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Strichartz estimates for the Schrödinger propagator for the Laguerre operator

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Abstract. We obtain homogeneous Strichartz estimate for the Schrödinger propagator e^{-itL_α} for the Laguerre operator L_α on $\mathbb{R}^n_+$. We follow regularization technique as introduced in *J. Funct. Anal.* **224**(**2**) (2005) 371–385. We also establish inhomogeneous Strichartz estimates for different admissible pairs.

Keywords. Laguerre operator; Laguerre functions; homogeneous and inhomogeneous Strichartz estimates.

1. Introduction

In this paper we will establish Strichartz estimate for the Schrödinger propagator e^{-itL_α} for the Laguerre operator on $\mathbb{R}^n_+$. Strichartz estimates are useful for establishing existence of solution for semilinear Schrödinger and wave equations, in which no derivatives are present in the nonlinearity. Strichartz estimates were first proved by Strichartz [15] for solutions of Schrödinger and wave equations on $\mathbb{R}^n$. They were generalized to non endpoint admissible pairs (q, p) by Ginibre and Velo [7] and by Lindblad and Sogge [10]. The end point estimates were proved by Keel and Tao [8]. End point estimates were also proved by D'Ancona *et al.* [3] for magnetic Schrödinger equation with some conditions on the potential A and V. In literature estimate (4.1) is known as the homogeneous Strichartz type estimate, whereas other estimates in Theorems 4.6 and 4.8 are known as inhomogeneous Strichartz type estimates (see [6], [18] and [16]).

Laguerre operator is closely related to the twisted Laplacian (special Hermite operator) in the following sense: If $f \in \mathcal{S}(\mathbb{C}^n)$ is radial then $\mathcal{L}f(z) = L_{n-1}f(r)$ where $\mathcal{L}$ is the twisted Laplacian on $\mathbb{C}^n$, L_{n-1} is the 1-dimensional Laguerre operator of type $n-1$ given by (1.1) and $r = |z|$. Moreover special Hermite functions $\Phi_{\mu+\tilde{\mu},\mu}$, $\Phi_{\mu,\mu+\tilde{\mu}}$ on $\mathbb{C}^n$ with $\tilde{\mu} \in \mathbb{Z}^n_{\geq 0}$ are related with n-dimensional Laguerre functions $\psi^{\tilde{\mu}}_\mu$ (see Theorems 1.3.4 and 1.3.5, pages 19–20 in [17]), where $\psi^{\tilde{\mu}}_\mu$ are given by (2.1). Ratnakumar and Sohani [14] proved well-posedness of nonlinear Schrödinger equation for the twisted Laplacian on $\mathbb{C}^n$, see Zhang and Zheng [20] for such a result.

Laguerre operator L_α on $\mathbb{R}_+ = (0, \infty)$ with $\alpha \in (-1, \infty)$ is given by

$$L_\alpha = -\frac{d^2}{dx^2} - \frac{2\alpha + 1}{x}\frac{d}{dx} + \frac{x^2}{4}. \tag{1.1}$$

The paper is organized as follows. In §2 we discuss the spectral theory of the Laguerre operator. In §3 we define the Schrödinger propagator e^{itL_α} as a unitary operator on $L^2(\mathbb{R}^n_+, \mathrm{d}\nu)$ and express as an integral operator on $L^1 \cap L^2(\mathbb{R}^n_+, \mathrm{d}\nu)$. We establish operator norm estimate for e^{itL_α} in Lemma 3.5. In §4 we prove Strichartz estimates in Theorem 4.6 and for different admissible pairs in Theorem 4.8.

2. Spectral theory of the Laguerre operator

The one-dimensional Laguerre polynomial $L_k^\alpha(x)$ of type $\alpha > -1$ are defined by the generating function identity

$$\sum_{k=0}^{\infty} L_k^\alpha(x) t^k = (1-t)^{-\alpha-1} e^{-\frac{xt}{1-t}}, \quad |t| < 1.$$

Here $x > 0$ and $k \in \mathbb{Z}_{\geq 0}$. Each L_k^α is a polynomial of degree k explicitly given by

$$L_k^\alpha(x) = \sum_{j=0}^{k} \frac{\Gamma(k+\alpha+1)}{\Gamma(k-j+1)\Gamma(j+\alpha+1)} \frac{(-x)^j}{j!}.$$

The Laguerre functions $\psi_k^\alpha(x) = \left(\frac{2^{-\alpha}k!}{\Gamma(k+\alpha+1)}\right)^{\frac{1}{2}} L_k^\alpha(\frac{x^2}{2}) e^{-\frac{x^2}{4}}$ form a complete orthonormal family in $L^2(\mathbb{R}_+, x^{2\alpha+1}\mathrm{d}x)$. Each ψ_k^α is an eigenfunction of the Laguerre operator L_α given by (1.1) with eigenvalue $(2k+\alpha+1)$, i.e.,

$$L_\alpha \psi_k^\alpha = (2k+\alpha+1)\psi_k^\alpha.$$

If $f, g \in C^2(\mathbb{R}_+) \cap L^2(\mathbb{R}_+, x^{2\alpha+1}\mathrm{d}x)$ such that $L_\alpha f, L_\alpha g \in L^2(\mathbb{R}_+, x^{2\alpha+1}\mathrm{d}x)$, then $\langle L_\alpha f, g\rangle_{x^{2\alpha+1}\mathrm{d}x} = \langle f, L_\alpha g\rangle_{x^{2\alpha+1}\mathrm{d}x}$, where inner product is with respect to the measure $x^{2\alpha+1}\mathrm{d}x$. Therefore the Laguerre operator L_α is self-adjoint with respect to the measure $x^{2\alpha+1}\mathrm{d}x$.

Thus every $f \in L^2(\mathbb{R}_+, x^{2\alpha+1}\mathrm{d}x)$ has the Laguerre expansion

$$f = \sum_{k=0}^{\infty} \langle f, \psi_k^\alpha\rangle_{x^{2\alpha+1}\mathrm{d}x}\, \psi_k^\alpha.$$

We call $\langle f, \psi_k^\alpha\rangle_{x^{2\alpha+1}\mathrm{d}x}$ as the k-th Fourier–Laguerre coefficient of f. Now for each multi index $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in (-1, \infty)^n$, the n-dimensional Laguerre functions are defined by the tensor product of the 1-dimensional Laguerre functions

$$\psi_\mu^\alpha(x) = \prod_{j=1}^{n} \psi_{\mu_j}^{\alpha_j}(x_j), \quad x \in \mathbb{R}^n_+ = (\mathbb{R}_+)^n. \tag{2.1}$$

The n-dimensional Laguerre operator L_α for $\alpha = (\alpha_1, \ldots, \alpha_n) \in (-1, \infty)^n$, is defined as the sum of 1-dimensional Laguerre operators L_{α_j},

$$L_\alpha = \sum_{j=1}^{n} L_{\alpha_j} = -\Delta - \sum_{j=1}^{n}\left(\frac{2\alpha_j+1}{x_j}\frac{\partial}{\partial x_j}\right) + \frac{|x|^2}{4}.$$

Therefore $L_\alpha \psi_\mu^\alpha = (2|\mu| + \sum_{j=1}^n \alpha_j + n)\psi_\mu^\alpha$, where $|\mu| = \sum_{j=1}^n \mu_j$. Hence, ψ_μ^α are eigenfunctions of L_α with eigenvalue $2|\mu| + \sum_{j=1}^n \alpha_j + n$ and they also form a complete orthonormal system in $L^2(\mathbb{R}_+^n, \mathrm{d}\nu(x))$. Laguerre operator L_α is self-adjoint with repect to measure $\mathrm{d}\nu$ where $\mathrm{d}\nu(x) = x_1^{2\alpha_1+1} \cdots x_n^{2\alpha_n+1} \mathrm{d}x_1 \cdots \mathrm{d}x_n$.

Thus every $f \in L^2(\mathbb{R}_+^n, \mathrm{d}\nu(x))$ has the Laguerre expansion

$$f = \sum_\mu \langle f, \psi_\mu^\alpha \rangle_\nu \psi_\mu^\alpha = \sum_{k=0}^\infty P_k f,$$

where the inner product is with respect to measure ν and P_k denotes the Laguerre projection operator corresponding to the eigenvalue $2k + \sum_{j=1}^n \alpha_j + n$ given by

$$P_k f = \sum_{|\mu|=k} \langle f, \psi_\mu^\alpha \rangle_\nu \psi_\mu^\alpha.$$

Remark 2.1. In view of estimate (3.2) throughout this paper we only consider $\alpha \in (-\frac{1}{2}, \infty)^n$.

Remark 2.2. Note that $L^\infty(\mathbb{R}_+^n, \mathrm{d}x) = L^\infty(\mathbb{R}_+^n, \mathrm{d}\nu)$ with equality of norms $\|f\|_{L^\infty(\mathbb{R}_+^n, \mathrm{d}x)} = \|f\|_{L^\infty(\mathbb{R}_+^n, \mathrm{d}\nu)}$, where $\mathrm{d}x$ denotes the usual Lebesgue measure on $\mathbb{R}_+^n$.

3. Schrödinger propagator e^{-itL_α}

If $f \in C^2 \cap L^2(\mathbb{R}_+^n, \mathrm{d}\nu)$ such that $L_\alpha f \in L^2(\mathbb{R}_+^n, \mathrm{d}\nu)$ then we observe that

$$\langle L_\alpha f, \psi_\mu^\alpha \rangle_\nu = \langle f, L_\alpha \psi_\mu^\alpha \rangle_\nu = \left(2|\mu| + n + \sum_{j=1}^n \alpha_j \right) \langle f, \psi_\mu^\alpha \rangle_\nu .$$

Therefore for $f \in L^2(\mathbb{R}_+^n, \mathrm{d}\nu)$, we define $\mathrm{e}^{-itL_\alpha} f$ as $L^2(\mathbb{R}_+^n, \mathrm{d}\nu)$ function by the following:

$$\mathrm{e}^{-itL_\alpha} f = \sum_{k=0}^\infty \mathrm{e}^{-it(2k+n+\sum_{j=1}^n \alpha_j)} \sum_{|\mu|=k} \langle f, \psi_\mu^\alpha \rangle_\nu \psi_\mu^\alpha.$$

It is easy to see that e^{-itL_α} is a unitary operator with adjoint operator e^{itL_α} on $L^2(\mathbb{R}_+^n, \mathrm{d}\nu)$.

Remark 3.1. $\mathrm{e}^{-itL_\alpha} f$ is periodic in t if and only if $\sum_{j=1}^n \alpha_j$ is rational whereas $\mathrm{e}^{it\sum \alpha_j} \mathrm{e}^{-itL_\alpha} f$ and $|\mathrm{e}^{-itL_\alpha} f|$ are always periodic in t with period a rational mutiple of 2π.

In view of equation (3.5) we say that Schrödinger propagator e^{-itL_α} is an integral operator on $L^1 \cap L^2(\mathbb{R}_+^n, \mathrm{d}\nu)$. We prove integral representation (3.5) by using regularization argument introduced in [11] (see also [12], [13]). Using this integral representation and kernel estimate in Lemma 3.4, we prove operator norm estimates of e^{-itL_α} in Lemma 3.5.

Since $\mathrm{e}^{-it(2k+n+\sum_{j=1}^n \alpha_j)}$ has modulus 1, for using Mehler's formula for Laguerre functions (see (4.17.6) of [9]), we consider an auxiliary complex semi group $\{\mathrm{e}^{-(r+it)L_\alpha}\}$ with $r > 0$.

Lemma 3.2. *Let $r > 0, \alpha \in (-\frac{1}{2}, \infty)^n$. Then $\mathrm{e}^{-(r+it)L_\alpha}$ is an integral operator on $L^2(\mathbb{R}^n_+, \mathrm{d}\nu)$. Moreover*

$$\mathrm{e}^{-(r+it)L_\alpha} f(x) = \int_{\mathbb{R}^n_+} f(y) K(x, y, r, t, \alpha) \mathrm{d}\nu(y),$$

$$K(x, y, r, t, \alpha) = \mathrm{e}^{-nr} \mathrm{e}^{-it\left(n+\sum \alpha_j\right)} \left(1-\mathrm{e}^{-2(r+it)}\right)^{-n} \mathrm{e}^{-\left(\frac{|x|^2+|y|^2}{4}\right)\left(\frac{1+\mathrm{e}^{-2(r+it)}}{1-\mathrm{e}^{-2(r+it)}}\right)}$$

$$\times \prod_{j=1}^{n} \left((x_j y_j)^{-\alpha_j} (\mathrm{e}^{-2it})^{\frac{-\alpha_j}{2}} I_{\alpha_j} \left(\frac{x_j y_j \mathrm{e}^{-r} (\mathrm{e}^{-2it})^{\frac{1}{2}}}{1 - \mathrm{e}^{-2(r+it)}} \right) \right),$$

where I_{α_j} is the modified Bessel function of first kind and $|\arg(\mathrm{e}^{-2it})| < \pi$.

Proof. We observe the following:

$$\begin{aligned}
\mathrm{e}^{-(r+it)L_\alpha} f(x) &= \sum_{k=0}^{\infty} \mathrm{e}^{-(r+it)\left(2k+n+\sum \alpha_j\right)} P_k f \\
&= \sum_{k=0}^{\infty} \mathrm{e}^{-(r+it)\left(2k+n+\sum \alpha_j\right)} \sum_{|\mu|=k} \left\langle f, \psi^\alpha_\mu \right\rangle_\nu \psi^\alpha_\mu(x) \\
&= \sum_{k=0}^{\infty} \mathrm{e}^{-(r+it)\left(2k+n+\sum \alpha_j\right)} \sum_{|\mu|=k} \int_{\mathbb{R}^n_+} f(y) \psi^\alpha_\mu(x) \psi^\alpha_\mu(y) \mathrm{d}\nu(y) \\
&= \sum_{k=0}^{\infty} \mathrm{e}^{-(r+it)\left(2k+n+\sum \alpha_j\right)} \int_{\mathbb{R}^n_+} f(y) \sum_{|\mu|=k} \psi^\alpha_\mu(x) \psi^\alpha_\mu(y) \mathrm{d}\nu(y) \\
&= \int_{\mathbb{R}^n_+} f(y) \sum_{k=0}^{\infty} \mathrm{e}^{-(r+it)\left(2k+n+\sum \alpha_j\right)} \sum_{|\mu|=k} \psi^\alpha_\mu(x) \psi^\alpha_\mu(y) \mathrm{d}\nu(y) \\
&= \int_{\mathbb{R}^n_+} f(y) K(x, y, r, t, \alpha) \mathrm{d}\nu(y).
\end{aligned}$$

In fifth equality series inside the integral converges in $L^2(\mathbb{R}^{2n}_+, \mathrm{d}\nu(x, y))$, see Remark 3.3. In view of Theorem 4.9 at page 94 of [1], fifth equality follows from dominated convergence theorem. Here

$$\begin{aligned}
K(x, y, r, t, \alpha) &= \sum_{k=0}^{\infty} \mathrm{e}^{-(r+it)\left(2k+n+\sum \alpha_j\right)} \sum_{|\mu|=k} \prod_{j=1}^{n} \psi^{\alpha_j}_{\mu_j}(x_j) \psi^{\alpha_j}_{\mu_j}(y_j) \\
&= \mathrm{e}^{-(r+it)\left(n+\sum \alpha_j\right)} \prod_{j=1}^{n} \sum_{\mu_j=0}^{\infty} \mathrm{e}^{-2(r+it)\mu_j} \psi^{\alpha_j}_{\mu_j}(x_j) \psi^{\alpha_j}_{\mu_j}(y_j)
\end{aligned}$$

$$= \mathrm{e}^{-nr}\mathrm{e}^{-it(n+\sum\alpha_j)}(1-\mathrm{e}^{-2(r+it)})^{-n}\mathrm{e}^{-\left(\frac{|x|^2+|y|^2}{4}\right)\left(\frac{1+\mathrm{e}^{-2(r+it)}}{1-\mathrm{e}^{-2(r+it)}}\right)}$$
$$\times\prod_{j=1}^{n}\left((x_jy_j)^{-\alpha_j}(\mathrm{e}^{-2it})^{\frac{-\alpha_j}{2}}I_{\alpha_j}\left(\frac{x_jy_j\mathrm{e}^{-r}(\mathrm{e}^{-2it})^{\frac{1}{2}}}{1-\mathrm{e}^{-2(r+it)}}\right)\right)$$

and $|\arg(\mathrm{e}^{-2it})| < \pi$. In the above, second equality follows because both functions are in $L^2\left(\mathbb{R}_+^{2n}, \mathrm{d}\nu(x,y)\right)$ and have the same Fourier–Laguerre coefficients. Last equality follows from Mehler's formula for Laguerre functions (see (4.17.6) of [9]). For Bessel functions of arbitrary order and modified Bessel functions of the first kind, we refer to §§ 5.3 and 5.7 of [9]. □

Remark 3.3. If $\mu \in \mathbb{Z}^n_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 1}$, then $\sum_{|\mu|=k} 1 = \binom{k+n-1}{k} \leq (2k)^{n-1}$ and for $r > 0$, we have

$$\sum_{k=1}^{\infty} k^{n-1}\mathrm{e}^{-4rk} < \infty.$$

Lemma 3.4. *Let $K(x,y,r,t,\alpha)$ be the kernel as in Lemma* 3.2. *Then we have uniform estimate for K in $r \in (0,1]$.*

$$|K(x,y,r,t,\alpha)| \leq \frac{C}{|\sin t|^{n+\sum_{j=1}^n\alpha_j}}, \tag{3.1}$$

where C only depends on n and α.

Proof. Let $\arg(\mathrm{e}^{-2it}) = -2\tilde{t}$ with $|\tilde{t}| < \frac{\pi}{2}$. Then $\mathrm{e}^{-2it} = \mathrm{e}^{-2i\tilde{t}}$, $(\mathrm{e}^{-2it})^{\frac{1}{2}} = \mathrm{e}^{-i\tilde{t}}$ and $\cos 2t = \cos 2\tilde{t}$. Now we observe the following:

$$\begin{aligned}
\left|1-\mathrm{e}^{-2(r+it)}\right| &= (1+\mathrm{e}^{-4r}-2\mathrm{e}^{-2r}\cos 2t)^{\frac{1}{2}}\\
\frac{x_jy_j\mathrm{e}^{-(r+i\tilde{t})}}{1-\mathrm{e}^{-2(r+it)}} &= x_jy_j\mathrm{e}^{-r}\left(\frac{(1-\mathrm{e}^{-2r})\cos\tilde{t}-i(1+\mathrm{e}^{-2r})\sin\tilde{t}}{1+\mathrm{e}^{-4r}-2\mathrm{e}^{-2r}\cos 2t}\right)\\
\left|\mathrm{Re}\left(\frac{x_jy_j\mathrm{e}^{-(r+i\tilde{t})}}{1-\mathrm{e}^{-2(r+it)}}\right)\right| &\leq \frac{x_jy_j\mathrm{e}^{-r}(1-\mathrm{e}^{-2r})}{1+\mathrm{e}^{-4r}-2\mathrm{e}^{-2r}\cos 2t}\\
\frac{1+\mathrm{e}^{-2(r+it)}}{1-\mathrm{e}^{-2(r+it)}} &= \frac{(1-\mathrm{e}^{-4r})-2i\mathrm{e}^{-2r}\sin 2t}{1+\mathrm{e}^{-4r}-2\mathrm{e}^{-2r}\cos 2t}.
\end{aligned}$$

We see that

$$|I_\beta(z)| \leq \frac{|z|^\beta}{2^\beta\Gamma(\beta+1)}\exp(|\mathrm{Re}(z)|), \quad \text{for } \beta > -\frac{1}{2} \tag{3.2}$$

which follows from inequality (1) in section 3.31, page 49 of [19] and equalities (5.7.4) and (5.7.6) of [9]. We have the following observation

$$\begin{aligned}
(1+\mathrm{e}^{-2r})(|x|^2+|y|^2)-4\mathrm{e}^{-r}\sum x_jy_j &= (1-\mathrm{e}^{-r})^2(|x|^2+|y|^2)+2\mathrm{e}^{-r}\sum_{j=1}^{n}(x_j-y_j)^2\\
&\geq (1-\mathrm{e}^{-r})^2(|x|^2+|y|^2).
\end{aligned}$$

Using the above observations, we see that

$$\begin{aligned}
&|K(x, y, r, t, \alpha)| \\
&\quad \leq C\mathrm{e}^{-r(n+\sum\alpha_j)}(1+\mathrm{e}^{-4r}-2\mathrm{e}^{-2r}\cos 2t)^{-\frac{(n+\sum\alpha_j)}{2}}\mathrm{e}^{-\frac{(1-\mathrm{e}^{-2r})(1-\mathrm{e}^{-r})^2(|x|^2+|y|^2)}{4(1+\mathrm{e}^{-4r}-2\mathrm{e}^{-2r}\cos 2t)}} \\
&\quad \leq C(1+\mathrm{e}^{-4r}-2\mathrm{e}^{-2r}\cos 2t)^{-\frac{(n+\sum\alpha_j)}{2}}. \qquad (3.3)
\end{aligned}$$

Now for $r \in (0, 1]$ we have

$$1+\mathrm{e}^{-4r}-2\mathrm{e}^{-2r}\cos 2t = (1-\mathrm{e}^{-2r})^2 + 4\mathrm{e}^{-2r}\sin^2 t \geq 4\mathrm{e}^{-2}\sin^2 t.$$

Therefore using this estimate in (3.3) we get the desired estimate. □

Lemma 3.5. *Let $t \notin \frac{\pi}{2}\mathbb{Z}$, $2 \leq p \leq \infty$ and $p' = \frac{p}{p-1}$. Then*

$$\|\mathrm{e}^{-itL_\alpha} f\|_{L^p(\mathrm{d}\nu)} \leq C|\sin t|^{-\left(1-\frac{2}{p}\right)(n+\sum\alpha_j)}\|f\|_{L^{p'}(\mathrm{d}\nu)},$$

where constant C depends only on n, p, α.

Proof. For $f \in L^2(\mathbb{R}^n_+, \mathrm{d}\nu)$ we have

$$\|\mathrm{e}^{-itL_\alpha} f\|^2_{L^2(\mathrm{d}\nu)} = \sum_{k=0}^{\infty} |\mathrm{e}^{-it(2k+n+\sum\alpha_j)}|^2 \,.\, \|P_k f\|^2_{L^2(\mathrm{d}\nu)} = \|f\|^2_{L^2(\mathrm{d}\nu)}. \quad (3.4)$$

For $f \in L^1 \cap L^2(\mathbb{R}^n_+, \mathrm{d}\nu)$ we observe from Lemma 3.4 and Remark 2.2 that

$$\|\mathrm{e}^{-(r+it)L_\alpha} f\|_{L^\infty(\mathbb{R}^n_+,\mathrm{d}\nu)} \leq C|\sin t|^{-(n+\sum\alpha_j)}\|f\|_{L^1(\mathbb{R}^n_+,\mathrm{d}\nu)}.$$

Since $\mathrm{e}^{-(r_m+it)L_\alpha} f \to \mathrm{e}^{-itL_\alpha} f$ in $L^2(\mathrm{d}\nu)$ as $r_m \to 0$, $\mathrm{e}^{-(r_{m_j}+it)L_\alpha} f(x) \to \mathrm{e}^{-itL_\alpha} f(x)$ a.e. x for some subsequence $\{r_{m_j}\}$. Also observe that

$$\int_{\mathbb{R}^n_+} f(y)K(x, y, r_{m_j}, t, \alpha)\mathrm{d}\nu(y) \to \int_{\mathbb{R}^n_+} f(y)K(x, y, 0, t, \alpha)\mathrm{d}\nu(y) \quad \text{a.e. } x.$$

Therefore for $f \in L^1 \cap L^2(\mathbb{R}^n_+, \mathrm{d}\nu)$ we get

$$\mathrm{e}^{-itL_\alpha} f(x) = \int_{\mathbb{R}^n_+} f(y)K(x, y, 0, t, \alpha)\mathrm{d}\nu(y). \quad (3.5)$$

From Remark 2.2 and Lemma 3.4 we observe that

$$\|\mathrm{e}^{-itL_\alpha} f\|_{L^\infty(\mathbb{R}^n_+,\mathrm{d}\nu)} \leq C|\sin t|^{-(n+\sum\alpha_j)}\|f\|_{L^1(\mathbb{R}^n_+,\mathrm{d}\nu)}.$$

This inequality can be extended to $L^1(\mathbb{R}^n_+, \mathrm{d}\nu)$ by density argument. From Riesz–Thorin interpolation theorem (see [5]), the lemma follows. □

4. Strichartz estimates

DEFINITION 4.1

Let $n \geq 1$ and $\alpha \in (-\frac{1}{2}, \infty)^n$. We say that a pair (q, p) is *admissible* if

$$1 \leq q \leq 2, \;\; 0 \leq \left(n + \sum_{j=0}^{n} \alpha_j\right)\left(1 - \frac{2}{p}\right) < 1$$

or

$$2 < q \leq \infty \text{ and } 0 \leq \left(n + \sum_{j=0}^{n} \alpha_j\right)\left(1 - \frac{2}{p}\right) \leq \frac{2}{q} < 1.$$

Remark 4.2.

(i) The admissibility condition on (q, p) implies that $0 \leq \left(n+\sum_{j=0}^{n} \alpha_j\right)\left(1 - \frac{2}{p}\right) < 1$.
(ii) If $1 \leq q \leq 2, n = 1, 1 + \alpha < 1$, then $p \in [2, \infty]$.
(iii) If $1 \leq q \leq 2, n = 1, 1 + \alpha = 1$, then $p \in [2, \infty)$.
(iv) If $1 \leq q \leq 2$, $(n + \sum_{j=0}^{n} \alpha_j) > 1$, then $p \in \left[2, \frac{2\left(n+\sum_{j=0}^{n} \alpha_j\right)}{\left(n+\sum_{j=0}^{n} \alpha_j\right)-1}\right)$.

Admissible condition is basically got from the following Lemma 4.3 and Remark 4.4 which are useful in proving Strichartz estimate (4.3). This lemma was proved in Lemma 2, pp. 293–294 of [12] for compact interval $[-\pi, \pi]$. We state here for arbitrary compact interval $[a, b]$. Same proof will work here, and hence we skip the proof.

Lemma 4.3. *Let (a, b) be a bounded interval and T be the operator given by*

$$Tf(t) = \int_a^b K(t-s)f(s)\mathrm{d}s.$$

Then the following inequality

$$\|Tf\|_q \;\; \leq \;\; C_K \|f\|_{q'}$$

holds for $q = \infty$ if $K \in L^{\infty}(a-b, b-a)$, for $q \in (2, \infty)$ if $K \in$ weak $L^{\frac{q}{2}}(a-b, b-a)$ and also for $1 \leq q \leq 2$ if $K \in L^1(a-b, b-a)$. Constant C_K is independent of f.

Remark 4.4. Let $p \in [2, \infty], a, b \in \mathbb{R}$. $|\sin t|^{-\left(1-\frac{2}{p}\right)\left(n+\sum \alpha_j\right)} \in$ weak $L^{\frac{q}{2}}(a-b, b-a)$ with $q \in (2, \infty)$ if $1 < \frac{q}{2} \leq \frac{1}{\left(n+\sum_{j=0}^{n} \alpha_j\right)(1-\frac{2}{p})}$ or $(n + \sum_{j=0}^{n} \alpha_j)(1 - \frac{2}{p}) \leq \frac{2}{q} < 1$. Also $|\sin t|^{-(1-\frac{2}{p})(n+\sum \alpha_j)} \in L^1(a-b, b-a)$ if $(n + \sum_{j=0}^{n} \alpha_j)(1 - \frac{2}{p}) < 1$. If we consider $p = 2$ then $|\sin t|^{-(1-\frac{2}{p})(n+\sum \alpha_j)} = 1 \in L^{\infty}(a-b, b-a)$.

Now we prove a lemma which is helpful in proving Strichartz estimates (Theorem 4.6).

Lemma 4.5. *Let* $[a, b]$ *be a bounded interval containing* t_0. *Let* $h_j(x, t) \in L^{q_j'}((a, b), L^2(\mathbb{R}_+^n, \mathrm{d}\nu(x)))$, *where* q_j' *is a conjugate exponent of* q_j *with* $1 \le q_j \le \infty$ *for* $j = 1, 2$. *Then the functions*

$$\mathrm{e}^{-i(t-t_0)L_\alpha} h_1(x, t) \mathrm{e}^{-i(s-t_0)L_\alpha} h_2(x, s), \quad h_1(x, t)\mathrm{e}^{i(t-s)L_\alpha} h_2(x, s)$$

belong to $L^1(\mathbb{R}_+^n \times (a, b) \times (a, b), \mathrm{d}\nu(x) \times \mathrm{d}t \times \mathrm{d}s)$.

Proof. For simplicity we are considering $h_1 = h_2 = h$ and $q_1 = q_2 = q$. Since $h \in L^{q'}((a, b), L^2(\mathrm{d}\nu))$, $h(\cdot, t) \in L^2(\mathbb{R}_+^n, \mathrm{d}\nu)$ for a.e. $t \in (a, b)$. Therefore $\mathrm{e}^{-i(t-t_0)L_\alpha} h(\cdot, t) \in L^2(\mathbb{R}_+^n, \mathrm{d}\nu)$ for a.e. $t \in (a, b)$. Then by Hölder's inequality $\mathrm{e}^{-i(t-t_0)L_\alpha} h(\cdot, t)\mathrm{e}^{-i(s-t_0)L_\alpha} h(\cdot, s) \in L^1(\mathbb{R}_+^n, \mathrm{d}\nu)$ for a.e. $t, s \in (a, b)$ and

$$\int_{\mathbb{R}_+^n} |\mathrm{e}^{-i(t-t_0)L_\alpha} h(x, t)\mathrm{e}^{-i(s-t_0)L_\alpha} h(x, s)|\mathrm{d}\nu(x) \le \|h(\cdot, t)\|_{L^2(\mathrm{d}\nu)} \|h(\cdot, s)\|_{L^2(\mathrm{d}\nu)}.$$

Integrating with respect to t and s over $(a, b) \times (a, b)$ and using Hölder's inequality in the t variable, we get

$$\int_a^b \int_a^b \int_{\mathbb{R}_+^n} |\mathrm{e}^{-i(t-t_0)L_\alpha} h(x, t)\mathrm{e}^{-i(s-t_0)L_\alpha} h(x, s)|\mathrm{d}\nu(x)\mathrm{d}t\mathrm{d}s \le \left(\int_a^b \|h(\cdot, t)\|_{L^2(\mathrm{d}\nu)}\mathrm{d}t\right)^2 \le (b-a)^{\frac{2}{q}} \|h\|^2_{L^{q'}((a,b),L^2(\mathrm{d}\nu))}.$$

Similarly $h_1(x, t)\mathrm{e}^{i(t-s)L_\alpha} h_2(x, s) \in L^1(\mathbb{R}_+^n \times (a, b) \times (a, b), \mathrm{d}\nu(x) \times \mathrm{d}t \times \mathrm{d}s)$ can be proved. □

The main Strichartz type estimates in this paper are compiled in the following two theorems.

Theorem 4.6 (Strichartz estimates). *Let* $[a, b]$ *be a bounded interval containing* t_0. *Let* $f \in L^2(\mathbb{R}_+^n, \mathrm{d}\nu)$ *and* $g \in L^{q'}((a, b), L^{p'}(\mathbb{R}_+^n, \mathrm{d}\nu))$ *where* (q, p) *is an admissible pair and* q', p' *are the corresponding conjugate indices.*

Then $\mathrm{e}^{-i(t-t_0)L_\alpha} f \in L^q((a, b), L^p(\mathrm{d}\nu)) \cap C(\mathbb{R}, L^2(\mathrm{d}\nu)) \cap L^\infty(\mathbb{R}, L^2(\mathrm{d}\nu))$ *and* $\int_{t_0}^t \mathrm{e}^{-i(t-s)L_\alpha} g(x, s)\mathrm{d}s \in L^q((a, b), L^p(\mathrm{d}\nu)) \cap C([a, b], L^2(\mathrm{d}\nu))$. *Furthermore the following estimates hold over* $\mathbb{R}_+^n \times (a, b)$:

$$\|\mathrm{e}^{-i(t-t_0)L_\alpha} f\|_{L^q((a,b),L^p(\mathrm{d}\nu))} \le C\|f\|_{L^2(\mathbb{R}_+^n,\mathrm{d}\nu)}, \tag{4.1}$$

$$\left\|\int_a^b \mathrm{e}^{i(t-t_0)L_\alpha} g(x, t)\mathrm{d}t\right\|_{L^2(\mathbb{R}_+^n,\mathrm{d}\nu)} \le C\|g\|_{L^{q'}((a,b),L^{p'}(\mathrm{d}\nu))}, \tag{4.2}$$

$$\left\|\int_{t_0}^t \mathrm{e}^{-i(t-s)L_\alpha} g(x, s)\mathrm{d}s\right\|_{L^q((a,b),L^p(\mathrm{d}\nu))} \le C\|g\|_{L^{q'}((a,b);\, L^{p'}(\mathbb{R}_+^n,\mathrm{d}\nu))}, \tag{4.3}$$

$$\left\|\int_{t_0}^t \mathrm{e}^{-i(t-s)L_\alpha} g(x, s)\mathrm{d}s\right\|_{L^\infty((a,b),L^2(\mathrm{d}\nu))} \le C\,\|g\|_{L^{q'}((a,b);\, L^{p'}(\mathbb{R}_+^n,\mathrm{d}\nu))}. \tag{4.4}$$

If $g \in L^1((a,b),\ L^2(\mathbb{R}^n_+, \mathrm{d}\nu))$ *then the following estimate holds*:

$$\left\| \int_{t_0}^{t} \mathrm{e}^{-i(t-s)L_\alpha} g(x,s)\mathrm{d}s \right\|_{L^q((a,b),L^p(\mathbb{R}^n_+,\mathrm{d}\nu))} \leq C\|g\|_{L^1((a,b);\ L^2(\mathbb{R}^n_+,\mathrm{d}\nu))}. \quad (4.5)$$

Here constant C is independent of f, g and t_0.

Remark 4.7.

(i) If $t < t_0$, then by integral $\int_{t_0}^{t}$ we mean $-\int_{t}^{t_0}$.
(ii) Estimate (4.3) also holds if we replace integral $\int_{t_0}^{t}$ by $\int_a^b$.
(iii) Since $|\mathrm{e}^{-i(t-t_0)L_\alpha} f(x)|$ is periodic in t (see Remark 3.1), it is not possible to obtain Strichartz estimate (4.1) for unbounded interval when $q < \infty$.
(iv) (q, p) be an arbitrary admissible pair in estimates (4.1), (4.5).
(v) In view of Lemma 3.5, Theorem 4.6 also holds if we replace $t - t_0$ by $-(t - t_0)$ in (4.1), (4.2) and $t - s$ by $-(t - s)$ in (4.3), (4.4) and (4.5).

Proof. We prove estimates in the following order: (4.3), (4.2), (4.1), (4.4), (4.5). Using Minkowski's inequality for integrals and from Lemma 3.5, we get

$$\begin{aligned} &\left\| \int_{t_0}^{t} \mathrm{e}^{-i(t-s)L_\alpha} g(x,s)\mathrm{d}s \right\|_{L^p(\mathrm{d}\nu)} \\ &\quad \leq C \int_a^b |\sin(t-s)|^{-(1-\frac{2}{p})(n+\sum \alpha_j)} \|g(\cdot,s)\|_{L^{p'}(\mathrm{d}\nu)}\mathrm{d}s. \end{aligned}$$

Since (q, p) is an admissible pair, estimate (4.3) follows from the Lemma 4.3 and Remark 4.4.

By density argument it is enough to prove estimate (4.2) for $g \in L^{q'}((a,b), L^{2p'}(\mathbb{R}^n_+, \mathrm{d}\nu))$, see section 8.18 of [4]. Since e^{-itL_α} is the adjoint operator of e^{itL_α} on $L^2(\mathbb{R}^n_+, \mathrm{d}\nu)$, from Lemma 4.5, Hölder's inequality for the mixed L^p spaces and estimate (4.3) with Remark 4.7 (ii), we get estimate (4.2):

$$\begin{aligned} &\left\| \int_a^b \mathrm{e}^{i(t-t_0)L_\alpha} g(\cdot,t)\mathrm{d}t \right\|^2_{L^2(\mathbb{R}^n_+,\mathrm{d}\nu)} \\ &\quad = \left\langle \int_a^b \mathrm{e}^{i(t-t_0)L_\alpha} g(\cdot,t)\mathrm{d}t, \int_a^b \mathrm{e}^{i(s-t_0)L_\alpha} g(\cdot,s)\mathrm{d}s \right\rangle_\nu \\ &\quad = \int_a^b \int_a^b \left\langle \mathrm{e}^{i(t-t_0)L_\alpha} g(\cdot,t), \mathrm{e}^{i(s-t_0)L_\alpha} g(\cdot,s)\right\rangle_\nu \mathrm{d}s\mathrm{d}t \\ &\quad = \int_a^b \int_a^b \left\langle g(\cdot,t), \mathrm{e}^{-i(t-s)L_\alpha} g(\cdot,s)\right\rangle_\nu \mathrm{d}s\mathrm{d}t \\ &\quad = \int_a^b \left\langle g(\cdot,t), \int_a^b \mathrm{e}^{-i(t-s)L_\alpha} g(\cdot,s)\mathrm{d}s \right\rangle_\nu \mathrm{d}t \\ &\quad \leq \|g\|_{L^{q'}((a,b),L^{p'}(\mathrm{d}\nu))} \left\| \int_a^b \mathrm{e}^{-i(t-s)L_\alpha} g(x,s)\mathrm{d}s \right\|_{L^q((a,b),L^p(\mathrm{d}\nu))} \\ &\quad \leq C\|g\|^2_{L^{q'}((a,b),L^{p'}(\mathrm{d}\nu))}. \end{aligned}$$

Estimate (4.1) follows if $(q, p) = (\infty, 2)$. Since $|\mathrm{e}^{-it(2k+n+\sum\alpha_j)} - 1| \leq 2$ and $\|P_k f\|_{L^2(\mathrm{d}\nu)} \in l^2(\mathbb{Z}_{\geq 0})$, $\mathrm{e}^{-itL_\alpha} f(x) \in C(\mathbb{R}, L^2(\mathbb{R}^n_+, \mathrm{d}\nu))$ follows from dominated convergence theorem. $\mathrm{e}^{-itL_\alpha} f(x) \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^n_+, \mathrm{d}\nu))$ follows from (3.4). For $q < \infty$, estimate (4.1) follows from duality argument, estimate (4.2), Lemma 4.5 and the fact that e^{-itL_α} is adjoint operator of e^{itL_α} on $L^2(\mathbb{R}^n_+, \mathrm{d}\nu)$.

By density argument it is enough to prove estimate (4.4) for $g \in L^{q'}((a, b), L^2 \cap L^{p'}(\mathbb{R}^n_+, \mathrm{d}\nu))$. Now we prove estimate (4.4) by using the duality argument in the x-variable. Let $h \in C_c^\infty(\mathbb{R}^n_+)$ with $\|h\|_{L^2(\mathrm{d}\nu)} = 1$. By Hölder's inequality, Lemma 4.5, estimate (4.1) and the fact that e^{-itL_α} is the adjoint operator of e^{itL_α} on $L^2(\mathbb{R}^n_+, \mathrm{d}\nu)$, we get

$$
\begin{aligned}
&\left|\left\langle \int_{t_0}^{t} \mathrm{e}^{-i(t-s)L_\alpha} g(\cdot, s)\mathrm{d}s, h\right\rangle_\nu\right| \\
&\quad = \left|\int_{t_0}^{t} \langle g(\cdot, s), \mathrm{e}^{i(t-s)L_\alpha} h\rangle_\nu \mathrm{d}s\right| \\
&\quad \leq \int_a^b |\langle g(\cdot, s), \mathrm{e}^{i(t-s)L_\alpha} h\rangle_\nu| \mathrm{d}s \\
&\quad \leq \|g\|_{L^{q'}((a,b),L^{p'}(\mathrm{d}\nu))} \left\|\mathrm{e}^{-i(s-t_0)L_\alpha}(\mathrm{e}^{i(t-t_0)L_\alpha} h)\right\|_{L^q((a,b)(\mathrm{d}s),L^p(\mathrm{d}\nu))} \\
&\quad \leq C\|g\|_{L^{q'}((a,b),L^{p'}(\mathrm{d}\nu))} \|\mathrm{e}^{i(t-t_0)L_\alpha} h\|_{L^2(\mathrm{d}\nu)}. \\
&\quad = C\|g\|_{L^{q'}((a,b),L^{p'}(\mathrm{d}\nu))} \|h\|_{L^2(\mathrm{d}\nu)}.
\end{aligned}
$$

Taking supremum over all h with $\|h\|_{L^2(\mathrm{d}\nu)} = 1$ and then supremum over $t \in (a, b)$, we get the required estimate.

Estimate (4.5) follows from estimate (4.4) if $(q, p) = (\infty, 2)$. So we assume that $(q, p) \neq (\infty, 2)$. To prove estimate (4.5), take $h \in L^{q'}((a, b), L^2 \cap L^{p'}(\mathrm{d}\nu))$. Now from Lemma 4.5, Remark 4.7(i) and the fact that e^{-itL_α} is the adjoint operator of e^{itL_α} on $L^2(\mathbb{R}^n_+, \mathrm{d}\nu)$, we observe the following:

$$
\begin{aligned}
&\int_a^b \left\langle \int_{t_0}^{t} \mathrm{e}^{-i(t-s)L_\alpha} g(\cdot, s)\mathrm{d}s, h(\cdot, t)\right\rangle_\nu \mathrm{d}t \\
&\quad = \left(\int_{t=a}^{t_0}\int_{s=t}^{t_0} + \int_{t=t_0}^{b}\int_{s=t_0}^{t}\right)\langle g(\cdot, s), \mathrm{e}^{i(t-s)L_\alpha} h(\cdot, t)\rangle_\nu \mathrm{d}s\ \mathrm{d}t \\
&\quad = \left(\int_{s=a}^{t_0}\int_{t=a}^{s} + \int_{s=t_0}^{b}\int_{t=s}^{b}\right)\langle g(\cdot, s), \mathrm{e}^{i(t-s)L_\alpha} h(\cdot, t)\rangle_\nu \mathrm{d}t\ \mathrm{d}s \\
&\quad = \int_a^{t_0} \left\langle g(\cdot, s), \int_a^s \mathrm{e}^{-i(s-t)L_\alpha} h(\cdot, t)\mathrm{d}t\right\rangle_\nu \mathrm{d}s \\
&\qquad + \int_{t_0}^{b} \left\langle g(\cdot, s), \int_s^b \mathrm{e}^{-i(s-t)L_\alpha} h(\cdot, t)\mathrm{d}t\right\rangle_\nu \mathrm{d}s.
\end{aligned}
$$

In view of estimate (4.4) and Hölder's inequality, estimate (4.5) follows.

Now we prove $\int_{t_0}^t e^{-i(t-s)L_\alpha} g(x,s)\mathrm{d}s \in C([a,b], L^2(\mathrm{d}\nu))$. Let $t_m \to t$. We take $h \in L^2(\mathbb{R}_+^n, \mathrm{d}\nu)$,

$$\begin{aligned}
&\left|\left\langle \int_{t_0}^t e^{-i(t_m-s)L_\alpha} g(\cdot,s)\mathrm{d}s - \int_{t_0}^t e^{-i(t-s)L_\alpha} g(\cdot,s)\mathrm{d}s, h\right\rangle_\nu\right| \\
&\quad = \left|\int_{t_0}^t \left\langle e^{-i(t_m-s)L_\alpha} g(\cdot,s) - e^{-i(t-s)L_\alpha} g(\cdot,s), h\right\rangle_\nu \mathrm{d}s\right| \\
&\quad = \left|\int_{t_0}^t \left\langle g(\cdot,s), \left(e^{i(t_m-s)L_\alpha} - e^{i(t-s)L_\alpha}\right)h\right\rangle_\nu \mathrm{d}s\right| \\
&\quad \le \int_a^b \|g(\cdot,s)\|_{L^{p'}(\mathrm{d}\nu)} \|e^{-i(s-t_0)L_\alpha}\left(e^{i(t_m-t_0)L_\alpha}h - e^{i(t-t_0)L_\alpha}h\right)\|_{L^p(\mathrm{d}\nu)}\mathrm{d}s \\
&\quad \le \|g\|_{L^{q'}((a,b),L^{p'}(\mathrm{d}\nu))} \\
&\qquad \times \|e^{-i(s-t_0)L_\alpha}\left(e^{i(t_m-t_0)L_\alpha}h - e^{i(t-t_0)L_\alpha}h\right)\|_{L^q((a,b)(\mathrm{d}s),L^p(\mathrm{d}\nu))} \\
&\quad \le C\|g\|_{L^{q'}((a,b),L^{p'}(\mathrm{d}\nu))}\|\left(e^{i(t_m-t_0)L_\alpha} - e^{i(t-t_0)L_\alpha}\right)h\|_{L^2(\mathrm{d}\nu)}.
\end{aligned}$$

This shows that $\int_{t_0}^t e^{-i(t_m-s)L_\alpha} g(\cdot,s)\mathrm{d}s \to \int_{t_0}^t e^{-i(t-s)L_\alpha} g(\cdot,s)\mathrm{d}s$ weakly in $L^2(\mathbb{R}_+^n, \mathrm{d}\nu)$. $L^2(\mathbb{R}_+^n, \mathrm{d}\nu)$ norm of this sequence is constant and equal to

$$\begin{aligned}
&\left\|\int_{t_0}^t e^{-i(t_m-s)L_\alpha} g(\cdot,s)\mathrm{d}s\right\|_{L^2(\mathbb{R}_+^n,\mathrm{d}\nu)} \\
&\quad = \left\|e^{-i(t_m-t)L_\alpha}\int_{t_0}^t e^{-i(t-s)L_\alpha} g(\cdot,s)\mathrm{d}s\right\|_{L^2(\mathbb{R}_+^n,\mathrm{d}\nu)} \\
&\quad = \left\|\int_{t_0}^t e^{-i(t-s)L_\alpha} g(\cdot,s)\mathrm{d}s\right\|_{L^2(\mathbb{R}_+^n,\mathrm{d}\nu)}.
\end{aligned}$$

Therefore we have the convergence in $L^2(\mathbb{R}_+^n, \mathrm{d}\nu)$:

$$\int_{t_0}^t e^{-i(t_m-s)L_\alpha} g(\cdot,s)\mathrm{d}s \to \int_{t_0}^t e^{-i(t-s)L_\alpha} g(\cdot,s)\mathrm{d}s.$$

Also since $\left\|\int_t^{t_m} e^{-i(t_m-s)L_\alpha} g(\cdot,s)\mathrm{d}s\right\|_{L^2(\mathrm{d}\nu)} \le C\|g\|_{L^{q'}([t,t_m],L^{p'}(\mathrm{d}\nu))} \to 0$ as $t_m \to t$, we conclude that $\int_{t_0}^t e^{-i(t-s)L_\alpha} g(x,s)\mathrm{d}s \in C([a,b], L^2(\mathrm{d}\nu))$. □

Now we prove Strichartz type estimates for different admissible pairs.

Theorem 4.8 (Strichartz estimates). *Let (q_1,p_1), (q_2,p_2), (q_3,p_3), (q_4,p_4) be admissible pairs such that $q_1, q_2 > 2$ and*

$$\frac{2}{q_1} = \left(n + \sum_{j=0}^n \alpha_j\right)\left(1 - \frac{2}{p_1}\right), \quad \frac{2}{q_2} = \left(n + \sum_{j=0}^n \alpha_j\right)\left(1 - \frac{2}{p_2}\right).$$

Let (a,b) be a bounded interval containing t_0. If $g \in L^{q_1'}((a,b);\ L^{p_1'}(\mathbb{R}^n_+, \mathrm{d}\nu))$ then following estimate holds:

$$\left\| \int_{t_0}^{t} \mathrm{e}^{-i(t-s)L_\alpha} g(x,s)\mathrm{d}s \right\|_{L^{q_2}((a,b),L^{p_2}(\mathrm{d}\nu))} \leq C \|g\|_{L^{q_1'}((a,b),\ L^{p_1'}(\mathbb{R}^n_+,\mathrm{d}\nu))}. \tag{4.6}$$

If $g \in L^{q_3'}((a,b);\ L^{p_3'}(\mathbb{R}^n_+, \mathrm{d}\nu))$ then following estimate holds:

$$\left\| \int_{t_0}^{t} \mathrm{e}^{-i(t-s)L_\alpha} g(x,s)\mathrm{d}s \right\|_{L^{q_4}((a,b),L^{p_4}(\mathrm{d}\nu))} \leq C(b-a)^{\kappa} \|g\|_{L^{q_3'}((a,b),\ L^{p_3'}(\mathrm{d}\nu))}. \tag{4.7}$$

Here constant C is independent of g and t_0. Here $\kappa = \kappa_1 + \kappa_2$ and

$$\begin{aligned} \kappa_1 &= \frac{1}{q_4} - \left(n + \sum_{j=0}^{n} \alpha_j\right)\left(\frac{1}{2} - \frac{1}{p_4}\right), \\ \kappa_2 &= \frac{1}{q_3} - \left(n + \sum_{j=0}^{n} \alpha_j\right)\left(\frac{1}{2} - \frac{1}{p_3}\right). \end{aligned}$$

Proof. Estimate (4.6) follows from bilinear Riesz–Thorin interpolation theorem and also estimates (4.3), (4.4) and (4.5) (see proof of estimate (23) in [20] and also Step 4, page 36 of [2]).

To prove estimate (4.7), let us define $\tilde{q}_3, \tilde{q}_4$ by

$$\begin{aligned} \frac{2}{\tilde{q}_3} &= \left(n + \sum_{j=0}^{n} \alpha_j\right)\left(1 - \frac{2}{p_3}\right) < 1, \\ \frac{2}{\tilde{q}_4} &= \left(n + \sum_{j=0}^{n} \alpha_j\right)\left(1 - \frac{2}{p_4}\right) < 1. \end{aligned}$$

Then $1 \leq q_3 \leq \tilde{q}_3$, $1 \leq q_4 \leq \tilde{q}_4$, $2 < \tilde{q}_3, \tilde{q}_4$. By Hölder's theorem in the t-variable and estimate (4.6) we obtain estimate (4.7):

$$\begin{aligned} &\left\| \int_{t_0}^{t} \mathrm{e}^{-i(t-s)L_\alpha} g\mathrm{d}s \right\|_{L^{q_4}((a,b),L^{p_4}(\mathrm{d}\nu))} \\ &\quad \leq (b-a)^{\kappa_1} \left\| \int_{t_0}^{t} \mathrm{e}^{-i(t-s)L_\alpha} g\mathrm{d}s \right\|_{L^{\tilde{q}_4}((a,b),L^{p_4}(\mathrm{d}\nu))} \\ &\quad \leq C(b-a)^{\kappa_1} \|g\|_{L^{\tilde{q}_3'}((a,b),L^{p_3'}(\mathrm{d}\nu))} \\ &\quad \leq C(b-a)^{\kappa_1+\kappa_2} \|g\|_{L^{q_3'}((a,b),L^{p_3'}(\mathrm{d}\nu))}. \end{aligned}$$

This completes the proof. □

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