Proc. Indian Acad. Sci. (Math. Sci.) Vol. 123, No. 4, November 2013, pp. 539–546.

Completely continuous and weakly completely continuous abstract Segal algebras

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MS received 20 June 2012; revised 30 May 2013

Abstract. Let $\mathcal{A}$ be a Banach algebra. It is obtained a necessary and sufficient condition for the complete continuity and also weak complete continuity of symmetric abstract Segal algebras with respect to $\mathcal{A}$, under the condition of the existence of an approximate identity for $\mathcal{B}$, bounded in $\mathcal{A}$. In addition, a necessary condition for the weak complete continuity of $\mathcal{A}$ is given. Moreover, the applications of these results about some group algebras on locally compact groups are obtained.

Keywords. Abstract Segal algebras; completely continuous; weakly completely continuous.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra. A bounded operator $T : \mathcal{A} \to \mathcal{A}$ is called a right (left) multiplier if $T(ab) = aT(b)$ (T(ab) = T(a)b) for all $a, b \in \mathcal{A}$. Recall that $a \in \mathcal{A}$ is called a right completely continuous (right weakly completely continuous, respectively) element of $\mathcal{A}$ if the operator ρ_a of right multiplication by a is compact (weakly compact, respectively). An algebra $\mathcal{A}$ is called right completely continuous (right weakly completely continuous) if any element $a \in \mathcal{A}$ is right completely continuous (right weakly completely continuous, respectively). Left completely continuous (left weakly completely continuous) elements and left completely continuous (weakly completely continuous) Banach algebras are defined in a similar way, via the compactness (weakly compactness) of operators λ_a of left multiplication by a. An algebra $\mathcal{A}$ is called completely continuous (weakly completely continuous) if it is both left and right completely continuous (weakly completely continuous).

The set of all right (resp. left) completely continuous elements of $\mathcal{A}$ will be denoted by RCC($\mathcal{A}$) (resp. LCC($\mathcal{A}$)). We use the notation RWCC($\mathcal{A}$) (resp. LWCC($\mathcal{A}$)) for weakly right (resp. left) completely continuous elements of $\mathcal{A}$. These sets are in fact norm closed subspaces of $\mathcal{A}$.

For a Banach algebra $\mathcal{A}$, let $S, T \subseteq \mathcal{A}$, and $U \subseteq \mathcal{A}^*$. Set $S{\cdot}T = \{st : s \in S\ ,\ t \in T\}$, $S \cdot U = \{s \cdot f : s \in S\ ,\ f \in U\}$ and $U \cdot S = \{f \cdot s : s \in S\ ,\ f \in U\}$, where the functionals $s \cdot f$ and $f \cdot s$ are defined by the formulas $\langle s \cdot f, a\rangle = \langle f, as\rangle$ and $\langle f \cdot s, a\rangle = \langle f, sa\rangle$, for each $a \in \mathcal{A}$.

A functional $f \in \mathcal{A}^*$ is said to be weakly almost periodic if the set

$$\{a \cdot f : a \in \mathcal{A}\ ,\ \|a\| \leq 1\}$$

is relatively weakly compact in $\mathcal{A}^*$. We denote the set of all weakly almost periodic functionals on $\mathcal{A}$ by $\mathrm{WAP}(\mathcal{A})$, that is in fact a norm closed two-sided Banach $\mathcal{A}$-submodule of $\mathcal{A}^*$.

The second dual space $\mathcal{A}^{**}$ of $\mathcal{A}$ can be equipped with two multiplications, denoted by $\Box$ and $\Diamond$, which makes $\mathcal{A}^{**}$ a Banach algebra. Indeed, for all $\Phi, \Psi \in \mathcal{A}^{**}$ and $f \in \mathcal{A}^*$, put

$$\langle \Phi \,\Box\, \Psi, f\rangle = \langle \Phi, \Psi \cdot f\rangle \quad \text{and} \quad \langle \Phi \,\Diamond\, \Psi, f\rangle = \langle \Psi, f \cdot \Phi\rangle,$$

where $\Psi \cdot f, f \cdot \Phi \in \mathcal{A}^*$ are defined by the equalities

$$\langle \Psi \cdot f, a\rangle = \langle \Psi, f \cdot a\rangle \qquad \text{and} \qquad \langle f \cdot \Phi, a\rangle = \langle \Phi, a \cdot f\rangle,$$

for each $a \in \mathcal{A}$. One can consider $\mathcal{A}$ as a subalgebra of $(\mathcal{A}^{**}, \Box)$ and also $(\mathcal{A}^{**}, \Diamond)$. More importantly

$$a \,\Box\, \Phi = a \,\Diamond\, \Phi \qquad \text{and} \qquad \Phi \,\Box\, a = \Phi \,\Diamond\, a,$$

for each $a \in \mathcal{A}$ and $\Phi \in \mathcal{A}^{**}$ (see [4], for more information in this field). It is a known result that the algebra $\mathcal{A}$ is weakly completely continuous if and only if it is an ideal in $\mathcal{A}^{**}$ (Lemma 3 of [5]).

An algebra $\mathcal{A}$ is said to be Arens regular if these two multiplications coincide. As an important result, it is remarkable to note that an algebra $\mathcal{A}$ is Arens regular if and only if $\mathrm{WAP}(\mathcal{A}) = \mathcal{A}^*$ (see [5]).

The aim of the present work is to establish the relations between (weakly) completely continuous elements of a Banach algebra $\mathcal{A}$ and symmetric abstract Segal algebras $\mathcal{B}$ with respect to $\mathcal{A}$, in the case where $\mathcal{B}$ admits an approximate identity, bounded in $\mathcal{A}$. It is in fact a generalization of the results connected to symmetric Segal algebras, given in §6 of [10]. Also it is given a necessary condition for the weak complete continuity of $\mathcal{A}$, not necessarily with an approximate identity, which makes a new proof for the result due to Ghahramani [8]. Moreover, some applications of these results in group algebras are presented.

2. Main results

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach algebra. A Banach algebra $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is called a symmetric abstract Segal algebra with respect to $\mathcal{A}$ if the following conditions are satisfied (see [3] for more details).

(1) $\mathcal{B}$ is a dense two-sided ideal of $\mathcal{A}$.
(2) There exists $M > 0$ such that $\|b\|_{\mathcal{A}} \leq M\|b\|_{\mathcal{B}}$, for each $b \in \mathcal{B}$.
(3) There exists $C > 0$ such that $\|ab\|_{\mathcal{B}} \leq C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}$ and $\|ba\|_{\mathcal{B}} \leq C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}$, for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

As the main results, we show that (weak) complete continuity of $\mathcal{A}$ is equivalent to the (weak) complete continuity of $\mathcal{B}$, provided with the existence of an approximate identity for $\mathcal{B}$, which is bounded in $\mathcal{A}$. Note that $\mathrm{RCC}_{\mathcal{A}}(\mathcal{B})$ (resp. $\mathrm{RWCC}_{\mathcal{A}}(\mathcal{B})$) is the set of all $a \in \mathcal{A}$ such that $\mathcal{P}_a : \mathcal{B} \to \mathcal{B}$ is compact (resp. weakly compact.) We also use the notations $\mathrm{LCC}_{\mathcal{A}}(\mathcal{B})$ and $\mathrm{LWCC}_{\mathcal{A}}(\mathcal{B})$ in a similar way.

PROPOSITION 2.1

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{B}$ be a symmetric abstract Segal algebra with respect to $\mathcal{A}$. Then

(i) $\mathcal{B} \cdot \mathrm{LCC}(\mathcal{A}) \subseteq \mathrm{LCC}(\mathcal{B})$.
(ii) $\mathrm{LCC}_{\mathcal{A}}(\mathcal{B}) \cdot \mathcal{B} \subseteq \mathrm{LCC}(\mathcal{A})$.
(iii) $\mathrm{RCC}(\mathcal{A}) \cdot \mathcal{B} \subseteq \mathrm{RCC}(\mathcal{B})$.
(iv) $\mathcal{B} \cdot \mathrm{RCC}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathrm{RCC}(\mathcal{A})$.

Proof.

(i) Let $a \in \mathrm{LCC}(\mathcal{A})$, $b \in \mathcal{B}$ and $(b_j)_{j\in\mathcal{I}}$ be a bounded net in $\mathcal{B}$. Thus $(b_j)_{j\in\mathcal{I}}$ is bounded in $\mathcal{A}$, as well and by the left complete continuity of a, there exist a subnet (b_{j_k}) of $(b_j)_{j\in\mathcal{I}}$ and $c \in \mathcal{A}$ such that $\|ab_{j_k} - c\|_{\mathcal{A}} \to 0$. Hence $\|bab_{j_k} - bc\|_{\mathcal{B}} \to 0$. It follows that $ba \in \mathrm{LCC}(\mathcal{B})$.
(ii) Let $b \in \mathrm{LCC}_{\mathcal{A}}(\mathcal{B})$, $c \in \mathcal{B}$ and $(a_j)_{j\in\mathcal{I}}$ be a bounded net in $\mathcal{A}$. Thus $(ca_j)_{j\in\mathcal{I}}$ is bounded in $\mathcal{B}$. By the left complete continuity of b, there exist a subnet (a_{j_k}) of $(a_j)_{j\in\mathcal{I}}$ and $d \in \mathcal{B}$ such that $\|bca_{j_k} - d\|_{\mathcal{B}} \to 0$. Hence $\|bca_{j_k} - d\|_{\mathcal{A}} \to 0$. It follows that $bc \in \mathrm{LCC}(\mathcal{A})$.
(iii) and (iv) follow from similar arguments to the proof of parts (i) and (ii). □

Let $(e_i)_{i\in I}$ be an approximate identity for $(\mathcal{B}, \|.\|_{\mathcal{B}})$ which is bounded in $\mathcal{A}$. In this case, density $\mathcal{B}$ in $\mathcal{A}$ implies that $(e_i)_{i\in I}$ is also a bounded approximate identity for $\mathcal{A}$. Moreover, Cohen factorization theorem [13] implies that $\mathcal{B} = \mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$. A result due to Burnham [3] implies that for a proper subset $\mathcal{B}$ of $\mathcal{A}$, $(e_i)_{i\in I}$ is not bounded in $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$.

Theorem 2.2. *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{B}$ be a symmetric abstract Segal algebra with respect to $\mathcal{A}$ with an approximate identity which is bounded in $\mathcal{A}$. Then $\mathcal{A}$ is left* (*right*) *completely continuous if and only if $\mathcal{B}$ is left* (*right*) *completely continuous.*

Proof. Let $\mathcal{A}$ be left completely continuous and $b \in \mathcal{B}$. By the explanation preceding the theorem, $b = ca$, for some $c \in \mathcal{B}$ and $a \in \mathcal{A}$. Suppose that $(b_j)_{j\in\mathcal{I}}$ is a bounded net in $\mathcal{B}$. Since $\mathcal{A}$ is left completely continuous then $a \in \mathrm{LCC}(\mathcal{A})$. Thus $\|ab_{j_k} - d\|_{\mathcal{A}} \to 0$, for a subnet (b_{j_k}) of $(b_j)_{j\in\mathcal{I}}$ and some $d \in \mathcal{A}$. Consequently $\|cab_{j_k} - cd\|_{\mathcal{B}} \to 0$ and so $\|bb_{j_k} - cd\|_{\mathcal{B}} \to 0$, which implies that $b \in \mathrm{LCC}(\mathcal{B})$. It follows that $\mathcal{B}$ is left completely continuous. For the converse, we first show that $\mathrm{LCC}_{\mathcal{A}}(\mathcal{B}) = \mathcal{A}$. Let $(e_i)_{i\in\mathcal{I}}$ be an approximate identity for $\mathcal{B}$ which is bounded in $\mathcal{A}$ and $a \in \mathcal{A}$. Since $ae_i \in \mathcal{B}$, for all e_i, by the hypothesis $\lambda_{ae_i} : \mathcal{B} \to \mathcal{B}$ is compact. Moreover

$$\begin{aligned} \|\lambda_{ae_i} - \lambda_a\| &= \sup_{\|c\|_{\mathcal{B}}\le 1} \|\lambda_{ae_i}(c) - \lambda_a(c)\|_{\mathcal{B}} \\ &\le \|ae_i - a\|_{\mathcal{A}} \\ &\to 0. \end{aligned}$$

Since the set of compact operators is closed under the operator norm, it follows that $\lambda_a : \mathcal{B} \to \mathcal{B}$ is compact and so $a \in \mathrm{LCC}_{\mathcal{A}}(\mathcal{B})$. It follows that $\mathrm{LCC}_{\mathcal{A}}(\mathcal{B}) = \mathcal{A}$. Now Proposition 2.1 implies that

$$\mathcal{B} = \mathcal{A} \cdot \mathcal{B} = \mathrm{LCC}_{\mathcal{A}}(\mathcal{B}) \cdot \mathcal{B} \subseteq \mathrm{LCC}(\mathcal{A}).$$

Since $\mathcal{B}$ is dense in $\mathcal{A}$ and LCC($\mathcal{A}$) is a norm closed subspace of $\mathcal{A}$, the result is fulfilled automatically. One can easily get the same result for the case of right complete continuity. □

One can easily prove the following proposition with similar arguments given in the proof of Proposition 2.1. Some other relations and properties can be found in [18].

PROPOSITION 2.3

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{B}$ be a symmetric abstract Segal algebra with respect to $\mathcal{A}$. Then

(i) $\mathcal{B} \cdot \mathrm{LWCC}(\mathcal{A}) \subseteq \mathrm{LWCC}(\mathcal{B})$.
(ii) $\mathrm{LWCC}_{\mathcal{A}}(\mathcal{B}) \cdot \mathcal{B} \subseteq \mathrm{LWCC}(\mathcal{A})$.
(iii) $\mathrm{RWCC}(\mathcal{A}) \cdot \mathcal{B} \subseteq \mathrm{RWCC}(\mathcal{B})$.
(iv) $\mathcal{B} \cdot \mathrm{RWCC}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathrm{RWCC}(\mathcal{A})$.

Theorem 2.4. *Let $\mathcal{A}$ be a Banach algebra and $\mathcal{B}$ be a symmetric abstract Segal algebra with respect to $\mathcal{A}$ with an approximate identity which is bounded in $\mathcal{A}$. Then $\mathcal{A}$ is left* (*right*) *weakly completely continuous if and only if $\mathcal{B}$ is left* (*right*) *weakly completely continuous.*

Proof. Let $\mathcal{A}$ be left weakly completely continuous and $b \in \mathcal{B}$ and $(b_j)_{j\in\mathcal{I}}$ be a bounded net in $\mathcal{B}$. The hypothesis implies that $b = da$, for some $d \in \mathcal{B}$ and $a \in \mathcal{A}$. Since $(b_j)_{j\in\mathcal{I}}$ is also bounded in A, there exists a subnet (b_{j_k}) of $(b_j)_{j\in\mathcal{I}}$ such that $(ab_{j_k})_{j\in\mathcal{I}}$ is convergent to $c \in \mathcal{A}$ in the weak topology of $\mathcal{A}$. Hence for each $f \in \mathcal{B}^*$,

$$f(dab_{j_k}) \to f(dc)$$

and so (dab_{j_k}) is convergent to dc in the weak topology of $\mathcal{B}$. It follows that $b = da \in \mathrm{LWCC}(\mathcal{B})$ and thus $\mathcal{B}$ is left weakly completely continuous.

For the converse, first by a similar argument to the proof of Theorem 2.2, one can easily show that $\mathrm{LWCC}_{\mathcal{A}}(\mathcal{B}) = \mathcal{A}$. Now Proposition 2.3 concludes that

$$\mathcal{B} = \mathcal{A} \cdot \mathcal{B} = \mathrm{LWCC}_{\mathcal{A}}(\mathcal{B}) \cdot \mathcal{B} \subseteq \mathrm{LWCC}(\mathcal{A}).$$

Since $\mathcal{B}$ is dense in $\mathcal{A}$ and LWCC($\mathcal{A}$) is a norm closed subspace of $\mathcal{A}$, it follows that $\mathcal{A} = \mathrm{LWCC}(\mathcal{A})$ and so $\mathcal{A}$ is left weakly completely continuous. One can easily get the same result for the case of right (resp. two-sided) weak complete continuity. □

As a consequence of Theorem 2.4 and also [5], the next result is obtained. It also has been pointed out in [16].

COROLLARY 2.5

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{B}$ be a symmetric abstract Segal algebra with respect to $\mathcal{A}$ with an approximate identity which is bounded in $\mathcal{A}$. Then $\mathcal{A}$ is a left (*right*, *two-sided*) *ideal in $\mathcal{A}^{**}$ if and only if $\mathcal{B}$ is a left* (*right*, *two-sided*) *ideal in $\mathcal{B}^{**}$.*

We end this section by giving a necessary condition for the weak complete continuity of a Banach algebra. This result has been proved also in [6], with the assumption of existence of an approximate identity for $\mathcal{A}$. We get the same result without this assumption.

Theorem 2.6. *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}$ is weakly completely continuous, then $\mathcal{A}\cdot\mathcal{A}^*\cdot\mathcal{A}\subseteq \mathrm{WAP}(\mathcal{A})$.*

Proof. Let $a, b \in \mathcal{A}$, $f \in \mathcal{A}^*$ and $(x_i)_{i\in\mathcal{I}}$ be a bounded net in $\mathcal{A}$. We show that the net $(x_i a\cdot f\cdot b)$ has a subnet that is convergent in the weak topology of $\mathcal{A}^*$. Since a is a right weakly completely continuous element of $\mathcal{A}$, then $(x_i a)$ admits a subnet $(x_{i_j} a)$ that is weakly convergent in $\mathcal{A}$. Define the complex-valued function g on $\mathcal{A}$ by

$$g(x) = \lim_j \langle f\cdot x, x_{i_j} a\rangle.$$

It is clear that $g \in \mathcal{A}^*$. Since $\mathcal{A}$ is weakly completely continuous, $\mathcal{A}$ is an ideal in $\mathcal{A}^{**}$ by Lemma 3 of [5]. Consequently for each $F \in \mathcal{A}^{**}$, we have $b\,\Box\, F \in \mathcal{A}$ and

$$\begin{aligned}\langle F, g\cdot b\rangle &= \langle b\diamondsuit F, g\rangle = \langle g, b\,\Box\, F\rangle = \lim_j \langle f\cdot(b\,\Box\, F), x_{i_j} a\rangle\\ &= \lim_j \langle b\,\Box\, F, x_{i_j} a\cdot f\rangle.\end{aligned}$$

Furthermore,

$$\lim_j \langle b\,\Box\, F, x_{i_j} a\cdot f\rangle = \lim_j \langle b\diamondsuit F, x_{i_j} a\cdot f\rangle = \lim_j \langle F, x_{i_j} a\cdot f\cdot b\rangle.$$

It follows that $(x_{i_j} a\cdot f\cdot b)$ is convergent to $g\cdot b$, in the weak topology of $\mathcal{A}^*$. It follows that $a\cdot f\cdot b \in \mathrm{WAP}(\mathcal{A})$, as claimed. $\square$

Theorem 2.6 and also [5] yield the next result. Some known results can be proved in more easier ways, by this result

COROLLARY 2.7

Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}$ is an ideal in $\mathcal{A}^{**}$, then $\mathcal{A}\cdot\mathcal{A}^*\cdot\mathcal{A}\subseteq \mathrm{WAP}(\mathcal{A})$.

3. Applications in group algebras

In this section, we investigate the results, given in the previous section, for the group algebras. Let us repeat and review some terminologies and preliminaries, used in the present section.

Let G be a locally compact group with a fixed left Haar measure λ. Given a complex-valued function f on G; the left (resp. right) translation of f by $x \in G$ will be denoted by $({}_x f)(y) = f(x^{-1}y)$ (resp. $(f_x)(y) = \Delta(x) f(yx)$, for all $y \in G$. We denote by $\mathrm{CB}(G)$ the space of all bounded continuous complex-valued functions on G with the supremum norm. Also $\mathrm{UC}(G)$ is the subspace of $\mathrm{CB}(G)$, consisting of all bounded left and right uniformly continuous functions on G; i.e. all $f \in \mathrm{CB}(G)$ such that the maps $x \mapsto {}_x f$ and $x \mapsto f_x$ from G into $\mathrm{CB}(G)$ are continuous. We also denote by $\mathrm{WAP}(G)$, the space of all bounded continuous weakly almost periodic functions on G; i.e all $f \in \mathrm{CB}(G)$ such that $\{{}_x f : x \in G\}$ is relatively weakly compact in $\mathrm{CB}(G)$. By Theorem 3.11 of [2], $\mathrm{WAP}(G)$ is a closed subspace of $\mathrm{UC}(G)$.

In the beginning of the present section, we investigate some aspects of Theorem 2.6 or equivalently Corollary 2.7. First, let us recall the basic definition of Segal algebras (see [17] for complete information).

A linear subspace $S(G)$ of $L^1(G)$ is said to be a Segal algebra, if it satisfies the following conditions:

(1) $S(G)$ is dense in $L^1(G)$.
(2) $S(G)$ is a Banach space under some norm $\|\cdot\|_s$ and $\|f\|_1 \leq \|f\|_s$; for all $f \in S(G)$.
(3) $S(G)$ is left translation invariant and the map $x \mapsto {}_x f$ of G into $S(G)$ is continuous.
(4) $\|{}_x f\|_s = \|f\|_s$; for all $f \in S(G)$ and $x \in G$.

A Segal algebra $S(G)$ is called symmetric if it is right translation invariant, and for each $f \in S(G)$ and $x \in G$, $\|f_x\|_s = \|f\|_s$, and also the map $x \mapsto f_x$ from G into $S(G)$ is continuous. Note that every symmetric Segal algebra is in fact a symmetric abstract Segal algebra with respect to $L^1(G)$. Moreover, it has an approximate identity which held in $L^1(G)$ and each term having norm equal to 1 in $L^1(G)$-norm (see [17] for more information).

Remark 3.1.

(a) It is known that $L^1(G)$ is an ideal in its second dual if and only if G is compact. There are numerous proofs of this fact such as [5] and [12]. Using Theorems 2.4 and 2.6, we give another proof that is, in our opinion, short. Let $L^1(G)$ be an ideal in its second dual. By [13], we have

$$L^1(G) \cdot L^\infty(G) \cdot L^1(G) = L^1(G) * L^\infty(G) * L^1(G)\tilde{} = \mathrm{UC}(G),$$

Corollary 1.7 implies that

$$\mathrm{UC}(G) \subseteq \mathrm{WAP}(L^1(G)).$$

Since $\mathrm{WAP}(L^1(G)) = \mathrm{WAP}(G)$ [19], it follows that $\mathrm{UC}(G) = \mathrm{WAP}(G)$. This is possible only if G is compact [11]. For the converse, let G be a compact group. Thus for each $1 < p < \infty$, the usual Lebesgue space $L^p(G)$, defined in [13], is a symmetric abstract Segal algebra with respect to $L^1(G)$ [17] and since $L^p(G)$ is reflexive, then $L^p(G)$ is an ideal in its second dual. Now Corollary 2.5 implies that $L^1(G)$ is also an ideal in its second dual.

(b) The converse of Theorem 2.6 is not in general true. For instance, let $\mathcal{A}$ be an Arens regular Banach algebra. Thus $\mathrm{WAP}(\mathcal{A}) = \mathcal{A}^*$ [5] and so $\mathcal{A} \cdot \mathcal{A}^* \cdot \mathcal{A} \subseteq \mathrm{WAP}(\mathcal{A})$. But $\mathcal{A}$ is not necessarily weakly completely continuous. We provide some examples. Note that every bounded operator from a C^*-algebra into its dual space is weakly compact [1]. It follows that all C^*-algebras are Arens regular. It is known that $L^\infty(G)$ is a C^*-algebra under pointwise product and so it is Arens regular. But since $L^\infty(G)$ is unital, then $L^\infty(G)$ is weakly completely continuous or equivalently an ideal in its second dual just whenever G is finite.

The following proposition is a direct consequence of Remark 3.1, Theorems 2.2 and 2.4 and also Corollaries 2.5 and 2.7. It also has been given in Proposition 6.1 of [10] and [15].

PROPOSITION 3.2

Let G be a locally compact group and $S(G)$ be a symmetric Segal algebra in G. Then the following assertions are equivalent:

(i) *$S(G)$ is completely continuous.*

(ii) *$S(G)$ is weakly completely continuous.*
(iii) *$S(G)$ is an ideal in its second dual.*
(iv) *G is compact.*

Ghahramani and Lau defined Lebesgue–Fourier algebra $\mathcal{L}A(G)$ [9], and showed that it is a Segal algebra under convolution product Proposition 2.2 of [9]. Also $\mathcal{L}A(G)$ is a symmetric Segal algebra, whenever G is unimodular. Theorems 2.2 and 2.4 and Corollary 2.5 yield the following result. It has been given in Corollary 6.2 of [10].

COROLLARY 3.3

Let G be a unimodular locally compact group. Then the following assertions are equivalent:

(i) *$\mathcal{L}A(G)$ is completely continuous, under convolution product.*
(ii) *$\mathcal{L}A(G)$ is weakly completely continuous*, *under convolution product.*
(iii) *$\mathcal{L}A(G)$ is an ideal in its second dual.*
(iv) *G is compact.*

Let ω be a submultiplicative weight function on G; i.e. a Borel measurable function $\omega : G \to (0, \infty)$ with $\omega(xy) \le \omega(x)\omega(y)$, for all $x, y \in G$. The space $L^p(G, \omega)$ with respect to λ is the set of all complex-valued measurable functions f on G such that $f\omega \in L^p(G)$, where $1 \le p < \infty$. Moreover $L^p(G, \omega)$ is a Banach space under the norm $\|f\|_{p,\omega} = \|f\omega\|_p$. It is known that for such a weight function, $L^1(G, \omega)$ is a Banach algebra under convolution product, with a bounded approximate identity. In this case, $L^1(G, \omega)$ is an ideal in its second dual if and only if G is compact [8]. If G is a unimodular locally compact group then $L^1(G, \omega) \cap L^p(G, \omega)$ with the norm

$$\|f\|_s = \|f\|_{1,\omega} + \|f\|_{p,\omega}, \quad f \in L^1(G, \omega) \cap L^p(G, \omega),$$

is a symmetric abstract Segal algebra with respect to $L^1(G, \omega)$.

Moreover, every submultiplicative weight function is bounded and bounded away from zero on compacta (Proposition 1.16 of [7]). Thus if G is compact then $L^p(G, \omega) = L^p(G)$ with the equivalent norms.

We end the paper with the following result that is obtained from Theorems 2.2 and 2.4, Corollary 2.5 and Remark 3.1 together with [8]. Note that the usual bounded approximate identity of $L^1(G, \omega)$ is also an approximate identity for $L^1(G, \omega) \cap L^p(G, \omega)$; see the proof of Theorem 4.1 in [14].

PROPOSITION 3.4

Let G be a unimodular locally compact group, ω be a submultiplicative weight function on G and $1 < p < \infty$. Then the following assertions are equivalent:

(i) *$L^1(G, \omega) \cap L^p(G, \omega)$ is completely continuous.*
(ii) *$L^1(G, \omega) \cap L^p(G, \omega)$ is weakly completely continuous.*
(iii) *$L^1(G, \omega) \cap L^p(G, \omega)$ is an ideal in its second dual.*
(iv) *G is compact.*

Acknowledgements

The author would like to thank the referee of the paper for his/her invaluable comments and suggestions that have helped to improve the paper. We also would like to thank the Banach Algebra Center of Excellence for Mathematics, University of Isfahan.

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