the existence of a spectral gap. That means if any one of the diagonal entries in the matrix representation of A is not an essential point, then there exists a gap in the essential spectrum as indicated in the above special choice of w_{nk}.

The following is an example where the first entry a_{11} is a transient point and the spectral gap prediction is valid.

Example 3.6. Define a bounded self-adjoint operator A on $l^2(\mathbb{N})$, as follows.

$$A(x_n) = (x_{n-1} + x_{n+1}) + (v_n x_n),\, x_0 = 0,$$

where the periodic sequence $v_n = (1, 2, 3, 1, 2, 3, \ldots)$. This is a discretized version of the well-known Schrödinger operator. The matrix representation of A results in the block Toeplitz operator with corresponding matrix valued symbol given by

$$\tilde{f}(\theta) = \begin{bmatrix} 1 & 1 & \mathrm{e}^{i\theta} \\ 1 & 2 & 1 \\ \mathrm{e}^{-i\theta} & 1 & 3 \end{bmatrix}.$$

As indicated in the special choice above, by Theorem 3.1, if $\langle A(e_1), e_1\rangle = 1$ is a transient point, then $\sigma_e(A)$ has a gap. This is evident in this example, since from [3],

$$\sigma_e(A) = \bigcup_{j=1}^{3} \left[\inf_\theta(\lambda_j(\tilde{f}(\theta)), \sup_\theta(\lambda_j(\tilde{f}(\theta))\right],$$

where $\lambda_j(\tilde{f}(\theta))$ are the eigenvalues of $\tilde{f}(\theta)$. A straightforward numerical computation of the eigenvalue functions gives

$$\sigma_e(A) = [-0.2143, 0.3249] \cup [1.4608, 2.5392] \cup [3.6751, 4.2143].$$

Also since A is in the Arveson's class (all band-limited matrices comes in this class), the point 1 lies in the gap, is a transient point. Hence the prediction of the existence of gap, in Theorem 3.1, is valid in this example.

Special choice II

By invoking Theorem 1.1, there exists a sequence of eigenvalues of truncations $\lambda_{n_l}, \lambda_{n_m}$ such that $\lim_{n_l\to\infty} \lambda_{n_l} = \nu$ and $\lim_{n_m\to\infty} \lambda_{n_m} = \mu$. Define

$$w_{nk} = \begin{cases} t, & if\ k = l, \\ 1-t, & if\ k = m, \\ 0, & \text{otherwise}, \end{cases}$$

where $t \in (0, 1)$. If there exist $\delta > 0$ and $K > 0$ such that

$$\#\{\lambda_{nj};\ |t\lambda_{nl} + (1-t)\lambda_{nm} - \lambda_{nj}| < \delta\} < K,$$

then $\sigma_e(A)$ has a gap of width larger than δ.

The advantage of this special choice is that we are able to avoid the assumptions on the bounds of $\sigma(A)$ and $\sigma_e(A)$. This shows that a more general result is possible, provided that we choose the sequence of numbers w_{nk} carefully. In the following theorem, we observe that the converse of Theorem 3.1 is true in the case of operators in the Arveson's class.

Theorem 3.7. *Let A be a bounded self-adjoint operator in the Arveson's class. And suppose that there exists a gap in the essential spectrum. Then there exists a sequence of numbers* $a_n = \sum_{k=1}^{n} w_{nk}\lambda_{nk}$ *in the numerical ranges of* A_n'*s and a* $\delta > 0$ *such that*

$$\#\{\lambda_{nj};\ |a_n - \lambda_{nj}| < \delta\} < K,$$

for some $K > 0$.

Proof. Let (a, b) be a gap in the essential spectrum. Then by Theorem 1.7, there exists sequences of eigenvalues of truncations $\lambda_{n_l}, \lambda_{n_m}$ such that

$$\lim_{n_l\to\infty} \lambda_{n_l} = a \quad \text{and} \quad \lim_{n_m\to\infty} \lambda_{n_m} = b.$$

Define

$$a_n = a_n(t) = t\lambda_{n_l} + (1-t)\,\lambda_{n_m},$$

for some fixed $t \in (0,1)$. Since $c_t = ta + (1-t)b \in (a,b)$, it is not an essential point. Also since A is in the Arveson's class, all such points are transient points by Theorem 1.8. Hence there exists a $\delta_1 > 0$ such that

$$\sup N_n(c_t - \delta_1, c_t + \delta_1) < K_1 \quad \text{for some } K_1 > 0.$$

Also

$$a_n = t\lambda_{n_l} + (1-t)\,\lambda_{n_m} \to ta + (1-t)b = c_t \quad \text{as } n \to \infty.$$

Therefore there exists an N such that $|c_t - a_n| < \delta_1/2$ for all $n > N$. Now if for some $n > N$, $|a_n - \lambda_{nj}| < \delta_1/2$ then $|c_t - \lambda_{nj}| < \delta_1$. Therefore,

$$\#\left\{\lambda_{nj};\, |a_n - \lambda_{nj}| < \frac{\delta_1}{2}\right\} < N_n(c_t - \delta_1, c_t + \delta_1) < K_1, \quad \text{for all } n > N.$$

Now choosing $K = \sup\{K_1, N\}$ and $\delta = \frac{\delta_1}{2}$, we complete the proof. □

Remark 3.8. In the above proof, numbers $\{w_{nk} : k = 1, 2, \ldots n\}$ and the bound K will depend on the particular $t \in (0,1)$ that we choose.

3.1 *Gaps under perturbation*

Now we look at the spectral gaps that may occur between the bounds of the essential spectrum of a norm continuous family of self-adjoint operators. Recall that the gaps remain invariant under a compact perturbation of the operator. The question that we address here is how stable these gaps are, under a more general perturbation. We need the following theorem to achieve some invariance for the gaps.

Theorem 3.9. *Let A and B are bounded operators and A is invertible. If the quantity $\|A^{-1}\|\,\|B\| < 1$, then $A+B$ is also invertible.*

Now the following theorem is an immediate consequence.

Theorem 3.10. *Let (a,b) be a gap in $\sigma_e(A(0))$ which contains no discrete spectral value in it. Then for all small enough $\varepsilon > 0$, there exists a $\delta > 0$ such that $(a+\varepsilon, b-\varepsilon)$ is a gap in the essential spectrum of the norm continuous family of operators $A(x)$, for every x with $|x| < \delta$.*

Proof. First we note that $A - \lambda I$ is invertible for every λ in the interval (a,b), since it contains no spectral value. Therefore,

$$\sup\{\|(A-\lambda I)^{-1}\|;\, \lambda \in (a+\varepsilon, b-\varepsilon)\} = M < \infty \text{ for a fixed } \varepsilon > 0.$$

Now using the continuity assumption corresponding to minimum of $\left\{\frac{1}{M}, \epsilon\right\}$, there exists a $\delta > 0$ such that

$$\|(A(x) - A(0)\| < \min\left\{\frac{1}{M}, \epsilon\right\} \quad \text{for every } x \text{ with } |x| < \delta.$$

Now for $|x| < \delta$, observe that

$$\|(A - \lambda I)^{-1}\| \|A(x) - A(0)\| < M \cdot \frac{1}{M} < 1$$

for every λ in the interval $(a + \varepsilon, b - \varepsilon)$.

Hence by Theorem 3.9, if $|x| < \delta$, then

$$A(x) - \lambda I = A(x) - A(0) + A(0) - \lambda I$$

is invertible for every λ in the interval $(a+\varepsilon, b-\varepsilon)$. Therefore the interval $(a+\varepsilon, b-\varepsilon)$ does not intersect with $\sigma(A(x))$, for every x with $|x| < \delta$.

Now, since $\|A(x) - A(0)\| < \epsilon$, $(a+\varepsilon, b-\varepsilon)$ will lie between the bounds of $\sigma_e(A(x))$, for every x with $|x| < \delta$. We conclude that $(a + \varepsilon, b - \varepsilon)$ is a spectral gap in $\sigma_e(A(x))$ for all x, with $|x| < \delta$. □

Remark 3.11. In Theorem 3.10, ϵ must be small enough so that the interval $(a+\varepsilon, b-\varepsilon)$ should make sense. This theorem indicates that to some extent, the gaps are stable under small norm perturbation. Once we get $(a+\varepsilon, b-\varepsilon)$ is a gap, we may remove that interval and look at the rest of the interval (a, b) and continue the search for gaps.

Let us look at an example to support the above theorem.

Example 3.12. Define a two parameter family of matrix valued symbols as follows:

$$f(x,\theta) = \begin{bmatrix} a_1(x) & 1 & & & & & e^{-i\theta} \\ 1 & a_2(x) & 1 & & & & \\ & 1 & a_3(x) & 1 & & & \\ & & 1 & a_4(x) & 1 & & \\ & & & & \ddots & \ddots & \ddots \\ e^{i\theta} & & & & & 1 & a_p(x) \end{bmatrix},$$

where $a_1(.), a_2(.) \ldots a_p(.)$ are real continuous functions and θ varying in the interval $[0, 2\pi]$. Note that

$$f(x,\theta) = A_0(x) + A_1 e^{i\theta} + A_{-1} e^{-i\theta}, \text{ where}$$

$$A_0(x) = \begin{bmatrix} a_1(x) & 1 & & & \\ 1 & a_2(x) & 1 & & \\ & 1 & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & 1 \\ & & & & 1 & a_p(x) \end{bmatrix},$$

$$A_1 = \begin{bmatrix} & & \\ & & \\ & & \\ 1 & & \end{bmatrix} = A_{-1}{}^T.$$

We consider the one-parameter family of block Toeplitz–Laurent operators arising from these symbols, which are represented by the following doubly infinite matrices:

$$A(x) = \begin{bmatrix} \ddots & \ddots & & & & & & \\ \ddots & A_0(x) & A_{-1} & & & & & \\ & A_1 & A_0(x) & A_{-1} & & & & \\ & & A_1 & A_0(x) & A_{-1} & & & \\ & & & A_1 & A_0(x) & A_{-1} & & \\ & & & & A_1 & A_0(x) & A_{-1} & \\ & & & & & A_1 & A_0(x) & \ddots \\ & & & & & & \ddots & \ddots \end{bmatrix}.$$

Thus we get a norm continuous family of bounded operators, $A(x)$ which are self-adjoint for all x in the domain. Now, by Borg's theorem for discrete Schrödinger operator [14, 16], the essential spectrum of $A(x_0)$, has no gaps if and only if $a_1(x_0) = a_2(x_0) \cdots = a_p(x_0)$. Hence if there is a gap in $\sigma_e(A(0))$, then $a_i(0) < a_{i+1}(0)$ for some i. Using the continuity of a_i and a_{i+1}, we can find a $\delta > 0$ such that $a_i(x) < a_{i+1}(x)$ for all x with $|x| < \delta$. Hence there is a gap for $A(x)$ for all such x.

Using Theorems 3.1 and 3.10, we arrive at the following conclusions. The gap predictions that we have done for a single operator will remain valid for a family of operators. The advantage is that we can predict gaps of a family of operators, with assumptions only on the unperturbed operator. We give the precise statement below.

COROLLARY 3.13

Let A(x) be a norm continuous family of operators with $A(0) = A$, *and* $\lambda_{n1}(A_n) \geq \lambda_{n2}(A_n) \geq \cdots \geq \lambda_{nn}(A_n)$ *be the eigenvalues of* A_n *arranged in decreasing order. For each positive integer n, let* $a_n = \sum_{k=1}^{n} w_{nk}\lambda_{nk}$ *be the convex combination of eigenvalues of* A_n. *If there exists a* $\delta > 0$ *and* $K > 0$ *such that*

$$\#\{\lambda_{nj};\, |a_n - \lambda_{nj}| < \delta\} < K,$$

and $\sigma_e(A)$ *coincides with* $\sigma(A)$, *then* $\sigma_e(A(x))$ *has a gap for each x in a sufficiently small neighborhood of* 0.

Proof. By Theorem 3.1, $A(0)$ has a spectral gap. By Theorem 3.10, there exists a neighborhood of 0, and $A(x)$ has gaps for all x in the neighborhood. Hence the proof. □

Remark 3.14. In the above case, we considered perturbation of operators and not the perturbations of their truncations. In Example 3.12 also the perturbed symbol is directly related to the perturbation of operators. The perturbation of truncations and their link with the spectrum of the original operator is another problem yet to be handled.

Remark 3.15. The stability of the predictions of gaps under the stronger assumptions such as Lipschitz continuous or holomorphic perturbations is another question to be addressed. We expect that Theorem 3.10 can be made better under such stronger conditions. If the perturbation is Lipschitz continuous, then the number δ that appears in Theorem 3.10 can be estimated sharply. If the perturbation is holomorphic, not only that it is Lipschitz continuous, but one may also get better information regarding the changes in the discrete spectrum inside the gaps (see, for instance, [18]).

For the sake of completion, definitions of holomorphic and Lipschitz continuous perturbations are given below.

DEFINITION 3.16

A family of operators $A(x)$, with complex domain D_0 is holomorphic if there exists a family of operators $A^{'}(x)$ such that for each $y \in D_0$, the following limit

$$\lim_{x \to y} \left\| \frac{A(x) - A(y)}{|x - y|} - A^{'}(y) \right\|$$

should exist and must be zero.

DEFINITION 3.17

A family of operators $A(x)$, with complex domain D_0 is Lipschitz continuous if there exists a $M > 0$, such that

$$\|A(x) - A(y)\| \leq M|x - y|, \text{ for all } x, y \in D_0.$$

4. Gap issues of block Toeplitz–Laurent operators

In this section, we look at the spectral gap issues of some block Toeplitz–Laurent operators. The operators under our concern are some perturbations of discrete Schrödinger operator on $l^2(Z)$.

The classical Borg's theorem states that the Schrödinger operator with periodic potential has a connected essential spectrum, if and only if the periodic potential reduces to a constant [8, 24]. The discretized version of this theorem can be found in [14], and a pure linear algebraic proof in [16]. Here we extend the result into the case of some perturbation of discrete Schrödinger operator with periodic potential. We refer to [16] for a detailed description of the discretization of Schrödinger operator and formulating the discrete version in terms of block Toeplitz–Laurent operators and the matrix valued symbol.

Theorem 4.1. *Let A be the bounded operator defined by the block Toeplitz–Laurent matrix*

$$A = \begin{bmatrix} \ddots & \ddots & & & & & & & & & & \\ \ddots & A_0 & A_{-1} & A_{-2} & & \dots & A_{-N} & \dots & & & & \\ & A_1 & A_0 & A_{-1} & A_{-2} & & \dots & A_{-N} & \dots & & & \\ & A_2 & A_1 & A_0 & A_{-1} & A_{-2} & & \dots & A_{-N} & \dots & & \\ & & A_2 & A_1 & A_0 & A_{-1} & A_{-2} & & \dots & A_{-N} & & \\ \dots & A_N & \dots & A_2 & A_1 & A_0 & A_{-1} & A_{-2} & & & & \\ & \dots & A_N & \dots & A_2 & A_1 & A_0 & A_{-1} & A_{-2} & & & \\ & & \dots & A_N & \dots & A_2 & A_1 & A_0 & A_{-1} & A_{-2} & & \\ & & & \dots & A_N & \dots & A_2 & A_1 & A_0 & \ddots & & \\ & & & & & & & & \ddots & \ddots & & \end{bmatrix},$$

where

$$A_0 = \begin{bmatrix} b_1 & 1 & & & & a_0 \\ 1 & b_2 & 1 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ a_0 & & & & 1 & b_p \end{bmatrix}, \qquad A_k = \begin{bmatrix} & & a_k \\ & & \\ & & \\ & & \\ & & \\ & & \end{bmatrix} = {A_{-k}}^T,$$

such that $b_1 \leq b_2 \leq \cdots \leq b_p$ *and* $\sum_k |a_k| < \infty$. *If A has connected essential spectrum, then* $b_1 = b_2 = \cdots = b_p$.

Proof. The matrix-valued symbol associated with the block Toeplitz–Laurent operator A is

$$\tilde{f}(\theta) = \begin{bmatrix} b_1 & 1 & & & & f(\theta) \\ 1 & b_2 & 1 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ \bar{f}(\theta) & & & & 1 & b_p \end{bmatrix},$$

where $f(\theta) = \sum_k a_k \mathrm{e}^{ik\theta}$. Therefore from [3] we have

$$\sigma_e(A) = \bigcup_{j=1}^{p} \left[\inf_{\theta}(\lambda_j(\tilde{f}(\theta)), \sup_{\theta}(\lambda_j(\tilde{f}(\theta)) \right]. \tag{4.1}$$

Now consider the sub matrices

$$P_1 = \begin{bmatrix} b_1 & 1 & & & & \\ 1 & b_2 & 1 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ & & & & 1 & b_{p-1} \end{bmatrix}, \qquad P_2 = \begin{bmatrix} b_2 & 1 & & & & \\ 1 & b_3 & 1 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ & & & & 1 & b_p \end{bmatrix}.$$

If any of their eigenvalues are different, say $\lambda_j(P_1) < \lambda_j(P_2)$, then by Cauchy interlacing theorem, $\lambda_j(\tilde{f}(\theta)) \leq \lambda_j(P_1) < \lambda_j(P_2) \leq \lambda_{j+1}(\tilde{f}(\theta))$, for all θ. But from (4.1), this will give us the contradiction that essential spectrum of A is not connected. Hence all the eigenvalues of P_1 and P_2 are the same. Therefore

$$\mathrm{trace}(P_1) - \mathrm{trace}(P_2) = b_1 - b_p = 0.$$

Hence $b_1 = b_2 = \cdots = b_p$. □

Remark 4.2. The converse of the above assertion is in general not true. We may have gaps even if the diagonal entries of the block Toeplitz–Laurent operator are the same. For, if A is the block Toeplitz–Laurent operator arising from the matrix valued symbol,

$$\tilde{f}(\theta) = \begin{bmatrix} b & 1 + f(\theta) \\ 1 + \bar{f(\theta)} & b \end{bmatrix},$$

where f is a non negative function. Then the eigenvalue functions of $\tilde{f}(\theta)$ are

$$\lambda_1(\theta) = b - 1 - f(\theta), \quad \lambda_2(\theta) = b + 1 + f(\theta).$$

Hence spectrum of A will have a gap, since f is non negative.

Remark 4.3. We remark that the diagonal entries correspond to the periodic potential of the discrete Schrödinger operator. Hence we have proved the discrete Borg-type theorem for a perturbed operator with some extra assumptions on the potential.

Example 4.4. The assumption $b_1 \leq b_2 \leq \cdots \leq b_p$ can not be dropped in the above theorem, if $p > 2$. For, if we consider the block Toeplitz–Laurent operator arising from the matrix-valued symbol,

$$\tilde{f}(\theta) = \begin{bmatrix} 1 & 1 & 0 & 10\cos(\theta) \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 10\cos(\theta) & 0 & 1 & 1 \end{bmatrix}.$$

The eigenvalue functions of $\tilde{f}(\theta)$ are

$$\lambda_{1,2}(\theta) = 2 + 5\cos(\theta) \pm \sqrt{25\cos^2(\theta) - 10\cos(\theta) + 2},$$

$$\lambda_{3,4}(\theta) = 1 - 5\cos(\theta) \pm \sqrt{25\cos^2(\theta) + 1}.$$

We list the values of these functions at certain points in the table below.

θ	$\lambda_1(\theta)$	$\lambda_2(\theta)$	$\lambda_3(\theta)$	$\lambda_4(\theta)$
0	11.123	2.877	1.099	−9.099
π	3.083	−9.083	11.099	0.901

From the table, it is clear that the ranges of the above continuous functions intersect. Hence their union is a connected interval. Therefore the essential spectrum of the operator has no gaps, even the periodic potential does not reduce to a constant.

4.1 *Perturbation of matrices*

Finally we use some known results on the bounds for the eigenvalues of perturbed matrices (see [2, 20] and references therein) to strengthen our results by viewing the matrix-valued symbol as a perturbation of some constant matrix.

Lemma 4.5. *Let* $H = \begin{pmatrix} H_1 & E \\ E^* & H_1 \end{pmatrix}$ *and* $\tilde{H} = \begin{pmatrix} H_1 & 0 \\ 0 & H_1 \end{pmatrix}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ *and* $\tilde{\lambda_1} \geq \tilde{\lambda_2} \geq \cdots \geq \tilde{\lambda_p}$ *be the eigenvalues respectively. Then*

$$|\lambda_j - \tilde{\lambda_j}| \leq \|E\|. \tag{4.2}$$

Theorem 4.6. *Let A be the operator considered in Theorem* 4.1. *If* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ *are eigenvalues of the matrix*

$$\begin{bmatrix} b_1 & 1 & & & & \\ 1 & b_2 & 1 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ & & & & 1 & b_p \end{bmatrix},$$

and $\sigma_e(A)$ has no gap, then $|\lambda_j - \lambda_{j+1}| \leq 2\,\|f\|_\infty$ for all $j = 1, 2, \ldots, p-1$. In addition, if we assume that $b_1 \leq b_2 \leq \cdots \leq b_p$, then $|\lambda_j - \lambda_j| \leq 2$ for all $j = 1, 2, \ldots, p$.

Proof. Apply the above lemma with

$$H(\cdot) = \begin{bmatrix} b_1 & 1 & & & & f(\theta) \\ 1 & b_2 & 1 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ \bar{f(\theta)} & & & & 1 & b_p \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} b_1 & 1 & & & & \\ 1 & b_2 & 1 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ & & & & 1 & b_p \end{bmatrix}$$

and

$$E = \begin{bmatrix} & & & f(\theta) \\ & & & \\ & & & \\ & & & \end{bmatrix}.$$

Then we get

$$|\lambda_j - \tilde{\lambda_j(\theta)}| \leq \|E\| = \|f\|_\infty$$

by (4.2). Combining with (4.1), we get

$$\sigma_e(A) = \bigcup_{j=1}^{p} \left[\inf_\theta(\lambda_j(f_s(\theta))), \sup_\theta(\lambda_j(f_s(\theta))) \right] \subseteq \bigcup_{j=1}^{p} [\lambda_j - \|f\|_\infty, \lambda_j + \|f\|_\infty].$$

Therefore if $|\lambda_j - \lambda_{j+1}| > 2\,\|f\|_\infty$ for some j. Then there exists a gap in the essential spectrum. Hence we proved the first assertion. Now in addition, if we assume that $b_1 \leq b_2 \leq \cdots \leq b_p$, then since the essential spectrum of A is connected, by Theorem 4.1, $b_1 = b_2 = \cdots = b_p$. This implies that H is a tridiagonal Toeplitz matrix with b_1 on diagonal and 1 as off-diagonal entries. The eigenvalues of such matrices are explicitly known and they are $b_1 + 2\cos\left(\frac{\pi k}{p+1}\right)$. So the second assertion follows by a simple computation. □

Remark 4.7. Using the last theorem, we can predict the nature of spectrum of the operator A by only looking at the eigenvalues of the $p \times p$ matrix $\tilde{H}$.

4.2 *Jacobi matrices*

We can apply the same technique in the case of the periodic Jacobi matrices (see sections 2, 3 in [13] and references therein) to predict gaps in the essential spectrum.

Define a double infinite, p-periodic, $p \geq 2$, real Jacobi matrix by

$$J = \begin{bmatrix} \ddots & \ddots & & & & & & \\ \ddots & b_1 & a_1 & & & & & \\ & a_1 & b_2 & a_2 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \ddots & b_p & a_p & & \\ & & & & a_p & b_1 & \ddots & \\ & & & & & \ddots & \ddots & \end{bmatrix}, \quad a_{n+p} = a_n > 0;\ b_{n+p} = b_n, \tag{4.3}$$

following the standard convention $a_n > 0$. An important observation is that J is the block Toeplitz–Laurent operator, where in the case $p \geq 3$, the symbols are given by

$$f_k(\theta) = \begin{bmatrix} b_{k+1} & a_{k+1} & 0 & & & \mathrm{e}^{i\theta} a_{k+p-1} \\ a_{k+1} & b_{k+2} & a_{k+2} & & & 0 \\ 0 & a_{k+2} & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & a_{k+p-1} \\ \mathrm{e}^{-i\theta} a_{k+p-1} & 0 & & & a_{k+p-1} & b_{k+p} \end{bmatrix},$$

$k = 0, 1, \ldots, p-1$.

Also the spectrum of J is given by the following identity:

$$\sigma(J) = \bigcup_{j=1}^{p} [\min_{\theta} \lambda_j(f_k(\theta)), \max_{\theta} \lambda_j(f_k(\theta))]. \tag{4.4}$$

COROLLARY 4.8

Let J be the Jacobi matrix defined by (4.3) and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ eigenvalues of the matrix

$$\begin{bmatrix} b_1 & a_1 & & & & \\ a_1 & b_2 & a_2 & & & \\ & a_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & a_{p-1} \\ & & & & a_{p-1} & b_p \end{bmatrix}.$$

Then $\sigma_e(J)$ has a gap, if $|\lambda_j - \lambda_{j+1}| > 2|a_{p-1}|$ for some j.

Proof. The proof is an imitation of the proof of Theorem 4.6, however all the details are provided here. Apply Lemma 4.5 with

$$H(.) = \begin{bmatrix} b_1 & a_1 & & & & a_{p-1}\mathrm{e}^{i\theta} \\ a_1 & b_2 & a_2 & & & \\ & a_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & a_{p-1} \\ a_{p-1}\mathrm{e}^{-i\theta} & & & & a_{p-1} & b_p \end{bmatrix},$$

$$\tilde{H} = \begin{bmatrix} b_1 & a_1 & & & & \\ a_1 & b_2 & a_2 & & & \\ & a_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & a_{p-1} \\ & & & & a_{p-1} & b_p \end{bmatrix}$$

and

$$E = \begin{bmatrix} & & a_{p-1}\mathrm{e}^{i\theta} \\ & & \\ & & \\ & & \end{bmatrix}.$$

Then we get

$$|\lambda_j - \lambda_j(H(\theta)| \leq \|E\| = \|a_{p-1}\mathrm{e}^{i\theta}\|_\infty = |a_{p-1}| \text{ by (4.2).}$$

Combining with (4.4), we get

$$\sigma_e(J) = \bigcup_{j=1}^{p}\left[\inf_\theta(\lambda_j(H(\theta))), \sup_\theta(\lambda_j(H(\theta)))\right] \subseteq \bigcup_{j=1}^{p}[\lambda_j - |a_{p-1}|, \lambda_j + |a_{p-1}|].$$

Therefore if $|\lambda_j - \lambda_{j+1}| > 2|a_{p-1}|$ for some j, then there exists a gap in the essential spectrum. Hence the proof. □

Remark 4.9. The last couple of theorems help us to reduce the computations in predicting spectral gaps, for operators arising from the matrix valued symbols. We need to check only the eigenvalues of a matrix with constant entries. The proof also gives us the spectral inclusion

$$\sigma_e(A) \subseteq \bigcup_{j=1}^{p}[\lambda_j - \|f\|_\infty, \lambda_j + \|f\|_\infty]$$

which is very important, since the right-hand side includes only the eigenvalues of a constant matrix. Whether equality holds in this inclusion, is still not clear to us.

5. Concluding remarks

We conclude this note by listing down some remarks and future problems:

- Using Theorem 3.1 and the special choice I, we could predict the existence of spectral gaps from the finite matrix entries. Theorem 4.6 and its corollary can be used to predict the spectral gaps of the corresponding operators, by looking at the eigenvalues of a finite matrix with constant entries.
- The Borg's theorem is a classical theorem in inverse spectral theory. The discrete versions are also folklore [14]. The techniques of the proof here are adapted from [16].
- The discrete spectral values lying between a gap in the essential spectrum can be computed using linear algebraic techniques. To see this, let (a, b) be a gap in the essential spectrum of A. Let $\lambda_0 = (a + b)/2$. Since λ_0 is in the gap $f(\lambda_0) > 0$, where $f(\lambda_0)$ is the lower bound of the essential spectrum of $(A - \lambda_0 I)^2$, all the discrete spectral values below that can be computed with the use of truncations by Theorem 1.1. If β is an eigenvalue in the gap, $(\beta - \lambda_0)^2$ will be an eigenvalue lying below the lower bound of the essential spectrum of $(A - \lambda_0 I)^2$. From these we can compute β.

  Looking at these observations under a holomorphic perturbation is an interesting problem.
- Also under compact perturbation, though the spectral gaps remain the same, discrete eigenvalues may appear or disappear inside such gaps. Another problem is to handle such situations linear algebraically.
- Another scope is to carry over these results to the case of unbounded operators. In particular, one may think of estimating the spectrum and spectral gaps of Schrödinger operators by the eigenvalues of its truncations.

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