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Finite groups with the set of the number of subgroups of possible order containing exactly two elements

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Abstract. Let G be a finite group, and $n(G)$ be the set of the number of subgroups of possible order of G. We investigate the structure of G satisfying that $n(G) = \{1, m\}$ for any positive integer $m > 1$. At first, we prove that the nilpotent length of G is less than 2. Secondly, we investigate nilpotent groups with $m = p + 1$ or $p^2 + p + 1$ (p is a prime), and we get the classification of such kinds of groups. At last, we investigate non-nilpotent groups with $m = p + 1$ and get the classification of the groups under consideration.

Keywords. Finite group; the number of subgroups of possible order.

1. Introduction

Throughout this paper, groups mentioned are finite and p is a prime. An important topic in the group theory is to investigate the number of subgroups of possible order, and conversely it is also an important subject to determine the structure of a finite group by considering the number of its subgroups of possible orders. In the theory of p-groups, many classical counting theorems of the number of subgroups of possible orders found. For example, the following propositions are famous.

PROPOSITION 1.1

Let G be a group of order p^n and $s_k(G)$ be the number of subgroups of order p^k of G, $0 \le k \le n$. Then $s_k(G) \equiv 1 (\mathrm{mod}\ p)$ (see [1]).

PROPOSITION 1.2

Assume that G is a non-cyclic group of order p^n, $p > 2$. If $1 \le k \le n-1$, then $s_k(G) \equiv 1 + p (\mathrm{mod}\ p^2)$ (see [5]).

PROPOSITION 1.3

Assume that G is a group of order p^n, $0 \le k \le n$. If $s_1(G) = 1$, then G is a cyclic group, or a general quaternion group (see [1]).

PROPOSITION 1.4

Assume that G is a group of order p^n, $0 \leq k \leq n$. *If* $s_k(G) = 1$, $2 \leq k \leq n-1$, *then* G *is a cyclic group* (*see* [1]).

By Propositions 1.3 and 1.4, we see that the structure of a p-group is strictly determined by the number of its subgroups of possible orders. Hence it is a meaningful topic to study the structure of a group with given numbers of subgroups of possible orders. In fact, a lot of research have been done in this topic. For example, if a group G has exactly one Sylow subgroup for every prime, then G is nilpotent. More generally, if G has only one Sylow p-subgroup for a prime p, then G is p-closed. Zhang [10] investigated the structure of group by Sylow number. Naoki in [6] proved a conjecture of Huppert [4] which shows the relationship of Sylow number and p-nilpotence. Chen and Cao [2] classified the p-groups in which the number of subgroups of possible order is less than or equal to $p+1$. Qu *et al.* [7] classified the p-groups in which the number of subgroups of possible order is less than or equal to $2p^2 + p + 1$. Recently, Chen *et al.* [3] determined the groups in which the number of subgroups of possible order is less than or equal to 3, but there exist some gaps in the proof of their theorem. If we denote by $n(G)$ the set of the number of subgroups of possible order of a group G, then we can investigate the structure of G by $n(G)$. Obviously, it follows that $|n(G)| = 1$ if and only if G is cyclic. What follows is to studying the structure of G satisfying $|n(G)| = 2$. Note that 1 must belong to $n(G)$, and that there exists only one number not equal to 1 in $n(G)$ in this case. In this paper, we focus on groups G with $n(G) = \{1, m\}$ for any positive integer $m > 1$ and investigate the structures of groups under consideration. In § 2, we show that the nilpotent length of G is less than 2. Furthermore, we obtain that G is solvable. In § 3, we investigate the nilpotent groups G and prove that $m = p+1$ or $p^2 + p + 1$, and then classify these groups, where p is a prime. In § 4, we classify the non-nilpotent groups G with $m = p+1$.

Let G be a group. For convenience, we use $\pi(n)$ to denote the set of prime divisors of a positive number n and let $\pi(G) = \pi(|G|)$, $s_k(G)$ denotes the number of subgroups of order p^k of a p-group G, $n_p(G)$ denotes the number of the Sylow p-subgroups of G, Z_{p^n} denotes the cyclic group of order p^n, $d(G)$ denotes the minimal number of generators of G, $\Phi(G)$ denotes the Frattini subgroup of G, $F(G)$ denotes the Fitting subgroup of G, $\Omega_i(G) = \langle g \in G | g^{p^i} = 1 \rangle$, $H * K$ denotes the central products H and K, $nl(G)$ denotes the nilpotent length of G, $n(G)$ denotes the set of the number of subgroups of possible order of G.

In addition, let $F_0(G) = 1$. For the positive integer i, $F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G))$. A group G is said to be of *nilpotent length* $nl(G) = n$ if $F_{n-1}(G) < G$ and $F_n(G) = G$.

Other notations and terminologies are standard and the reader is referred to [9] if necessary.

2. The nilpotency of groups G with $n(G) = \{1, m\}$

Lemma 2.1. Let $N \trianglelefteq G$ and $\bar{G} = G/N$. Then for any subgroup K of G,

$$|\bar{G} : N_{\bar{G}}(\bar{K})| \big| |G : N_G(K)|,$$

especially, for any $p \in \pi(G)$, $n_p(\bar{G}) \big| n_p(G)$.

Proof. It follows from (b) of Lemma 1 of [8].

Theorem 2.1. *Let G be a group with $n(G) = \{1, m\}$ and M the product of all normal* Sylow *subgroups of G. Then $\pi(m) \subseteq \pi(M)$, and $nl(G) \leq 2$. Furthermore, G is solvable.*

Proof. At first, we assert that for any $r \in \pi(m)$, the Sylow r-subgroup of G is normal. Otherwise, by Sylow theorem, $n_r(G) = m \equiv 1 (\mathrm{mod}\ r)$, a contradiction to $r|m$. Hence $\pi(m) \subseteq \pi(M)$.

Secondly, we assert that $nl(G) \leq 2$. By Zassenhaus theorem, there exists a complement subgroup H of M in G such that $\bar{G} = G/M \cong H$ and $\pi(H) = \pi(G) \setminus \pi(M)$.

If $\pi(H) = \emptyset$, then $H = 1$, and thus all the Sylow subgroups of G are normal. It follows that G is nilpotent and $nl(G) = 1$.

If $\pi(H) \neq \emptyset$, then for any $p \in \pi(H)$, the Sylow p-subgroup is not normal in G. Thus, by Lemma 2.1, we have that $n_p(\bar{G})\big|n_p(G) = m$. But $n_p(\bar{G})\big||\bar{G}|$, and so $n_p(\bar{G})|(m, |\bar{G}|)$. Recall that $\pi(m) \subseteq \pi(M)$ and $(|M|, |\bar{G}|) = 1$. Then $(m, |\bar{G}|) = 1$. Hence $n_p(\bar{G}) = 1$ for any $p \in \pi(H)$. It follows that all the Sylow subgroups of $\bar{G}$ are normal, and so $\bar{G}$ is a nilpotent group. Therefore, G has a normal series $1 \trianglelefteq M \trianglelefteq G$ such that M and G/M are nilpotent, and so $nl(G) \leq 2$.

3. The classification of nilpotent groups G with $n(G) = \{1, m\}$

First, we give some lemmas, which are necessary for our classification.

Lemma 3.1. *Let G be a p-group, and $N \trianglelefteq G$. If there exists a positive integer t such that $s_k(G) \leq t$ holds for every positive integer k, then $s_k(G/N) \leq t$.*

Proof. This follows from Lemma 2.4 of [7].

Lemma 3.2. *Let G be a group of order p^n, and n is a positive integer. If $s_k(G)$ is invariable for any integer $1 \leq k < n$, then $d(G) \leq 3$.*

Proof. Assume that $d(G) > 3$. Then $p^{d(G)} - 1 > p^2 - 1$, and $G/\Phi(G)$ is an elementary abelian p-group of order $p^{d(G)}$. By hypothesis, for an integer $1 \leq k < n$,

$$s_k(G) = s_{n-1}(G) = s_{d(G)-1}(G/\Phi(G)) = p^{d(G)-1} + \cdots + p + 1.$$

Therefore, by Lemma 3.1, for an integer $1 \leq k \leq d(G)$,

$$s_k(G/\Phi(G)) \leq p^{d(G)-1} + \cdots + p + 1.$$

Recall that $p^{d(G)} - 1 > p^2 - 1$. Then

$$s_2(G/\Phi(G)) = \frac{(p^{d(G)} - 1)(p^{d(G)-1} - 1)}{(p^2 - 1)(p - 1)} > \frac{p^{d(G)} - 1}{p - 1} = s_{n-1}(G),$$

a contradiction. Hence $d(G) \leq 3$, as desired.

Lemma 3.3. *Let G be a nilpotent group, and $n(G) = \{1,\ m\}$. Then there exists some prime p in $\pi(G)$ such that $m = p + 1$ or $p^2 + p + 1$.*

Proof. We divide the proof into two cases according to whether G is a group of prime power order or not.

Case 1. Assume that G is a p-group of order p^n, $n \geq 2$. By hypothesis, we have that G is not a cyclic group, and so $d(G) > 1$.

We assert that $s_k(G) = m$ for any positive k, $n > k \geq 2$. Otherwise, G is cyclic by Proposition 1.4, a contradiction.

If $s_1(G) = 1$, then, by Proposition 1.3, G is a generalized quaternion group. Hence G must be isomorphic to quaternion group Q_8 by Theorem 1 of [4]. It follows that $p = 2$ and $m = 3$, as desired.

If the number of nontrivial subgroups of possible order of G is equal to m, then

$$m = s_{n-1}(G) = p^{d(G)-1} + \cdots + p + 1.$$

Therefore, by Lemma 3.2, $d(G) = 2$ or 3, which implies that $m = p + 1$ or $p^2 + p + 1$, as claimed.

Case 2. Assume that G is not a prime power group. By hypothesis, we have that all Sylow subgroups of G are normal, and there exists at least one Sylow subgroup of G which is not cyclic. Without loss of generality, let the p-Sylow subgroup is non-cyclic, then it follows that there exists a possible order t of subgroup of P such that P has at least 2 subgroups of order t, which implies by the hypothesis that the number of subgroups of order t is m, so $n(P) = \{1,\ m\}$. By Case 1, $m = p + 1$ or $p^2 + p + 1$.

COROLLARY 3.1

If G is a p-group except the quaternion group Q_8, and $n(G) = \{1,\ m\}$, then the number of every nontrivially subgroup of possible order is equal to m.

Proof. It follows from the proof of Lemma 3.3.

Theorem 3.1. *Assume G is a nilpotent group, and $n(G) = \{1,\ p+1\}$, where $p \in \pi(G)$.*

(I) *G is a group of prime power order if and only if G is one of the following groups*:

  (1) *Quaternion group*: Q_8;
  (2) $Z_{p^{n-1}} \times Z_p$, *where* $n \geq 2$;
  (3) $\langle a, b|\ a^{p^{n-1}} = 1, b^p = 1, b^{-1}ab = a^{1+p^{n-2}}\rangle$, *where p is an odd prime and* $n \geq 3$;
  (4) $\langle a, b|\ a^{2^{n-1}} = 1, b^2 = 1, b^{-1}ab = a^{1+2^{n-2}}\rangle$, *where $p = 2$ and $n \geq 4$.*

(II) *G is not a prime power order group if and only if $G = P \times \langle u\rangle$, where $P \in Syl_p(G)$, $p \nmid |u|$, and P is isomorphic to a group in Case I.*

Proof. Case I follows from Theorem 3.1 of [5]. Now we assume that G is not a prime power order group. By hypothesis, for some $q \in \pi(G)$, we have that there exists a Sylow q-subgroup Q of order q^n satisfying $n(Q) = \{1,\ p+1\}$, where $q \in \pi(G)$ and $n \geq 2$. By

Lemma 3.3, it follows that $p+1=q+1$ or q^2+q+1, and thus $q=p$, which means that all Sylow subgroups of G are cyclic except the Sylow p-subgroup. Therefore, Q is a group in Case I, and Case II follows by checking trivially.

Theorem 3.2. *Assume that G is a nilpotent group, and $n(G)=\{1,\ p^2+p+1\}$.*

(I) *G is a group of prime power order if and only if G is one of the following groups*:

(1) $Z_p \times Z_p \times Z_p$;
(2) $\langle a,\ b,\ c \mid a^{p^2}=b^p=c^p=1, [b,c]=a^p, [a,b]=[a,c]=1\rangle$.

(II) *G is not a prime power order group if and only if $G = P \times \langle v\rangle$, where $P \in Syl_p(G)$, $p \nmid |v|$, and P is isomorphic to a group in* Case I.

Proof. Case I is Corollary 2.2 of [6], if G is not a prime power order group. Similar to Theorem 3.1, for some $q \in \pi(G)$, G has a Sylow q-subgroup Q with $|Q|=q^n$ satisfying $n(Q)=\{1,\ p^2+p+1\}$. By Lemma 3.3, it follows that $p^2+p+1=q+1$ or q^2+q+1. Since both q and p are primes, we can easily get $p=q$. Thus all Sylow subgroups of G are cyclic except Sylow p-subgroup. It follows that Q is a group in Case I, and Case II follows by easily checking.

By Theorems 3.1 and 3.2, we can get the following corollary immediately.

COROLLARY 3.2

Let G be a p-group of order p^n, $n \geq 1$. If $s_k(G)=m$, where $1 \leq k < n$ and $1 \leq m < 11$, then G is isomorphic to one of the following groups:

(1) $Z_{p^n},\ n \geq 1$;
(2) $Z_{p^{n-1}} \times Z_p$, *where* $p=2,\ 3,\ 5,\ 7$ *and* $n \geq 2$;
(3) $\langle a,b \mid a^{2^{n-1}}=1,\ b^2=1,\ b^{-1}ab=a^{1+2^{n-2}}\rangle$, *where* $p=2$ *and* $n \geq 4$;
(4) $\langle a,b \mid a^{p^{n-1}}=1,\ b^p=1,\ b^{-1}ab=a^{1+p^{n-2}}\rangle$, *where* $p=3,\ 5,\ 7$ *and* $n \geq 3$;
(5) $Z_2 \times Z_2 \times Z_2$;
(6) $Q_8 * Z_4$.

4. The classification of non-nilpotent groups G with $n(G)=\{1,\ p+1\}$

To draw a conclusion, we list the following lemmas, which are all from [9].

Lemma 4.1. *Assume G is not an abelian group and all its Sylow subgroups are cyclic. Then*

$$G=\langle a,b\rangle, a^m=b^n=1, b^{-1}ab=a^r, ((r-1)n,m)=1, r^n \equiv 1 (\mathrm{mod}\ m),$$

where m, n and r are positive integers.

Lemma 4.2. *Let G be a group, and $n_p(G) \not\equiv 1 (\mathrm{mod}\ p^2)$. Then there exist two different Sylow p-subgroups P_1 and P_2 of G such that $|P_1 : P_1 \cap P_2| = p$.*

Lemma 4.3. *Let G be a p-group and let σ be a p'-automorphism of G. If σ acts trivially on $\Omega_1(G)$ ($\Omega_2(G)$ if $p=2$), then σ is the identity.*

Theorem 4.1. *Let G be a non-nilpotent group. Then $n(G) = \{1,\ p+1\}$ if and only if $G = H \times \langle w\rangle$, $(6, |w|) = 1$, and H is either S_3 or Q_{12}, where S_3 and Q_{12} denote the symmetric group of degree 3 and the generalized quaternion group of order 12, respectively.*

Proof. By hypothesis, there exists a Sylow subgroup of G which is not normal, and so for some prime $q \in \pi(G)$, G has $p+1$ Sylow q-subgroups. By Sylow theorem, we have $q = p$, and thus all Sylow subgroups of G are normal except Sylow p-subgroup. Therefore, G is a p-nilpotent group. Let $G = P \ltimes T$, where P is a Sylow p-subgroup of G of order p^n, $n \geq 1$, and T is the normal p-complement and

$$T = P_1 \times P_2 \times \cdots \times P_t, \quad t \geq 1,$$

where P_i is a Sylow p_i-subgroup of G, and p_i is a prime different from p for $i = 1, 2, \ldots, t$. Assume that P acts nontrivially on P_1. Then $\langle P,\ P_1\rangle = P \ltimes P_1$ has at least $p+1$ Sylow p-subgroups. By $n_p(G) = p+1$, we have that $n_p(P \ltimes P_1) = p+1$, and thus $(p+1)\big||P_1|$. Hence we get the following two cases, i.e., P_1 is a Sylow 3-subgroup of G while $p = 2$, and P_1 is a Sylow 2-subgroup of G while $p \neq 2$.

We assert that all the Sylow subgroups of G are cyclic.

At first, we show that every normal Sylow subgroup of G is cyclic. Otherwise, suppose that the normal Sylow q-subgroup Q is not cyclic for some prime $q \in \pi(G)\backslash\{p\}$, and then $n(Q) = \{1,\ p+1\}$. It follows that $q = p$, a contradiction. Hence all the normal Sylow subgroups of G are cyclic.

Secondly, we prove that each Sylow p-subgroup of G is cyclic. Otherwise, every Sylow p-subgroup of G is non-normal and non-cyclic. Assume that all the Sylow p-subgroups of G are $P_1,\ P_2, \ldots$ and P_{p+1}. Since $s_{n-1}(P_i) = kp+1 = 1$ or $p+1$ by hypothesis, we have that $k = 0,\ 1$.

If $k = 0$, then $s_{n-1}(P_i) = 1$, and thus P_i is cyclic for $i = 1, \ldots, p+1$, as desired.

If $k = 1$, then $s_{n-1}(P_i) = p+1$ for $i = 1, 2, \ldots, p+1$. By $n(G) = \{1, p+1\}$, we get that the subgroups of order p^{n-1} of $P_1,\ P_2, \ldots$ and P_{p+1} are identical, which concludes that $P_1 = P_2 = \cdots = P_{p+1}$. Therefore all the Sylow p-subgroups of G are cyclic, which implies that every Sylow subgroup of G is cyclic.

We assert that P_1 is not a Sylow 2-subgroup of G. Otherwise, if P_1 is a normal and cyclic Sylow 2-subgroup, then G has a normal 2-complement, and so P_1 acts trivially on P, a contradiction. It follows that P and P_1 are a Sylow 2-subgroup and a Sylow 3-subgroup of G, respectively. Furthermore, P just acts nontrivially on P_1, and trivially on $P_2, \ldots, P_t$, which means that

$$G = (P \ltimes P_1) \times P_2 \times \cdots \times P_t.$$

Now, consider the structure of $P \ltimes P_1$, which satisfies that

$$n(P \ltimes P_1) = \{1, 3\}.$$

For convenience, let

$$H = P \rtimes P_1,\ P = \langle a\rangle,\ P_1 = \langle b\rangle,$$

where $|a| = 2^n$, $|b| = 3^m$, $n,\ m \geq 1$. Obviously, it follows that

$$a^{-1}ba = b^r,\ r \not\equiv 1 (\mathrm{mod}\ 3^m).$$

Recall that $n_2(H) = 3$. We have that

$$|H : N_H(P)| = 3,$$

and so

$$|N_H(P)| = 2^n 3^{m-1},$$

which implies that

$$N_H(P) = P \times \langle b^3 \rangle.$$

We assert that $m = 1$. Otherwise if $m > 1$, then

$$b^3 = a^{-1} b^3 a = (a^{-1} b a)^3 = b^{3r},$$

which means that $b^{3(r-1)} = 1$. Therefore,

$$r \equiv 1 (\bmod\ 3^{m-1}).$$

By Lemma 3.1, we have that

$$((r-1)2^n, 3^m) = 1.$$

Hence $m = 1$, a contradiction.

Now we claim that $r = -1$. Since $r \not\equiv 1 (\bmod\ 3)$, we get that

$$r \equiv 0 \qquad \text{or} \qquad -1 (\bmod\ 3).$$

If $r \equiv 0 (\bmod\ 3)$, then $a^{-1} b a = b^r = 1$, a contradiction. Therefore,

$$r \equiv -1 (\bmod\ 3),$$

and so we can choose that $r = -1$.

We assert that $n \leq 2$. According to arguments above, we have

$$H = \langle a, b |\ a^{2^n} = 1, b^3 = 1, a^{-1} b a = b^{-1}, n \geq 1 \rangle.$$

Since

$$n_2(H) = 3 \not\equiv 1 (\bmod\ 2^2),$$

by Lemma 3.2, there exists two different Sylow 2-subgroups Q_1 and Q_2 such that

$$|Q_1 : Q_1 \cap Q_2| = 2.$$

Let $D = Q_1 \cap Q_2$, then D is a normal subgroup of Q_1 and Q_2. It follows that D is a normal subgroup of all Sylow 2-subgroups of H and $|N_H(D)| = |H|$, and so $H = N_H(D)$. Hence $D \trianglelefteq H$ and b acts trivially on $D = \langle a^2 \rangle$. Suppose that $n \geq 3$, we have that $|D| \geq 2^2$ and $\Omega_2(\langle a \rangle) \leq D$. It follows that b acts trivially on $\Omega_2(\langle a \rangle)$. By Lemma 4.3, we have that b acts trivially on $\langle a \rangle$, a contradiction. Therefore, $n \leq 2$.

If $n = 1$, then H is a non-abelian group of order 6, and thus $H \cong S_3$.

If $n = 2$, then H is a non-abelian group of order 12 with an element of order 4, and then $H \cong Q_{12}$.

Since P_i ($2\leq i \leq t$) are cyclic and normal subgroups of G, T is cyclic. Let $T = \langle w\rangle$, then $G = H \times \langle w\rangle$, where $(6, |w|) = 1$, and H is either S_3 or Q_{12}, as desired.

We can easily check that all groups in this theorem are non-nilpotent and satisfy $n(G) = \{1, p+1\}$ for some $p \in \pi(G)$.

Remark 4.1. Let G be a non-nilpotent group, and $n(G) = \{1, p+1\}$. In the proof of Theorem 4.1, we have that $p = 2$, and thus $n(G) = \{1, p+1\}$ and $n(G) = \{1, 3\}$ are equivalent. Under the condition $n(G) = \{1, 3\}$, some results have been obtained in [8], but some gaps in the proofs. We overcome the gaps in the proof of Theorem 4.1.

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