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Blowing-up semilinear wave equation with exponential nonlinearity in two space dimensions

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Abstract. We investigate the initial value problem for some semi-linear wave equation in two space dimensions with exponential nonlinearity growth.

Keywords. Nonlinear Schrödinger equation; nonlinear wave equation; stability; blow-up.

1. Introduction

Consider the initial value problem for a semilinear wave equation

$$\begin{cases} \ddot{u} - \Delta u + u = g(u) := u g_1(|u|^2), \\ (u(0), \dot{u}(0)) = (u_0, u_1) \in (H^1 \times L^2)(\mathbb{R}^2), \end{cases} \tag{1.1}$$

where $u(t, x) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ and $g_1 \in C^1(\mathbb{R}, \mathbb{R}_+)$ is a positive real function satisfying $g_1(0) = 0$.

Before going further, let recall a few historic facts about this problem. We begin with the monomial defocusing semi-linear wave equation in space dimensions $d \geq 3$,

$$\Box u + |u|^{p-1} u = 0, \quad p > 1. \tag{1.2}$$

The well-posedness of (1.2) in the scale of the Sobolev spaces H^s has been widely investigated (see for instance [3, 4, 6, 10, 23, 24]). It is well-known that the Cauchy problem associated to 1.2 is locally well-posed in the usual Sobolev space $H^s(\mathbb{R}^d)$ if $s > \frac{d}{2}$, or when $\frac{1}{2} \leq s < \frac{d}{2}$ and $p \leq 1 + \frac{4}{d-2s}$ [9, 13, 18]. Moreover if $p = 1 + \frac{4}{d-2s}$ and $\frac{1}{2} \leq s < \frac{d}{2}$, then we have global H^s-solutions for small Cauchy data [13, 17].

The global solvability in the energy space $(H^1 \times L^2)(\mathbb{R}^d)$ has attracted a great deal of works. A critical value of the power p appears, namely $p_c := \frac{d+2}{d-2}$ and there are mainly three cases.

In the subcritical case ($p < p_c$), Ginibre and Velo proved in [4] the global existence and uniqueness in the energy space.

In the critical case ($p = p_c$), the global existence was first proved by Struwe in the radially symmetric case [25], then by Grillakis [5] in the general case and later on by Shatah–Struwe [24] in other dimensions.

In the supercritical case ($p > p_c$), the question remains open except for some partial results (see [11, 12]).

In two space dimensions any polynomial nonlinearity is subcritical with respect to the H^1-norm. Hence, it is legitimate to consider an exponential nonlinearity. Moreover, the choice of an exponential nonlinearity emerges from a possible control of solutions via a Moser–Trudinger type inequality [1, 16, 19]. In fact, Nakamura and Ozawa [17] proved global well-posedness and scattering for small Cauchy data in any space dimension $N \geq 2$. Later on, Atallah [2] showed a local existence result to the 2D wave equation

$$\ddot{u} - \Delta_x u + u\, \mathrm{e}^{\alpha u^2} = 0 \tag{E_α}$$

for $0 < \alpha < 4\pi$ and with radially symmetric initial data $(0, u_1)$ having compact support. Later on, Ibrahim *et al.* [7] obtained global well-posedness of the previous equation in the energy space for small data. Recently, Struwe [26, 27] has constructed global solution for smooth data. In a recent work [14, 15], the authors obtained similar results without any smallness conditions. In the Schrödinger context, corresponding results hold [20, 21]. They showed global well-posedness and linearization in the energy space.

Our aim is to give a class of blowing up solutions in the focusing case associated to the equations considered in [7, 15]. The rest of the paper is organized as follows. In §2 we give some tools needed in the sequel and in §3 we prove the main result about instability of solution to (1.1).

We mention that C will be used to denote a constant which may vary from line to line. We also use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some absolute constant C and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. Finally, we denote the derivative operator $(Df)(x) := xf'(x)$.

2. Background material

In this section we give some technical tools needed in the sequel. First, let us fix the set of nonlinearity considered in this paper. G denotes the primitive of g which vanishes on zero.

(1) *Behavior on zero*

$$g_1(0) = g_1'(0) = 0. \tag{2.1}$$

(2) *Subcritical case*

$$\begin{cases} \forall\, \alpha > 0,\ \exists\, C_\alpha > 0/|g(s)| \leq C_\alpha \mathrm{e}^{\alpha s^2},\ \forall\, s \in \mathbb{R}, \\ (D-2)G(r) > 0 \quad \text{and} \quad (D-2)^2 G(r) \geq 0, \quad \forall\, r > 0. \end{cases} \tag{2.2}$$

(3) *Critical case*

$$\begin{cases} \lim_{|u|\to\infty} \frac{G_L(u)}{uG_L'(u)} = 0, \\ \exists\, \kappa_0 > 0 \ \text{ s.t. } \lim_{|u|\to\infty} G_L''(u)\mathrm{e}^{-\kappa|u|^2} = \{0 \text{ if } \kappa > \kappa_0,\ \infty \text{ if } \kappa < \kappa_0\}, \\ \exists\, \varepsilon > 0 \text{ s. t. } (D-4-\varepsilon)G(r) \geq 0 \quad \text{and} \quad (D-2)(D-4-\varepsilon)G(r) \geq 0, \quad \forall\, r > 0, \end{cases} \tag{2.3}$$

where we denote $G_L(u) := (1 - \chi(u))G(u)$ for some $\chi \in C_0^\infty(|x| < 1)$.

We will say that the nonlinearty g is subcritical (respectively critical) if it satisfies (2.1) and (2.2) (respectively (2.1) and (2.3)).

Remark 2.1. We give explicit examples.

(1) *Subcritical case*: $g(u) := u(1+|u|^2)^{\frac{-1}{2}}(\mathrm{e}^{(1+|u|^2)^{\frac{1}{2}}} - e(1+|u|^2)^{\frac{1}{2}})$.
(2) *Critical case*: $g(u) \in \{u(\mathrm{e}^{|u|^2} - 1 - |u|^2 - \frac{1}{2}|u|^4),\ u(3+|u|^2)|u|^4\mathrm{e}^{|u|^2}\}$.

Proof.
(1) Take the real function $f(x) := \mathrm{e}^x - 1 - x - \frac{x^2}{2}$. Then $f'(x) = \mathrm{e}^x - 1 - x$, $(D-2)f(x) = (x-2)\mathrm{e}^x + x + 2$ and $(D-2)^2 f(x) = (x^3 - 3x + 4)\mathrm{e}^x - x - 4$. Then $\min((D-2)f(x), (D-2)^2 f(x)) \geq 0$ if $x \geq 0$. Let $\phi(r) := -1 + \sqrt{1+r^2}$ and $G = f \circ \phi$. Then, $(D-2)G(r) = \phi(r) f'(\phi(r))\frac{r\phi'(r)}{\phi(r)} - 2f(\phi(r)) \geq Df(\phi(r)) \geq 0$ because $\frac{r\phi'(r)}{\phi(r)} = \frac{r^2}{1+r^2-\sqrt{1+r^2}} \geq 1$. Moreover, $(D-2)^2 G(r) = r^2 f''(\phi(r)) - 3rf'(\phi(r))\phi'(r) + 4f(\phi(r)) + r^2\phi'^2(r) f''(\phi(r))$ and $(D-2)^2 f(r) = -3rf'(r) + 4f(r) + r^2 f''(r)$. Since $\phi'(r) \leq 1$ and $f''(r) \geq 0$, we have $(D-2)^2 G(r) \geq 0$.
(2) See [8]. $\square$

Let us recall a few results about the existence of ground state of the stationary problem associated to (1.1). Define for $\phi \in H^1(\mathbb{R}^2)$, $(\alpha, \beta) \in \mathbb{R}^2_+ - \{(0,0)\}$ and $c > 0$, the quantities

$$\begin{aligned}
K_{\alpha,\beta,c}(\phi) &:= \alpha\|\nabla\phi\|^2_{L^2} + c(\alpha+\beta)\|\phi\|^2_{L^2} \\
&\quad - \int_{\mathbb{R}^2} (\alpha|\phi|g(|\phi|) + 2\beta G(\phi))\,\mathrm{d}x, \\
M(\phi) &= \frac{1}{2}\|\phi\|^2_{L^2}, \quad S_c(\phi) := cM(\phi) + \frac{1}{2}\|\nabla\phi\|^2_{L^2} - \int_{\mathbb{R}^2} G(\phi)\,\mathrm{d}x, \\
H_{\alpha,\beta}(\phi) &= \frac{1}{2(\alpha+\beta)}\left[\beta\|\nabla\phi\|^2_{L^2} + \alpha\int_{\mathbb{R}^2}(|\phi|g(|\phi|) - 2G(\phi))\,\mathrm{d}x\right], \\
m_c &= m_{\alpha,\beta,c} := \inf_{0\neq\phi\in H^1}\{S_c(\phi), \quad \text{s. t. } K_{\alpha,\beta,c}(\phi) = 0\}. \qquad (2.4)
\end{aligned}$$

The following result is a direct consequence of works [8, 22].

PROPOSITION 2.2

Let $c > 0$ and two real numbers $(\alpha, \beta) \in \mathbb{R}^2_+ - \{(0,0)\}$. Then, in the critical case (2.1) *and* (2.3) *or the subcritical case* (2.1) *and* (2.2),

(1) *the following number $m_c = m_{\alpha,\beta,c}$ is nonzero and independent of (α, β),*
(2) *there is a minimizer of the problem* (2.4), *which is a solution to*

$$\Delta\psi - c\psi + g(\psi) = 0, \quad 0 \neq \psi \in H^1(\mathbb{R}^2), \quad m = S_c(\psi). \qquad (2.5)$$

ψ is called the ground state.

Proof. We denote in this proof $\phi_\lambda := \phi(\lambda.)$. We have $\|\nabla\phi_\lambda\|_{L^2} = \|\nabla\phi\|_{L^2}$ and $\|\phi_\lambda\|_{L^2} = \lambda^{-1}\|\phi\|_{L^2}$. Then, for $\lambda > 0$,

$$m_c = \inf_{0\neq\phi\in H^1}\{S_c(\phi_\lambda), \quad \text{s. t. } K_{\alpha,\beta,c}(\phi_\lambda) = 0\}. \tag{2.6}$$

Take $\lambda^{-2} = c$. For $\frac{g}{c}$ rather than g and $\frac{G}{c}$ rather than G, we have

$$m_c = \inf_{0\neq\phi\in H^1}\{S_1(\phi) \quad \text{s. t. } K_{\alpha,\beta}(\phi) = 0\}. \tag{2.7}$$

There exists [8, 22] a certain minimizer nonzero ψ, such that $\Delta\psi_\lambda - \psi_\lambda + [\frac{g}{c}](\psi_\lambda) = 0$, so $\lambda^2\Delta\psi(\lambda.) - \psi(\lambda.) + [\frac{g}{c}](\psi(\lambda.)) = 0$. The proof is achieved since $c = \lambda^{-2}$. □

We return to the Klein–Gordon equation (1.1). Take $\omega \in (-1, 1)$ and $(1-\omega^2) := a^2 \in (0, 1)$. The change $v := \mathrm{e}^{-it\omega}u$ in (1.1) yields the perturbed Schrödinger equation

$$\ddot{v} + 2i\omega\dot{v} - \Delta v + a^2 v = g(v). \tag{2.8}$$

By [7] (respectively [14, 15]), we have local well-posedness of (1.1) in $(H^1 \times L^2)(\mathbb{R}^2)$ for the critical case, with small data (respectively subcritical case, for arbitrary data). Moreover, if we denote

$$\begin{aligned} E(u, v) &:= \frac{1}{2}\left(\|v\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + a^2\|u\|_{L^2}^2 - 2\int_{\mathbb{R}^2} G(u)\mathrm{d}x\right), \\ Q(u, v) &:= \omega\|u\|_{L^2}^2 + \Im\left(\int_{\mathbb{R}^2} v\bar{u}\mathrm{d}x\right). \end{aligned}$$

Then, a solution to (2.8) satisfies conservation of the energy and the charge

$$E(0) = E(t) := E(v, \dot{v}), \quad Q(0) = Q(t) := Q(v, \dot{v}). \tag{2.9}$$

Let us give some stable sets by the flow of (2.8).

Lemma 2.3. *For* $(0, 0) \neq (\alpha, \beta) \in \mathbb{R}_+^2$, *the set*

$$\Sigma_{\alpha,\beta} := \{(u, v) \in H^1 \times L^2, \ \textit{s. t. } E(u, v) < d \textit{ and } K_{\alpha,\beta,a^2}(u) < 0\} \tag{2.10}$$

is stable under the flow of (1.1), *where* $d := m_c$ *for* $c = a^2$.

Proof of Lemma 2.3. With conservation of the energy, $E < d$. Assume that for some $t_0 > 0$, we have $K(u(t_0)) = 0$ and $K(u(t)) < 0$, $\forall\ t \in [0, t_0)$. Since [8, 22] $d = \inf_{0\neq\phi\in H^1}\{H(\phi),\ K(\phi) \leq 0\}$, we have $\frac{1}{2}\|\nabla u(t)\|_{L^2}^2 = H_{0,1}(u(t)) \geq d > 0$ for all $t \in [0, t_0)$. So $K(u(t_0)) = 0$ and $u(t_0) \neq 0$. Then we have the absurdity $\inf_{0\neq\phi\in H^1}\{H(\phi),\ K(\phi) = 0\} = d \leq S_{a^2}(u(t_0)) \leq E < d$. □

3. Instability for a critical focusing wave equation

The main result of this paper is the following:

Theorem 3.1. *For $\lambda > 0$, we denote by u_λ the solution to (1.1) with data $(\psi_\lambda, i\omega\psi_\lambda)$ where ψ is a ground state to (2.5) with $c = a^2 = 1-\omega^2$ and $\psi_\lambda := \lambda\psi$. Assume that g satisfies the subcritical or critical case with the supplementary condition*

$$\exists\, \varepsilon_0 > 0 \text{ s. t. } \left(D - 2\left[\varepsilon_0 + \frac{1+\omega^2}{1-\omega^2}\right]\right) G(r) \geq 0, \quad \forall\, r > 0. \tag{3.1}$$

Then, for $\lambda > \max\{1, \lambda_0 := \frac{\sqrt{d(1+\varepsilon_0)}}{\omega\|\psi\|_{L^2}}\}$, u_λ blows up in finite time, precisely

$$\lim_{t\to T^*} \|u_\lambda(t)\|_{L^2} = \infty.$$

Remark 3.2. For $\omega \in (0,1)$ small enough, the condition (3.1) is satisfied by the critical case.

The previous theorem is a direct consequence of the following result.

PROPOSITION 3.3

Take the same hypothesis of the previous theorem. Let v_λ be the solution of (2.8) with data $(\lambda\psi, 0)$. Then, for $\lambda > \max\{1, \lambda_0\}$, v_λ blows up in finite time, precisely

$$\lim_{t\to T^*} \|v_\lambda(t)\|_{L^2} = \infty.$$

The proof of Proposition 3.3 is based on the following auxiliary result.

Lemma 3.4. *Take the same hypothesis of Proposition 3.3. Then, for $I_\lambda(t) := \frac{1}{2}\|v_\lambda(t)\|_{L^2}^2$ and $\lambda > \max\{1, \lambda_0\}$, a constant $a_\lambda > 0$ exists such that*

$$I_\lambda'' \geq (2+\varepsilon_0)\|\dot{v}_\lambda\|_{L^2}^2 + a_\lambda. \tag{3.2}$$

Proof. With a direct computation, denoting the conserved quantities $Q_\lambda := Q(v_\lambda, \dot{v}_\lambda)$ and $E_\lambda := E(v_\lambda, \dot{v}_\lambda)$, we have

$$\begin{aligned} I_\lambda''(t) &= \left[\Re \int_{\mathbb{R}^2} \dot{v}_\lambda \bar{v}_\lambda \, \mathrm{d}x\right]' \\ &= \|\dot{v}_\lambda(t)\|_{L^2}^2 + \Re \int_{\mathbb{R}^2} \ddot{v}_\lambda \bar{v}_\lambda \, \mathrm{d}x \\ &= \|\dot{v}_\lambda(t)\|_{L^2}^2 - K_{1,0,a^2}(v_\lambda) + 2\omega \Im \int_{\mathbb{R}^2} \dot{v}_\lambda \bar{v}_\lambda \, \mathrm{d}x \\ &= \|\dot{v}_\lambda(t)\|_{L^2}^2 - K_{1,0,a^2}(v_\lambda) + 2\omega Q_\lambda - 2\omega^2 \|v_\lambda(t)\|_{L^2}^2. \end{aligned}$$

Now since $2E_\lambda = \|\dot{v}_\lambda(t)\|_{L^2}^2 + K_{1,0,a^2} + \int_{\mathbb{R}^2} (|v_\lambda| g(v_\lambda) - 2G(v_\lambda))\,\mathrm{d}x$ yields, for any $\varepsilon > 0$,

$$\begin{aligned} I_\lambda''(t) &= (2+\varepsilon)\|\dot{v}_\lambda(t)\|_{L^2}^2 + \varepsilon K_{1,0,a^2} - 2(1+\varepsilon)E_\lambda + 2\omega Q_\lambda \\ &\quad - 2\omega^2 \|v_\lambda(t)\|_{L^2}^2 + (1+\varepsilon)\int_{\mathbb{R}^2} (|v_\lambda| g(v_\lambda) - 2G(v_\lambda))\,\mathrm{d}x. \end{aligned}$$

Since $K(\psi)=0$, for $\lambda>1$, we have

$$\begin{aligned}K_{0,1,c}(\psi_\lambda) &= c\|\psi_\lambda\|_{L^2}^2-2\int_{\mathbb{R}^2}G(\psi_\lambda)\mathrm{d}x\\ &= \lambda^2\int_{\mathbb{R}^2}\left(|\psi|^2-\frac{G(\lambda\psi)}{\lambda^2}\right)\mathrm{d}x.\end{aligned}$$

Now, using the fact that $(D-2)G>0$ on $\mathbb{R}_+^*$, the derivative of the real function $\zeta:\lambda\mapsto\int_{\mathbb{R}^2}\frac{G(\lambda\psi)}{\lambda^2}\mathrm{d}x$ satisfies $\zeta'(\lambda)=\frac{1}{\lambda^3}\int_{\mathbb{R}^2}(\lambda\psi g(\lambda\psi)-2G(\lambda\psi))\mathrm{d}x=\frac{1}{\lambda^3}\int_{\mathbb{R}^2}(D-2)G(\lambda\psi)\mathrm{d}x>0$. Thus, $\zeta(\lambda)>\zeta(1)=c\|\psi\|_{L^2}^2$ and $K_{0,1,c}(\psi_\lambda)<0$. So, by the previous lemma, for any $\lambda>1$, $K_{0,1}(v_\lambda)<0$ and $E_\lambda<d$. Then, for any $\varepsilon>0$, since $K_{1,0}>-\int_{\mathbb{R}^2}v_\lambda g(v_\lambda)$, we have

$$\begin{aligned}B_\varepsilon(t) &:= I_\lambda''(t)-(2+\varepsilon)\|\dot v_\lambda(t)\|_{L^2}^2\\
&= \varepsilon K_{1,0}-(2+2\varepsilon)E_\lambda+2\omega Q_\lambda-2\omega^2\|v_\lambda(t)\|_{L^2}^2\\
&\quad +(1+\varepsilon)\int_{\mathbb{R}^2}(|v_\lambda|g(v_\lambda)-2G(v_\lambda))\mathrm{d}x\\
&> \varepsilon K_{1,0}-(2+2\varepsilon)E_\lambda+2\omega Q_\lambda-4\frac{\omega^2}{c}\int_{\mathbb{R}^2}G(v_\lambda)\mathrm{d}x\\
&\quad +(1+\varepsilon)\int_{\mathbb{R}^2}(|v_\lambda|g(v_\lambda)-2G(v_\lambda))\mathrm{d}x\\
&> \varepsilon K_{1,0}-2(1+\varepsilon)d+2\omega Q_\lambda\\
&\quad +(1+\varepsilon)\int_{\mathbb{R}^2}\left(|v_\lambda|g(v_\lambda)-2(1+\frac{2\omega^2}{c(1+\varepsilon)})G(v_\lambda)\right)\mathrm{d}x\\
&> -2(1+\varepsilon)d+2\omega Q_\lambda\\
&\quad +\int_{\mathbb{R}^2}\left(|v_\lambda|g(v_\lambda)-2\left[\varepsilon+\frac{1+\omega^2}{1-\omega^2}\right]G(v_\lambda)\right)\mathrm{d}x\\
&> -2(1+\varepsilon)d+2\omega^2\lambda^2\|\psi\|_{L^2}^2\\
&\quad +\int_{\mathbb{R}^2}\left(|v_\lambda|g(v_\lambda)-2\left[\varepsilon+\frac{1+\omega^2}{1-\omega^2}\right]G(v_\lambda)\right)\mathrm{d}x.\end{aligned}$$

It is sufficient to take $\lambda_0:=\frac{\sqrt{d(1+\varepsilon_0)}}{\omega\|\psi\|_{L^2}}$. □

We are ready to prove the main result.

Proof of Proposition 3.3. With absurdity, assume that the life span of v_λ is denoted by $T_\lambda=\infty$. For $\lambda>\max\{1,\lambda_0\}$ we denote $I_\lambda(t):=\frac{1}{2}\|v_\lambda(t)\|_{L^2}^2$. By the previous lemma, for some $\varepsilon>0$,

$$\begin{aligned}I_\lambda''I_\lambda &\geq I_\lambda(a_\lambda+(2+\varepsilon)\,\|\dot v_\lambda\|_{L^2}^2)\\
&\geq a_\lambda I_\lambda+\left(1+\frac{\varepsilon}{2}\right)\|v_\lambda\|_{L^2}^2\|\dot v_\lambda\|_{L^2}^2\\
&\geq a_\lambda I_\lambda+\left(1+\frac{\varepsilon}{2}\right)\|v_\lambda\dot v_\lambda\|_{L^1}^2\\
&\geq a_\lambda I_\lambda+\left(1+\frac{\varepsilon}{2}\right)\left(I_\lambda'\right)^2.\end{aligned}$$

For $\alpha := \frac{\varepsilon}{2} > 0$, we have

$$(I_\lambda^{-\alpha})'' = -\alpha I_\lambda^{-\alpha-2}(I_\lambda'' I_\lambda - (\alpha+1)(I_\lambda')^2) < 0. \tag{3.3}$$

Moreover, since $I_\lambda'' \geq a_\lambda > 0$, there exists $t_1 > 0$ such that $I_\lambda' > 0$ on (t_1, ∞). Thus, for any $t > t_1$,

$$I_\lambda^{-\alpha}(t) \leq I_\lambda^{-\alpha}(t_1) - \alpha I_\lambda^{-\alpha-1}(t_1) I_\lambda'(t_1)(t - t_1)$$

which implies that for t large enough $I_\lambda(t) < 0$. This contradiction achieves the proof. □

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