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Remark on an infinite semipositone problem with indefinite weight and falling zeros

G A AFROUZI[1], S SHAKERI[1] and N T CHUNG[2]

[1]Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
[2]Department of Mathematics and Informatics, Quang Binh University, 312 Ly Thuong Kiet, Dong Hoi, Quang Binh, Vietnam
E-mail: afrouzi@umz.ac.ir; s.shakeri@umz.ac.ir; ntchung82@yahoo.com

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Abstract. In this work, we consider the positive solutions to the singular problem

$$\begin{cases} -\Delta u = am(x)u - f(u) - \dfrac{c}{u^\alpha} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \alpha < 1, a > 0$ and $c > 0$ are constants, Ω is a bounded domain with smooth boundary $\partial\Omega$, Δ is a Laplacian operator, and $f : [0, \infty] \longrightarrow \mathbb{R}$ is a continuous function. The weight functions $m(x)$ satisfies $m(x) \in C(\Omega)$ and $m(x) > m_0 > 0$ for $x \in \Omega$ and also $\|m\|_\infty = l < \infty$. We assume that there exist $A > 0, M > 0, p > 1$ such that $alu - M \leq f(u) \leq Au^p$ for all $u \in [0, \infty)$. We prove the existence of a positive solution via the method of sub-supersolutions when $m_0 a > \frac{2\lambda_1}{1+\alpha}$ and c is small. Here λ_1 is the first eigenvalue of operator $-\Delta$ with Dirichlet boundary conditions.

Keywords. Infinite semipositone problems; indefinite weight; falling zeros; sub-supersolution method.

1. Introduction

In this paper, we consider the positive solution to the boundary value problem

$$\begin{cases} -\Delta u = am(x)u - f(u) - \dfrac{c}{u^\alpha} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1, a > 0$ and $c > 0$ are constants, Ω is a bounded domain with smooth boundary $\partial\Omega$, Δ is a Laplacian operator, and $f : [0, \infty] \longrightarrow \mathbb{R}$ is a continuous function. The weight function $m(x)$ satisfies $m(x) \in C(\Omega)$ and $m(x) > m_0 > 0$ for $x \in \Omega$ also $\|m\|_\infty = l < \infty$.

We consider problem (1.1) under the following assumptions.

(H_1) There exist $A > 0$ and $p > 1$ such that $f(u) \leq Au^p$ for all $u \in [0, \infty)$.
(H_2) There exist a constant $M > 0$ such that $f(u) \geq alu - M$ for all $u \in [0, \infty)$.

Let $F(u) := am(x)u - f(u) - \frac{c}{u^\alpha}$. Then $\lim_{u\to 0} F(u) = -\infty$. Hence we refer to (1.1) as an infinite semipositone problem. In [6] the authors have studied the case when $F(u) :=$

$g(u) - \frac{c}{u^\alpha}$ where g is nonnegative and nondecreasing and $\lim_{u\to\infty} g(u) = \infty$. In fact a simple example of $g(u)$ satisfying our hypotheses is $g(u) = u - u^p$, $p > 1$. Note that this $g(u)$ has a falling zero at $u = 1$ and is negative for $u > 1$. Recently, Shivaji *et al* [4] studied Problem (1.1) when $m(x) = 1$. The purpose of this paper is to improve the result of [4] with weight m. We shall establish our existence results via the method of sub and supersolutions.

DEFINITION 1.1

We say that a real-valued function ψ (resp. z) in $C^2(\Omega) \cap C(\bar{\Omega})$ is called a subsolution (supersolution) to (1.1) if they satisfy $\psi(x) \le z(x)$ and

$$\begin{cases} -\Delta\psi \le \lambda a m(x)\psi - f(\psi) - \dfrac{c}{\psi^\alpha} & \text{in } \Omega, \\ \psi > 0 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

and

$$\begin{cases} -\Delta z \ge \lambda a m(x) z - f(z) - \dfrac{c}{z^\alpha} & \text{in } \Omega, \\ z > 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Then we have the following lemma.

Lemma 1.2 [2]. *If there exist sub-supersolutions ψ and z, respectively, such that $\psi \le z$ on $\bar{\Omega}$, then* (1.1) *has at least a positive solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying $\psi \le u \le z$ on $\bar{\Omega}$.*

In the next section, we will state and prove the existence of a positive solution to the problem (1.1) when $m_0 a > \frac{2\lambda_1}{1+\alpha}$ and c is sufficiently small.

2. Main result

Here we state and prove our main result for the problem (1.1).

Theorem 2.1. *Assume* (H$_1$) *and* (H$_2$). *If $m_0 a > \frac{2\lambda_1}{1+\alpha}$, then there exists $c^* = c^*(a, \Omega, A, p, \alpha)$ such that for $c < c^*$,* (1.1) *has a positive solution. Here λ_1 is the first eigenvalue of operator $-\Delta$ with zero Dirichlet boundary condition.*

Proof. From anti-maximum principle (see [1,5]), there exists $\sigma(\Omega) > 0$ such that the solution z_λ of

$$\begin{cases} -\Delta z - \lambda z = -1 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega \end{cases}$$

for $\lambda \in (\lambda_1, \lambda_1 + \sigma)$ is positive in Ω and is such that $\frac{\partial z}{\partial \nu} < 0$ on $\partial\Omega$, where ν is the outward normal vector at $\partial\Omega$. Fix

$$\lambda^* \in \left(\lambda_1, \min\left\{\lambda_1 + \sigma, \frac{1+\alpha}{2} m_0 a\right\}\right)$$

and let

$$K := \min\left\{\left(\frac{2}{(1+\alpha)A\|z_{\lambda^*}\|_\infty^{\frac{2p-1+\alpha}{1+\alpha}}}\right)^{\frac{1}{p-1}}, \left(\frac{m_0 a - \frac{2\lambda^*}{1+\alpha}}{2A\|z_{\lambda^*}\|_\infty^{\frac{2p-2}{1+\alpha}}}\right)^{\frac{1}{p-1}}\right\}.$$

We prove the existence of solution by comparison method (see [3]). It is easy to see that any subsolution of

$$\begin{cases} -\Delta u = am_0 u - f(u) - \dfrac{c}{u^\alpha} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

is a subsolution of (1.1). Also any supersolution of

$$\begin{cases} -\Delta u = alu - f(u) - \dfrac{c}{u^\alpha} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

is a supersolution of (1.1), where l is as defined above. Define

$$\psi := K z_{\lambda^*}^{\frac{2}{1+\alpha}}.$$

Then

$$\begin{aligned} \nabla\psi &= K\left(\frac{2}{1+\alpha}\right) z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} \nabla z_{\lambda^*}, \\ -\Delta\psi &= -K\left(\frac{2}{1+\alpha}\right)\left(\frac{1-\alpha}{1+\alpha}\right) z_{\lambda^*}^{\frac{-2\alpha}{1+\alpha}} |\nabla z_{\lambda^*}|^2 - K\left(\frac{2}{1+\alpha}\right) z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} \Delta z_{\lambda^*} \\ &= -K\left(\frac{2}{1+\alpha}\right)\left(\frac{1-\alpha}{1+\alpha}\right) z_{\lambda^*}^{\frac{-2\alpha}{1+\alpha}} |\nabla z_{\lambda^*}|^2 \\ &\quad -K\left(\frac{2}{1+\alpha}\right) z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} (1 - \lambda^* z_{\lambda^*}) \\ &= K\left(\frac{2}{1+\alpha}\right) \lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}} - K\left(\frac{2}{1+\alpha}\right) z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} \\ &\quad -K\left(\frac{2}{1+\alpha}\right)\left(\frac{1-\alpha}{1+\alpha}\right) \frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}} \end{aligned} \tag{2.3}$$

and

$$am_0\psi - f(\psi) - \frac{c}{\psi^\alpha} = am_0 K z_{\lambda^*}^{\frac{2}{1+\alpha}} - f(K z_{\lambda^*}^{\frac{2}{1+\alpha}}) - \frac{c}{K^\alpha z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}. \tag{2.4}$$

Let $\delta > 0$, $\mu > 0$, $m > 0$ be such that

$$|\nabla z_{\lambda^*}| \geq m \text{ in } \bar{\Omega}_\delta$$

and

$$z_{\lambda^*} \geq \mu \text{ in } \Omega \setminus \bar{\Omega}_\delta,$$

where $\bar{\Omega}_\delta = \{x \in \Omega : d(x, \partial\Omega) \le \delta\}$. Let

$$c^* := K^{1+\alpha} \min\left\{\frac{2}{1+\alpha}\frac{1-\alpha}{1+\alpha}m^2, \frac{1}{2}\mu^2\Big(m_0 a - \frac{2\lambda^*}{1+\alpha}\Big)\right\}.$$

In $\bar{\Omega}_\delta$, we compare (2.3) and (2.4) term by term to see that for $c < c^*$,

$$-\Delta\psi \le a m_0 \psi - f(\psi) - \frac{c}{\psi^\alpha} \quad \text{in } \bar{\Omega}_\delta.$$

Since $\frac{2}{1+\alpha}\lambda^* \le m_0 a$, we have

$$K\Big(\frac{2}{1+\alpha}\Big)\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}} \le K m_0 a z_{\lambda^*}{}^{\frac{2}{1+\alpha}} \tag{2.5}$$

and from the choice of K, we know that

$$AK^{p-1}\|z_{\lambda^*}\|_\infty^{\frac{2p-1+\alpha}{1+\alpha}} \le \frac{2}{1+\alpha}. \tag{2.6}$$

By (2.6) and (H_1) we have

$$-K\Big(\frac{2}{1+\alpha}\Big)z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} \le -AK^p z_{\lambda^*}^{\frac{2p}{1+\alpha}} \le -f(K z_{\lambda^*}^{\frac{2}{1+\alpha}}). \tag{2.7}$$

Next we know that

$$K^{1+\alpha}\Big(\frac{2}{1+\alpha}\Big)\Big(\frac{1-\alpha}{1+\alpha}\Big)m^2 \ge c$$

for $c < c^*$ and

$$|\nabla z_{\lambda^*}| \ge m \quad \text{in } \bar{\Omega}_\delta.$$

Thus

$$K\Big(\frac{2}{1+\alpha}\Big)\Big(\frac{1-\alpha}{1+\alpha}\Big)|\nabla z_{\lambda^*}|^2 \ge \frac{c}{K^\alpha}$$

and

$$-K\Big(\frac{2}{1+\alpha}\Big)\Big(\frac{1-\alpha}{1+\alpha}\Big)\frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}{}^{\frac{2\alpha}{1+\alpha}}} \le \frac{-c}{K^\alpha z_{\lambda^*}{}^{\frac{2\alpha}{1+\alpha}}}. \tag{2.8}$$

Hence for $c < c^*$ combining (2.5), (2.7) and (2.8) we have

$$-\Delta\psi \le a m_0 \psi - f(\psi) - \frac{c}{\psi^\alpha} \quad \text{in } \bar{\Omega}_\delta.$$

Now in $\Omega \setminus \bar{\Omega}_\delta$, from the fact that

$$c \leq \frac{1}{2} K^{\alpha+1} \frac{1}{2} \mu^2 \Big(m_0 a - \frac{2}{1+\alpha} \lambda^* \Big)$$

for $c < c^*$ and $z_{\lambda^*} \geq \mu$, we have

$$\frac{c}{K^\alpha} \leq \frac{1}{2} K {z_{\lambda^*}}^2 \Big(m_0 a - \frac{2}{1+\alpha} \lambda^* \Big). \tag{2.9}$$

Also from the choice of K, we have

$$A K^{p-1} {z_{\lambda^*}}^{\frac{2p-2}{1+\alpha}} \leq \frac{1}{2} \Big(m_0 a - \frac{2}{1+\alpha} \lambda^* \Big). \tag{2.10}$$

By using (2.9) and (2.10),

$$\begin{aligned}
-\Delta \psi &= K \Big(\frac{2}{1+\alpha} \Big) \lambda^* {z_{\lambda^*}}^{\frac{2}{1+\alpha}} - K \Big(\frac{2}{1+\alpha} \Big) {z_{\lambda^*}}^{\frac{1-\alpha}{1+\alpha}} \\
&\quad - K \Big(\frac{2}{1+\alpha} \Big) \Big(\frac{1-\alpha}{1+\alpha} \Big) \frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}} \\
&\leq K \Big(\frac{2}{1+\alpha} \Big) \lambda^* {z_{\lambda^*}}^{\frac{2}{1+\alpha}} \\
&= \frac{1}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}} \Big[\frac{1}{2} \frac{2}{1+\alpha} \lambda^* K {z_{\lambda^*}}^2 + \frac{1}{2} \frac{2}{1+\alpha} \lambda^* K {z_{\lambda^*}}^2 \Big] \\
&\leq \frac{1}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}} \Big[\Big(\frac{1}{2} K {z_{\lambda^*}}^2 m_0 a - \frac{c}{K^\alpha} \Big) + K z_{\lambda^*}^2 \Big(\frac{1}{2} m_0 a - A K^{p-1} z_{\lambda^*}^{\frac{2p-2}{1+\alpha}} \Big) \Big] \\
&= a m_0 K {z_{\lambda^*}}^{\frac{2}{1+\alpha}} - A K^p {z_{\lambda^*}}^{\frac{2p}{1+\alpha}} - \frac{c}{K^\alpha} z_{\lambda^*}^{\frac{-2\alpha}{1+\alpha}} \\
&\leq a m_0 K {z_{\lambda^*}}^{\frac{2}{1+\alpha}} - f(K {z_{\lambda^*}}^{\frac{2}{1+\alpha}}) - \frac{c}{K^\alpha} {z_{\lambda^*}}^{\frac{-2\alpha}{1+\alpha}} \\
&= a m_0 \psi - f(\psi) - \frac{c}{\psi^\alpha}.
\end{aligned}$$

Hence ψ is a subsolution of (1.1). Next, we construct a supersolution. From (H_2) we know that there is a large $\bar{M} > 0$ such that

$$a l u - f(u) - \frac{c}{u^\alpha} \leq \bar{M}$$

for all $u > 0$ and

$$\bar{M} e \geq \psi \quad \text{ in } \Omega,$$

where e is the unique positive solution of

$$\begin{cases} -\Delta e = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.11}$$

Let $z := \bar{M} e$. Then

$$-\Delta z = \bar{M} \geq a l z - f(z) - \frac{c}{z^\alpha} \text{ in } \Omega.$$

Thus z is a positive supersolution of (1.1) for $c < c^*$ satisfying $z \geq \psi$ and Theorem 2.1 is proved. $\square$

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