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Sign changing solutions of the $p(x)$-Laplacian equation

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Abstract. This paper deals with the variational and Nehari manifold method for the $p(x)$-Laplacian equations in a bounded domain or in the whole space. We prove existence of sign changing solutions under certain conditions.

Keywords. $p(x)$-Laplacian; elliptic equation; Nehari manifold; nodal solution.

1. Introduction

The study of differential equations and variational problems with variable exponent has been a new and interesting topic. Its interest is widely justified with many physical examples, such as nonlinear elasticity theory, electrorheological fluids, etc. (see [21, 22]). It also has wide applications in different research fields, such as image processing model (see e.g. [5, 13]), stationary thermorheological viscous flows (see [2]) and the mathematical description of the processes filtration of an idea barotropic gas through a porous medium (see [3]).

The study on variable exponent problems is attracting more and more interest in recent years, for example, there have been many contributions to nonlinear elliptic problems associated with the $p(x)$-Laplacian (see [14] for a thorough overview of the recent advantages) from various view points.

In this paper, we investigate the following Dirichlet problem of $p(x)$-Laplacian

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary, $p(x) \in C(\bar{\Omega})$ with $1 < p^- := \min_{x\in\bar{\Omega}} p(x) \le p^+ := \max_{x\in\bar{\Omega}} p(x) < +\infty$, $f(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function satisfying the conditions given in § 3.

The operator $-\Delta_{p(x)}$ is called $p(x)$-Laplacian, which becomes p-Laplacian when $p(x) \equiv p$ (a constant). The solvability of problem (P) can be studied by several approaches, for example, variational method [11, 15, 16, 20], topological method [4, 16], sub-supersolution method [9], Nehari manifold method [18] and monotone mapping theory [19]. The goal of this paper is to give existence of sign changing solutions for (P) using variational and Nehari manifold method, which is a new research topic. And under some assumptions, there exist three different nontrivial solutions of (P). Moreover, these solutions are one positive, one negative and the other one has non-constant sign. The results obtained are generalizations of well-known results for p-Laplacian problems.

This paper is composed of three sections. In § 2, we recall the definition of variable exponent Lebesgue spaces, $L^p(x)(\Omega)$, as well as Sobolev spaces, $W^{1,p(x)}(\Omega)$. Moreover, some properties of these spaces will be also exhibited to be used later. In § 3, we prove the existence of positive and negative solutions withour Ambrosetti–Rabinowitz condition. Moreover, we also give the existence of a sign-changing solution having exactly two nodal domain which are new results.

2. Preliminary results

Here, we introduce some definitions and results which will be used in the next section.

Firstly, we introduce some theories of the Lebesgue–Sobolev space with variable exponent. The detailed can be found in [6–8, 10, 12, 17].

Set $C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) : h(x) > 1 \text{ for any } x \in \bar{\Omega}\}$. In this paper, for any $h \in C_+(\bar{\Omega})$, we will denote

$$h^- = \min_{x\in\bar{\Omega}} h(x), \quad h^+ = \max_{x\in\bar{\Omega}} h(x)$$

and denote by $h_1 \ll h_2$ the fact that $\inf_{x\in\Omega}(h_2(x) - h_1(x)) > 0$.

For $p(x) \in C_+(\bar{\Omega})$, we define the variable exponent Lebesgue space:
$L^{p(x)}(\Omega) = \{u : u \text{ is a measurable real value function } \int_\Omega |u(x)|^{p(x)}\mathrm{d}x < +\infty\}$,
with the norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\mathrm{d}x \le 1\}$,
and define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm $\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}$.

We remember that spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces. Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Denote by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. Then the Hölder type inequality

$$\int_\Omega |uv|\mathrm{d}x \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{L^{p(x)}(\Omega)}|v|_{L^{q(x)}(\Omega)}, \; u \in L^{p(x)}(\Omega), v \in L^{q(x)}(\Omega) \tag{1}$$

holds. Furthermore, define the mapping $\varrho : W^{1,p(x)} \to \mathbb{R}$ by

$$\varrho(u) = \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\mathrm{d}x.$$

Then the following relations hold:

$$\|u\| < 1 (= 1, > 1) \Leftrightarrow \varrho(u) < 1 (= 1, > 1), \tag{2}$$

$$\|u\| > 1 \Rightarrow \|u\|^{p^-} \le \varrho(u) \le \|u\|^{p^+}, \tag{3}$$

$$\|u\| < 1 \Rightarrow \|u\|^{p^+} \le \varrho(u) \le \|u\|^{p^-}. \tag{4}$$

Remark 2.1. If $h \in C_+(\bar{\Omega})$ and $h(x) \le p^*(x)$ for any $x \in \bar{\Omega}$, by Theorem 2.3 in [17], we deduce that $W_0^{1,p(x)}(\Omega)$ is continuously embedded in $L^{h(x)}(\Omega)$. When $h(x) < p^*(x)$, the embedding is compact.

3. The main results and proof of the theorem

In this section we will discuss the existence of weak solutions of (P).

DEFINITION 3.1

We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (P), if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \mathrm{d}x = \int_\Omega f(x,u) v \mathrm{d}x,$$

for every $v \in W_0^{1,p(x)}(\Omega)$.

Define

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x,\ \Psi(u) = \int_\Omega F(x,u)\mathrm{d}x,$$

where $F(x,u) = \int_0^u f(x,t)\mathrm{d}t$. The energy functional $\varphi = \Phi - \Psi : W_0^{1,p(x)}(\Omega) \to \mathcal{R}$ associated with problem (P) is well defined. Then it is easy to see that $\varphi \in C^1(W_0^{1,p(x)}(\Omega))$ is weakly lower semi-continuous and $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (P) if and only if u is a critical point of φ. Indeed, we have

$$\begin{aligned} \varphi'(u)v &= \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \mathrm{d}x - \int_\Omega f(x,u)v\mathrm{d}x \\ &= \Phi'(u)v - \Psi'(u)v,\ \forall u, v \in W_0^{1,p(x)}(\Omega). \end{aligned}$$

Firstly, we use the Nehari method to study the existence of solutions. Our hypotheses on nonsmooth potential $f(x,t)$ are as follows:

H(f): $f : \Omega \times \mathcal{R} \to \mathcal{R}$ is a continuous function, satisfying

(i) f is C^1 in t;
(ii) $f(x,t) = 0(|t|^{p^+-1})$ as $|t| \to 0$ uniformly in x;
(iii) there exist $\mu > p^+$ and $p^+ \ll q \ll p^*$, such that

$$\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^\mu} = +\infty, \qquad \lim_{|t|\to\infty} \frac{|f(x,t)|}{|t|^{q(x)-1}} = 0$$

uniformly in $x \in \Omega$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$, $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$;
(iv) for each $x \in \Omega$, $\frac{\partial}{\partial t}\left(\frac{f(x,t)}{|t|^{p^+-1}}\right) > 0$ for $|t| > 0$.

Solutions (P) correspond to critical points of the C^1 functional

$$\varphi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x, \quad u \in W_0^{1,p(x)}(\Omega).$$

For a function $u(x)$ we use $u_+(x) = \max\{u(x), 0\}$ and $u_-(x) = \min\{u(x), 0\}$.

Theorem 3.1. *If hypotheses* H(f) *hold, then the problem* (P) *has a weak solution* $u \in W_0^{1,p(x)}(\Omega)$ *such that*

$$\varphi(u) = \max_{t>0} \varphi(tu) = \inf_{v \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \max_{t>0} \varphi(tv) > 0.$$

Proof. The proof is divided into the following three steps:

Step 1. We will show that 0 is a strict local minimum of φ. By conditions H(f)(i)–(iii), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(x,t)| \leq \varepsilon |t|^{p^+} + C_\varepsilon |t|^{q(x)}.$$

So,

$$\begin{aligned}\varphi(u) &= \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x \\ &\geq \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)} \mathrm{d}x - \varepsilon \int_\Omega |u|^{p^+} \mathrm{d}x - C_\varepsilon \int_\Omega |u|^{q(x)} \mathrm{d}x.\end{aligned}$$

Note that $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, so there exists a $c_0 > 0$ such that $|u|_{q(x)} \leq c_0 \|u\|$. Hence, for $\|u\| = \rho (\leq \frac{1}{c_0})$, we have $|u|_{q(x)} < 1$,

$$|u|_{q(x)}^{q^+} \leq \int_\Omega |u|^{q(x)} \mathrm{d}x \leq |u|_{q(x)}^{q^-}.$$

Thus,

$$\begin{aligned}\varphi(u) &\geq \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)} \mathrm{d}x - \varepsilon |u|_{p^+}^{p^+} - C_\varepsilon \|u\|^{q^-} \\ &\geq \frac{1}{p^+} \|u\|^{p^+} - \bar{c}_0^{p^+} \varepsilon \|u\|^{p^+} - C_\varepsilon \|u\|^{q^-}.\end{aligned}$$

Here we used the Sobolev embedding with constant $\bar{c}_0$, choose $\bar{c}_0^{p^+} \varepsilon = \frac{1}{2p^+}$, then

$$\varphi(u) \geq \|u\|^{p^+} \left(\frac{1}{2p^+} - C_\varepsilon \|u\|^{q^- - p^+} \right),$$

which shows that

$$\varphi(u) > 0 \ \text{ if } 0 < \|u\| < \min \left\{ \left(\frac{1}{2p^+ C_\varepsilon} \right)^{\frac{1}{q^- - p^+}}, \frac{1}{c_0}, 1 \right\}.$$

Step 2. We will show that for any $u \neq 0$, $\varphi(tu) \to -\infty$, as $t \to +\infty$.

By H(f)(ii)–(iii), there exists $l > 0$ such that $F(x,t) \geq l|t|^\mu - C$ for any $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Hence, for $u \neq 0$,

$$\begin{aligned}\varphi(tu) &\leq \frac{1}{p^-} \int_\Omega |t|^{p(x)} |\nabla u|^{p(x)} \mathrm{d}x - l \int_\Omega t^\mu |u|^\mu \mathrm{d}x - C\mathrm{meas}(\Omega) \\ &\leq \frac{t^{p^+}}{p^-} \int_\Omega |\nabla u|^{p(x)} \mathrm{d}x - l \int_\Omega t^\mu |u|^{p^+} \mathrm{d}x - C\mathrm{meas}(\Omega) \\ &\to -\infty, \text{ as } t \to +\infty.\end{aligned}$$

Thus, by Step 1 and Step 2,

$$c = \inf_{v \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \max_{t>0} \varphi(tv) > 0$$

is well-defined. Let $\{u_n\}$ be a minimizing sequence of c such that

$$\varphi(u_n) = \max_{t>0} \varphi(tu_n) \to c$$

as $n \to \infty$. We first prove that $\{u_n\}$ is bounded. If not, we consider $v_n = \frac{u_n}{\|u_n\|}$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in $W_0^{1,p(x)}(\Omega)$, $v_n \to v$ in $L^{p(x)}(\Omega)$ and $v_n(x) \to v(x)$ a.e. $x \in \Omega$.

(ℵ) If $v(x) \neq 0$, we have $|u_n(x)| \to +\infty$ a.e. $x \in \Omega$, then using H(f)(iii), we obtain

$$\frac{F(x, u_n(x))}{|u_n(x)|^{\mu}} |v_n(x)|^{\mu} \to +\infty \text{ a.e. } x \in \Omega.$$

Since $\|u_n\| > 1$ for n large, then by H(f)(iii) and Fatou's lemma we have

$$\begin{aligned}
\frac{1}{p^-} &\geq \lim_{n\to\infty} \frac{1}{p^- \|u_n\|^{p^+}} \left(\int_\Omega |\nabla u_n|^{p(x)} \mathrm{d}x + \int_\Omega |u_n|^{p(x)} \mathrm{d}x \right) \\
&\geq \lim_{n\to\infty} \frac{1}{\|u_n\|^{p^+}} \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} \mathrm{d}x \\
&\geq \lim_{n\to\infty} \frac{1}{\|u_n\|^{\mu}} \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} \mathrm{d}x \\
&= \lim_{n\to\infty} \frac{1}{\|u_n\|^{\mu}} \left(\varphi(u_n) + \int_\Omega F(x, u_n) \mathrm{d}x \right) \\
&\geq \int_\Omega \lim_{n\to\infty} \frac{F(x, u_n)}{|u_n|^{\mu}} |v_n|^{\mu} \mathrm{d}x - 1 \\
&\to +\infty, \text{ as } n \to +\infty,
\end{aligned}$$

which is impossible.

(ℵℵ) If $v(x) = 0$, then fixing an $R > \max\{1, (p^+c)^{\frac{1}{p^-}}\}$, we have

$$\begin{aligned}
c \leftarrow \varphi(u_n) &\geq \varphi\left(\frac{Ru_n}{\|u_n\|}\right) \\
&= \varphi(Rv_n) \\
&= \int_\Omega \frac{1}{p(x)} |R\nabla v_n|^{p(x)} \mathrm{d}x - \int_\Omega F(x, Rv_n) \mathrm{d}x \\
&\geq \frac{1}{p^+} R^{p^-} \int_\Omega |\nabla v_n|^{p(x)} \mathrm{d}x - \int_\Omega F(x, Rv_n) \mathrm{d}x \\
&= \frac{1}{p^+} R^{p^-} - \int_\Omega F(x, Rv_n) \mathrm{d}x \\
&\to \frac{1}{p^+} R^{p^-}, \text{ as } n \to +\infty.
\end{aligned}$$

So, $c \geq \frac{1}{p^+} R^{p^-} \Rightarrow R \leq (p^+ c)^{\frac{1}{p^-}}$, which is impossible. Thus, $\{u_n\}$ is bounded. Then we can assume, without loss, $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$. Since, for some $\eta > 0$,

$$\eta < \int_\Omega |\nabla u|^{p(x)} \mathrm{d}x = \int_\Omega u_n f(x, u_n) \mathrm{d}x \to \int_\Omega u f(x, u) \mathrm{d}x,$$

as $n \to \infty$, $u \neq 0$. There is $s > 0$ such that $\varphi(su) = \max_{t>0} \varphi(tu)$. Then

$$c \leq \varphi(su) \leq \liminf_{n\to\infty} \varphi(su_n) \leq \liminf_{n\to\infty} \varphi(u_n) = c.$$

Step 3. We will show that for any $u \neq 0$, $\exists! s = s_u > 0$, such that $\varphi(su) = \max_{t>0} \varphi(tu)$.

Set $g(t) = \varphi(tu)$ for $t > 0$. We prove next that $g(t)$ has an unique critical point for $t > 0$. Consider a critical point

$$\begin{aligned} g'(t) =& \langle \varphi'(tu), u \rangle \\ =& \int_\Omega t^{p(x)-1} |\nabla u|^{p(x)} \mathrm{d}x - \int_\Omega f(x, tu) u \mathrm{d}x \\ =& 0. \end{aligned}$$

From H(f)(iv), for all $t > 0$, we have

$$\frac{t^2 f'(x, t) - (p^+ - 1) t f(x, t)}{t^{p^+ + 1}} > 0,$$

which implies that

$$\begin{aligned} g''(t) =& \langle \varphi''(tu), u \rangle \\ =& \int_\Omega (p(x) - 1) t^{p(x)-2} |\nabla u|^{p(x)} \mathrm{d}x - \int_\Omega f'(x, tu) u^2 \mathrm{d}x \\ \leq& \frac{p^+ - 1}{t} \int_\Omega t^{p(x)-1} |\nabla u|^{p(x)} \mathrm{d}x - \int_\Omega f'(x, tu) u^2 \mathrm{d}x \\ =& \frac{p^+ - 1}{t} \int_\Omega f(x, tu) u \mathrm{d}x - \int_\Omega f'(x, tu) u^2 \mathrm{d}x \\ =& \frac{p^+ - 1}{t^2} \int_\Omega (f(x, tu) tu - f'(x, tu)(tu)^2) \mathrm{d}x \\ <& 0, \end{aligned}$$

i.e. if t is a critical point g, then it must be a strict local maximum. This implies the uniqueness.

Step 4. We will show that su is a critical point φ.

Since $\max_{t>0} \varphi(tu)$ is achieved at only one point $t = s$, it is also the unique point at which $\langle \varphi'(su), u \rangle = 0$. Next we claim that su is a critical point φ. Without loss of generality, we can assume that $s = 1$. If u is not a critical point, then there is $v \in C_0^\infty(\Omega)$ such that $\langle \varphi'(u), v \rangle = -2$. There is $\varepsilon_0 > 0$ such that

$$\langle \varphi'(tu + \varepsilon v), v \rangle \leq -1,$$

for $|t - 1| + |\varepsilon| \leq \varepsilon_0$.

Consider the two dimensional plane spanned by u and v. For $\varepsilon > 0$ small, let $t_\varepsilon > 0$ be the unique number such that $\max_{t>0} \varphi(t(u+\varepsilon v)) = \varphi(t_\varepsilon(u+\varepsilon v))$. Then $t_\varepsilon \to 1$ as $\varepsilon \to 0$. For ε small such that $|t_\varepsilon - 1| + t_\varepsilon \varepsilon \le \varepsilon_0$, we have a contradiction as follows.

On the one hand,

$$\varphi(t_\varepsilon(u+\varepsilon v)) \ge c.$$

On the other hand,

$$\varphi(t_\varepsilon(u+\varepsilon v)) = \varphi(t_\varepsilon u) + \int_0^1 \langle \varphi'(t_\varepsilon(u+st_\varepsilon\varepsilon v)), t_\varepsilon \varepsilon v\rangle \mathrm{d}s \le c - t_\varepsilon\varepsilon < c.$$

□

Theorem 3.2. *If hypotheses* H(f) *hold, then the problem* (P) *has a weak sign-changing solution* $w \in W_0^{1,p(x)}(\Omega)$ *such that* w *has exactly two nodal domains and*

$$\varphi(w) = \max_{t>0} \varphi(tw_+) + \max_{t>0} \varphi(tw_-) = \inf_{v_\pm \ne 0} \{\max_{t>0} \varphi(tv_+) + \max_{t>0} \varphi(tv_-)\} > 0.$$

Proof. It is easy to see that

$$c_1 \triangleq \inf_{v_\pm \ne 0} \left\{ \max_{t>0} \varphi(tv_+), \max_{t>0} \varphi(tv_-) \right\} \ge 2c > 0.$$

Let $\{u_n\}$ be a minimizing sequence for c_1 such that $\lim_{n\to\infty} \varphi(u_n) = c_1$ and

$$\varphi((u_n)_+) = \max_{t>0}\{\varphi(t(u_n)_+)\}, \varphi((u_n)_-) = \max_{t>0}\{\varphi(t(u_n)_-)\}.$$

We can prove as in the proof of Theorem 3.1 that $\{u_n\}$ is bounded. Then for a subsequence

$$u_n \rightharpoonup u, (u_n)_+ \rightharpoonup u_+ \text{ and } (u_n)_- \rightharpoonup u_-.$$

Then using the weakly semi-continuity of φ we show that there are $a > 0$ and $b > 0$ such that

$$\varphi(au_+ + bu_-) = \max_{t>0} \varphi(tau_+) + \max_{t>0} \varphi(tbu_-) = c_1.$$

Let $w = au_+ + bu_-$. Next we show that w is the solution desired. If w is not a critical point, there is $\eta \in C_0^\infty(\mathbb{R}^N)$ such that $\langle \varphi'(w), \eta\rangle = -2$. Then there is a $\delta > 0$ such that if $|t-1| + |s-1| \le \delta$ and $0 \le \varepsilon \le \delta$,

$$\langle \varphi'(tw_+ + sw_- + \varepsilon\eta), \eta\rangle \le -1 \tag{5}$$

holds.

Let $D = \{(t,s) \in \mathbb{R}^2 : |t-1| \le \delta, |s-1| \le \delta)\}$. Choose a continuous function $h : D \to [0,1]$ such that

$$h(t,s) = \begin{cases} 1, & \text{if } |t-1| \le \dfrac{\delta}{4}, |s-1| \le \dfrac{\delta}{4}, \\ 0, & \text{if } |t-1| \ge \dfrac{\delta}{2}, |s-1| \ge \dfrac{\delta}{2}. \end{cases}$$

Denote, for $(t,s) \in D$,

$$G(t,s) = tw_+ + sw_- + \delta h(t,s)\eta.$$

Then $G \in C(D, W_0^{1,p(x)}(\Omega))$. Define $H : D \to \mathbb{R}^2$ as

$$H(t,s) = (K([G(t,s)]_+), K([G(t,s)]_-)),$$

where $K(u) =< \varphi'(u), u >$ for $u \in W_0^{1,p(x)}(\Omega)$. Thus, $H \in C(D, \mathbb{R}^2)$.

If $|t-1| = \delta$ or $|s-1| = \delta$, $h(t,s) = 0$, therefore $H(t,s) = (K(tw_+), K(tw_-)) \neq (0,0)$. As a consequence, the degree $\deg(H, \operatorname{int}(D), 0)$ is well defined and $\deg(H, \operatorname{int}(D), 0) = 1$. Thus there is a $(t,s) \in \operatorname{int}(D)$ such that $H(t,s) = 0$. In the following we fix (t,s). Then we have

$$\varphi(G(t,s)) \geq c_1.$$

On the other hand, from (5) we arrive at

$$\begin{aligned}\varphi(G(t,s)) =& \varphi(tw_+ + sw_-) \\ &+ \int_0^1 \langle \varphi'(tw_+ + sw_- + \theta\delta h(t,s)\eta, \delta h(t,s)\eta)\rangle \mathrm{d}\theta \\ &\leq \varphi(tw_+) + \varphi(tw_-) - \delta h(t,s).\end{aligned}$$

If t or s is not equal to 1 we have the right-hand side strictly less than c_1. If $t = s = 1$, by $h(t,s) = 1$ we also have a contradiction. Thus w is a weak solution of (P).

If w has more than two nodal domain, say, there are Ω_i for $i = 1, 2, 3$, open sets made up from nodal domain. Let $w_i = w|_{\Omega_i}$ and assume $w_1 \geq 0$, $w_2 \leq 0$ and $w_3 \leq 0$. Then we may consider $v = w_1 + w_2$ and $v_\pm \neq 0$. But $\max_{t>0} \varphi(tv_+) + \max_{t>0} \varphi(tv_-) = \varphi(w_1 + w_2) < \varphi(w) = c_1$, a contradiction. The proof is complete. □

COROLLARY 3.1

If hypotheses H(f) *hold, then the problem* (P) *has at least three solutions: one positive, one negative, and one sign-changing solution having exactly two nodal domains.*

Specially, when $p(x) = p$, problem (P) *reduces to the following nonlinear problem:*

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x,u), & in\ \Omega, \\ u = 0, & on\ \partial\Omega. \end{cases} \tag{P$_1$}$$

The next theorem concerns problems where the potential is only a Caratheodory function. The hypotheses on the potential function are as follows:

H(f)$_1$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, satisfying

(i) $f(x,t) = 0(|t|^{p-1})$ as $|t| \to 0$ uniformly in x;

(ii) there exist $\mu > p^+$ and $p^+ < q < p^*$, such that

$$\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^\mu} = +\infty, \qquad \lim_{|t|\to\infty} \frac{|f(x,t)|}{|t|^{q-1}} = 0$$

uniformly in $x \in \Omega$, where $p^* = \frac{Np}{N-p}$, $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$;

(iii) $f(x,t)t - pF(x,t)$ is nondecreasing in $|t|$ and increasing for $|t| > 0$ small.

Theorem 3.3. *If hypotheses* H(f)$_1$ *hold, then the problem* (P$_1$) *has at least three solutions: one positive, one negative, and one sign-changing solution having exactly two nodal domains.*

Proof. The steps are similar to those of Theorem 3.1. In fact, we only need to modify Step 3 as follows:

Step 3′. We show that for any $u \neq 0$, $\exists! s = s_u > 0$, such that $\varphi(su) = \max_{t>0} \varphi(tu)$ under the condition H(f)$_1$(iii). Then from Steps 1, 2, 3′, 4 above, the problem (P_1) has at least three solutions.

Set $g(s) = \varphi(su)$ for $s > 0$. Assume $g(s)$ has max at t, so $g'(t) = 0$, that is,

$$\begin{aligned} g'(t) =& \langle \varphi'(tu), u\rangle \\ =& \int_\Omega t^{p(x)-1}|\nabla u|^{p(x)}\mathrm{d}x - \int_\Omega f(x,tu)u\mathrm{d}x \\ =& 0. \end{aligned}$$

Let $c = g(t)$ be the max value. $c = \frac{1}{p}[\int_\Omega (f(x,su)su - pF(x,su))\mathrm{d}x]$ when $s = t$ and the right hand side is monotone increasing in s. So there is only one t.

COROLLARY 3.2

If hypotheses H(f)(i), (ii), (iii) *and the following condition* (iv)′ *hold, then the problem* (P_1) *has at least three solutions*: *one positive*, *one negative*, *and one sign-changing solution having exactly two nodal domains.*

Step iv′. *For each* $x \in \Omega$, $\frac{\partial}{\partial t}\left(\frac{f(x,t)}{|t|^{p-1}}\right) > 0$ *for* $|t| > 0$.

Remark 3.1. It is easy to see our arguments generalize to the following nonlinear Schrödinger equations in the entire space when the potential functions possess a certain compactness condition

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + V(x)|u|^{p(x)-2}u = f(x,u), \\ u \in W^{1,p(x)}(\mathbb{R}^N). \end{cases}$$

Here we always assume that

$$(V)\ V \in C(\mathbb{R}^N),\ \ V_- := \inf_{\mathbb{R}^N} V(x) > 0 \text{ and}$$
$$\mu(\{x \in \mathbb{R}^N : V(x) \leq M\}) < +\infty$$

for all $M > 0$. Here μ denotes the Lebesque measure in $\mathbb{R}^N$. Note that if $V \in C(\mathbb{R}^N, (0,+\infty))$ is coercive, namely

$$\lim_{|x|\to\infty} V(x) = +\infty,$$

then V is satisfied.

In the case, $E := \left\{u \in W^{1,p(x)}(\mathbb{R}^N) \middle| \int_{\mathbb{R}^N} (|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\mathrm{d}x < +\infty\right\}$ is compactly embedded into $L^{q(x)}(\mathbb{R}^N)$ with $p < q \ll p^*$ (see Lemma 2.6 of [1]). Again, we get a version of the above main theorems for this case.

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