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Uncertainty inequalities for the Heisenberg group

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Abstract. We establish the Heisenberg–Pauli–Weyl uncertainty inequalities for Fourier transform and the continuous wavelet transform on the Heisenberg group.

Keywords. Heisenberg group; Fourier transform; wavelet transform; uncertainty inequality; heat kernel.

1. Introduction

The classical Heisenberg–Pauli–Weyl uncertainty inequality for the Fourier transform on $\mathbb{R}^n$ is

$$\|f\|_2^4 \le c_n \int_{\mathbb{R}^n} |x|^2 |f(x)|^2 \mathrm{d}x \int_{\mathbb{R}^n} |\xi|^2 |\hat{f}(\xi)|^2 \mathrm{d}\xi, \tag{1.1}$$

which can also be written in the form:

$$\|f\|_2^2 \le c_n \||x| f\|_2 \|(-\Delta)^{1/2} f\|_2,$$

where Δ denotes the Laplacian.

By means of the classical one-dimensional uncertainty inequality (1.1), Singer [13] obtained an uncertainty inequality for the continuous wavelet transform on a real line:

$$k_\phi \|f\|_2^2 \le \|\tau W_\phi(f)(a,\tau)\|_{L^2(\mathrm{d}\tau \mathrm{d}a/a^2)} \|\xi \hat{f}(\xi)\|_2, \tag{1.2}$$

where ϕ is an admissible wavelet and k_ϕ is an appropriate positive constant. For more on the history and the relevance of the uncertainty inequality, we refer the readers to the survey [5], the books [6,8], and the papers [2,10,11].

For the Heisenberg group H^n, Thangavelu [16] proved the following theorem.

Theorem 1.1. *For a normalized L^2 function f on H^n, one has*

$$\sqrt{n}\left(\frac{\pi}{2}\right)^{\frac{n+1}{2}} \le \left(\int_{H^n} |z|^2 |f(z,t)|^2 \mathrm{d}z\mathrm{d}t\right)^{1/2} \left(\int_{H^n} |\mathscr{L}^{1/2} f(z,t)|^2 \mathrm{d}z\mathrm{d}t\right)^{1/2},$$

where $\mathscr{L}$ is the Heisenberg sub-Laplacian.

Later, Sitaram *et al.* [14] obtained a generalized theorem.

Theorem 1.2. *For* $f \in L^2(H^n)$, $0 \le \gamma < n+1$, *one has*

$$\|f\|_2^2 \le K\left(\int_{H^n} |(z,t)|^{2\gamma}|f(z,t)|^2\mathrm{d}z\mathrm{d}t\right)^{1/2}\left(\int_{H^n} |\mathscr{L}^{\gamma/2} f(z,t)|^2\mathrm{d}z\mathrm{d}t\right)^{1/2},$$

where K *is a constant.*

In this paper, we will prove the following two extensions:

Theorem 1.3. *For* $f \in L^2(H^n)$, $a, b > 0$, *one has*

$$\|f\|_2^2 \le C\left(\int_{H^n} |(z,t)|^{2a}|f(z,t)|^2\mathrm{d}z\mathrm{d}t\right)^{\frac{b}{a+b}}\left(\int_{H^n} |\mathscr{L}^{b/2} f(z,t)|^2\mathrm{d}z\mathrm{d}t\right)^{\frac{a}{a+b}}, \tag{1.3}$$

where C *is a constant.*

Theorem 1.4. *For* $f \in H_\alpha^\sigma$, $\phi \in AW_\alpha^\sigma$ *and* $a, b > 0$, *one has*

$$\begin{aligned}\|f\|_2^2 \le\ & C\left(\int_0^\infty \int_{H^n} |(z,t)|^{2a}|W_\phi f(z,t)|^2 \frac{\mathrm{d}z\mathrm{d}t\mathrm{d}\rho}{\rho^{n+2}}\right)^{\frac{b}{a+b}} \\ & \times\left(\int_{H^n} |\mathscr{L}^{b/2} f(z,t)|^2\mathrm{d}z\mathrm{d}t\right)^{\frac{a}{a+b}},\end{aligned}$$

where C *is a constant.*

The idea that we modify Theorem 1.2 to Theorem 1.3 originates from [1], where the analogue of inequality (1.3) is established for a locally compact space. The main approach of the proof of Theorem 1.3 is based on the estimate of the heat kernel together with the relation between the sub-Laplacian and the group Fourier transform. Our method is different from that in [1]. Furthermore, as an application of (1.3) we deduce an uncertainty inequality for the continuous wavelet transform on the Heisenberg group, which is an extension of (1.2).

2. Preliminaries

The Heisenberg group, denoted by H^n, is a nilpotent Lie group of step two whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z,t)(z',t') = \left(z+z', t+t'+\frac{1}{2}\mathrm{Im}\, z\bar{z}'\right).$$

For $(z,t) \in H^n$, the homogeneous norm of (z,t) is given by $|(z,t)| = (|z|^4+|t|^2)^{1/4}$. Then the ball of radius r centered at (z,t) is defined by $B_r(z,t) = \{(z',t') \in H^n : |(z,t)^{-1}(z',t')| < r\}$. Let S^n be the unit sphere in H^n and suppose f is a measurable function, one has (see [3])

$$\int_{H^n} f(z,t)\mathrm{d}z\mathrm{d}t = \int_{S^n}\int_0^\infty f(r\zeta)r^{2n+1}\mathrm{d}r\mathrm{d}\zeta. \tag{2.1}$$

Let $\pi_\lambda(z,t)$ $(z = x + iy,\ \lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\})$ be the Schrödinger representations acted on $\varphi \in L^2(\mathbb{R}^n)$ by

$$\pi_\lambda(z,t)\varphi(\xi) = \mathrm{e}^{i\lambda t}\mathrm{e}^{i\lambda(x\cdot\xi + \frac{1}{2}x\cdot y)}\varphi(\xi + y).$$

Given a function $f \in L^1(H^n)$, its group Fourier transform $\hat{f}$ is defined to be the operator-valued function and

$$\hat{f}(\lambda) = \int_{H^n} f(z,t)\pi_\lambda(z,t)\mathrm{d}z\mathrm{d}t.$$

Let $\mathrm{d}\mu(\lambda) = (2\pi)^{-n-1}|\lambda|^n\mathrm{d}\lambda$. Then one has the inversion of Fourier transform

$$f(z,t) = \int_{-\infty}^{\infty} \mathrm{tr}(\pi_\lambda^*(z,t)\hat{f}(\lambda))\mathrm{d}\mu(\lambda)$$

and the Plancherel formula

$$\|f\|_2^2 = \int_{\mathbb{R}^*} \|\hat{f}(\lambda)\|_{HS}^2\mathrm{d}\mu(\lambda).$$

Suppose f and g are measurable functions on H^n, then their convolution is defined by

$$f * g(z,t) = \int_{H^n} f((z,t)(-w,-s))g(w,s)\mathrm{d}w\mathrm{d}s.$$

It follows from the definition of the Fourier transform that $\widehat{f * g}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda)$. In addition, one has the generalized Yong inequality:

$$\|f * g\|_r \le \|f\|_p\|g\|_q,$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$.

Now consider the Heisenberg sub-Laplacian

$$\mathscr{L} = -\sum_{j=1}^{n}(X_j^2 + Y_j^2),$$

where $X_j = \frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial t}$, $Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2}x_j\frac{\partial}{\partial t}$. For the Schrödinger representations π_λ one has

$$\pi_\lambda^*(X_j) = i\lambda\xi_j,\ \pi_\lambda^*(Y_j) = \frac{\partial}{\partial\xi_j}.$$

So that $\pi_\lambda^*(\mathscr{L}) = -\Delta + \lambda^2|\xi|^2 = H(\lambda)$ is the Hermite operator. Let Φ_α $(\alpha \in \mathbb{N}^n)$ stand for the normalized Hermite functions on $\mathbb{R}^n$. For $\lambda \in \mathbb{R}^*$, define $\Phi_\alpha^\lambda(\xi) = |\lambda|^{\frac{n}{4}}\Phi_\alpha(|\lambda|^{\frac{1}{2}}\xi)$. Then one has

$$H(\lambda)\Phi_\alpha^\lambda = (2|\alpha| + n)|\lambda|\Phi_\alpha^\lambda.$$

One important relevance of sub-Laplacian $\mathscr{L}$ is the heat semigroup defined by

$$(\mathrm{e}^{-s\mathscr{L}}f)(z,t) = q_s * f(z,t),$$

where q_s is the heat kernel given by

$$q_s(z,t) = c_n \int_{-\infty}^{\infty} \mathrm{e}^{-i\lambda t} \left(\frac{\lambda}{\sinh \lambda s} \right)^n \mathrm{e}^{-\frac{1}{4}(\lambda \coth \lambda s)|z|^2} \mathrm{d}\lambda$$

with the positive constant c_n. Note that the heat kernel is a C^∞ function on $H^n \times (0,\infty)$ and its Fourier transform is (see p. 86 of [18])

$$\hat{q}_s(\lambda) = \mathrm{e}^{-sH(\lambda)}.$$

More details about the sub-Laplacian and the heat kernel on Heisenberg group can be found in [4,15,17].

Now we are going to recall some facts about the continuous wavelet transform on the Heisenberg group. Let $P_\alpha (\alpha \in \mathbb{N}^n)$ be the projection from $L^2(\mathbb{R}^n)$ to one-dimensional subspace spanned by Φ_α. For $\sigma = +$ or $-$, define

$$H_\alpha^\sigma = \{ f \in L^2(H^n) : \hat{f}(\lambda) = \hat{f}(\lambda) P_\alpha \quad \text{and} \quad \hat{f}(\lambda) = 0 \text{ if } \lambda \notin \mathbb{R}^\sigma \}.$$

By Theorem 1 of [9] one has

$$L^2(H^n) = \bigoplus_{\alpha \in \mathbb{N}^n} (H_\alpha^+ \oplus H_\alpha^-).$$

For a nonzero function $\phi \in H_\alpha^\sigma$, if it satisfies

$$C_\phi = \int_{\mathbb{R}^*} \frac{\|\hat{\phi}(\lambda)\|_{HS}^2}{|\lambda|} \mathrm{d}\lambda < +\infty,$$

we then call ϕ an admissible wavelet and write $\phi \in AW_\alpha^\sigma$. Given a function $f \in H_\alpha^\sigma$, the continuous wavelet transform of f with respect to ϕ is defined by

$$W_\phi f(z,t,\rho) = \langle f, U(z,t,\rho)\phi \rangle,$$

where

$$U(z,t,\rho)\phi(z',t') = \rho^{-(n+1)/2} \phi \left(\frac{z'-z}{\sqrt{\rho}}, \frac{t'-t-\frac{1}{2}\mathrm{Im}\, z\bar{z}'}{\rho} \right).$$

Note the following facts: for $f \in H_\alpha^\sigma$ and $\phi \in AW_\alpha^\sigma$, one has

$$\int_{H^n} W_\phi f(z,t,\rho) \pi_\lambda(z,t) \mathrm{d}z \mathrm{d}t = \rho^{(n+1)/2} \hat{f}(\lambda) \hat{\phi}(\rho\lambda)^* \tag{2.2}$$

and

$$\int_0^\infty \int_{H^n} \left| W_\phi f(z,t,\rho) \right|^2 \frac{\mathrm{d}z \mathrm{d}t \mathrm{d}\rho}{\rho^{n+2}} = C_\phi \|f\|_2^2. \tag{2.3}$$

For further details of wavelet transform on H^n, we refer to [7,9,12].

3. Proof of Theorems 1.3 and 1.4

In order to prove our main theorems, we first need some lemmas for preparation. Throughout the paper, we will use C to denote the positive constant, which is not necessarily same at each occurrence.

Lemma 3.1. *The heat kernel $q_s(z,t)$ satisfies the estimate*

$$\|q_s\|_2^2 \le Cs^{-n-1}$$

with some positive constants C.

Proof. By Proposition 2.8.2 of [18] one has that

$$|q_s(z,t)| \le Cs^{-n-1}\mathrm{e}^{-\frac{A}{s}(|z|^2+|t|)}$$

holds for some positive constants C and A. Since $(|z|^4+|t|^2)^{\frac{1}{2}} \le |z|^2+|t|$, we have

$$|q_s(z,t)| \le Cs^{-n-1}\mathrm{e}^{-\frac{A}{s}|(z,t)|^2}.$$

Thus by (2.1) we get

$$\begin{aligned}
\|q_s\|_2^2 &\le C\int_{H^n} s^{-2n-2}\mathrm{e}^{-\frac{2A}{s}|(z,t)|^2}\mathrm{d}z\mathrm{d}t\\
&= C\int_0^\infty s^{-2n-2}\mathrm{e}^{-\frac{2A}{s}r^2}r^{2n+1}\mathrm{d}r\\
&= Cs^{-n-1},
\end{aligned}$$

which proves this lemma. □

Lemma 3.2. *Let $f \in L^2(H^n)$. Then for $0 \le \gamma < n+1$, one has*

$$\|f * q_s\|_2 \le Cs^{-\gamma/2}\left(\int_{H^n}|(z,t)|^{2\gamma}|f(z,t)|^2\mathrm{d}z\mathrm{d}t\right)^{1/2}$$

with some positive constants C.

Proof. We assume the non-trivial case that the integral of the right-hand side is finite. Let $B_r = B_r(0,0)$ and set $f_r = f\chi_{B_r}$, $f^r = f\chi_{H^n\setminus B_r}$. Note that

$$\|\hat{f}(\lambda)\|_{HS}^2 = \sum_a \|\hat{f}(\lambda)\Phi_\alpha^\lambda\|_2^2.$$

Then by the Plancherel formula we have

$$\begin{aligned}
\|f^r * q_s\|_2^2 &= \int_{\mathbb{R}^*}\sum_\alpha \|\hat{f}^r(\lambda)\mathrm{e}^{-sH(\lambda)}\Phi_\alpha^\lambda\|_2^2\mathrm{d}\mu(\lambda)\\
&= \int_{\mathbb{R}^*}\sum_\alpha \|\hat{f}^r(\lambda)\mathrm{e}^{-s(2|\alpha|+n)|\lambda|}\Phi_\alpha^\lambda\|_2^2\mathrm{d}\mu(\lambda)\\
&\le \int_{\mathbb{R}^*}\sum_\alpha \|\hat{f}^r(\lambda)\Phi_\alpha^\lambda\|_2^2\mathrm{d}\mu(\lambda)\\
&= \int_{H^n}|f^r|^2\mathrm{d}z\mathrm{d}t\\
&\le \frac{1}{r^{2\gamma}}\int_{H^n\setminus B_r}|(z,t)|^{2\gamma}|f(z,t)|^2\mathrm{d}z\mathrm{d}t.
\end{aligned}$$

On the other hand, by the generalized Yong inequality and Lemma 3.1 we have

$$
\begin{aligned}
\|f_r * q_s\|_2 &= \|q_s\|_2 \|f_r\|_1 \\
&\le C s^{-\frac{n+1}{2}} \left(\int_{H^n} |(z,t)|^{2\gamma} |f(z,t)|^2 \mathrm{d}z\mathrm{d}t \right)^{1/2} \left(\int_{B_r} |(z,t)|^{-2\gamma} \mathrm{d}z\mathrm{d}t \right)^{1/2}.
\end{aligned}
$$

Note that the second integral of the right-hand side is controlled by $Cr^{-\gamma+n+1}$ since $0 \le \gamma < n+1$. Therefore,

$$
\begin{aligned}
\|f * q_s\|_2 &\le \|f^r * q_s\|_2 + \|f_r * q_s\|_2 \\
&\le C r^{-\gamma} (1 + s^{-(n+1)/2} r^{n+1}) \left(\int_{H^n} |(z,t)|^{2\gamma} |f(z,t)|^2 \mathrm{d}z\mathrm{d}t \right)^{1/2}.
\end{aligned}
$$

Choosing $r = s^{1/2}$ then gives the desired estimate. □

Lemma 3.3. *For* $f \in L^2(H^n)$, *if* $0 < a < n+1, 0 < b \le 2$, *one has*

$$
\begin{aligned}
\|f\|_2^2 \;\le\; & C \left(\int_{H^n} |(z,t)|^{2a} |f(z,t)|^2 \mathrm{d}z\mathrm{d}t \right)^{\frac{b}{a+b}} \\
& \times \left(\int_{H^n} |\mathscr{L}^{b/2} f(z,t)|^2 \mathrm{d}z\mathrm{d}t \right)^{\frac{a}{a+b}},
\end{aligned}
\tag{3.1}
$$

where C is a constant.

Proof. By Lemma 3.2, we have

$$
\begin{aligned}
\|f\|_2 \;\le\; & \|f * q_s\|_2 + \|f - f * q_s\|_2 \\
\le\; & C s^{-a/2} \left(\int_{H^n} |(z,t)|^{2a} |f(z,t)|^2 \mathrm{d}z\mathrm{d}t \right)^{1/2} \\
& + \left(\int_{\mathbb{R}^*} \sum_\alpha \|\hat{f}(\lambda)\big(1 - \mathrm{e}^{-s(2|\alpha|+n)|\lambda|}\big) \Phi_\alpha^\lambda\|_2^2 \mathrm{d}\mu(\lambda) \right)^{1/2} \\
\le\; & C s^{-a/2} \left(\int_{H^n} |(z,t)|^{2a} |f(z,t)|^2 \mathrm{d}z\mathrm{d}t \right)^{1/2} \\
& + \left(\int_{\mathbb{R}^*} \sum_\alpha \|\hat{f}(\lambda) \frac{1 - \mathrm{e}^{-s(2|\alpha|+n)|\lambda|}}{(s(2|\alpha|+n)|\lambda|)^{b/2}} \right. \\
& \left. \quad \times (s(2|\alpha|+n)|\lambda|)^{b/2} \Phi_\alpha^\lambda\|_2^2 \mathrm{d}\mu(\lambda) \right)^{1/2}.
\end{aligned}
$$

Note that if $s \ge 1$, then for $b \ge 0$,

$$
\frac{1 - \mathrm{e}^{-s}}{s^{b/2}} \le 1;
$$

if $0 < s < 1$, then for $0 \le b \le 2$,

$$
\frac{1 - \mathrm{e}^{-s}}{s^{b/2}} \le C \mathrm{e}^{-s} s^{1-b/2} \le C,
$$

since $\dfrac{\frac{1-\mathrm{e}^{-s}}{s^{b/2}}}{\mathrm{e}^{-s}s^{1-b/2}} \to 1 (s \to 0^+)$. Thus we obtain

$$\begin{aligned} \|f\|_2 \ \leq\ & Cs^{-a/2}\left(\int_{H^n} |(z,t)|^{2a}|f(z,t)|^2 \mathrm{d}z\mathrm{d}t\right)^{1/2} \\ & + Cs^{b/2}\left(\int_{\mathbb{R}^*} \sum_{\alpha} \|\widehat{\mathscr{L}^{b/2}f}(\lambda)\Phi_\alpha^\lambda\|_2^2 \mathrm{d}\mu(\lambda)\right)^{1/2}, \end{aligned} \tag{3.2}$$

where we have applied the fact $\hat{f}(\lambda)H(\lambda)^{b/2} = \widehat{\mathscr{L}^{b/2}f}(\lambda)$ to get the last term. Minimizing the right-hand side of (3.2) then gives the desired result. □

Proof of Theorem 1.3. If $a \geq n+1$, let $a' < n+1$, then we have for all $\epsilon > 0$,

$$\frac{|(z,t)|^{a'}}{\epsilon^{a'}} \leq 1 + \frac{|(z,t)|^a}{\epsilon^a},$$

which implies that

$$\||(z,t)|^{a'} f\|_2 \leq \epsilon^{a'}\|f\|_2 + \epsilon^{a'-a}\||(z,t)|^a f\|_2.$$

Optimizing in ϵ then gives the Landau–Kolmogorov inequality:

$$\||(z,t)|^{a'} f\|_2 \leq C\|f\|_2^{1-a'/a}\||(z,t)|^a f\|_2^{a'/a}. \tag{3.3}$$

Now for $b > 2$, we set $b' \leq 2$. Analogues to the above case we get

$$\|\mathscr{L}^{b'/2} f\|_2 \leq C\|f\|_2^{1-b'/b}\|\mathscr{L}^{b/2} f\|_2^{b'/b}. \tag{3.4}$$

Plugging (3.3) and (3.4) into (3.1) with a replaced by a', b replaced by b' then suggests the desired result. □

Proof of Theorem 1.4. First, by the Plancherel formula and (2.2) we have

$$\begin{aligned} &\int_0^\infty \int_{H^n} |\mathscr{L}^{b/2} W_\phi f(z,t,\rho)|^2 \frac{\mathrm{d}z\mathrm{d}t\mathrm{d}\rho}{\rho^{n+2}} \\ &\quad = \int_0^\infty \int_{\mathbb{R}^*} \|\hat{f}(\lambda)\hat{\phi}^*(\rho\lambda)H(\lambda)^{b/2}\|_{HS}^2 \mathrm{d}\mu(\lambda)\frac{\mathrm{d}\rho}{\rho} \\ &\quad \leq \int_0^\infty \int_{\mathbb{R}^*} \|\hat{f}(\lambda)H(\lambda)^{b/2}\|_{HS}^2 \|\hat{\phi}^*(\rho\lambda)\|_{HS}^2 \mathrm{d}\mu(\lambda)\frac{\mathrm{d}\rho}{\rho} \\ &\quad = C_\phi \int_{H^n} |\mathscr{L}^{b/2} f(z,t)|^2 \mathrm{d}z\mathrm{d}t. \end{aligned}$$

Next from Theorem 1.3, we get

$$\begin{aligned} \int_{H^n} |W_\phi f(z,t,\rho)|^2 \mathrm{d}z\mathrm{d}t \ \leq\ & C\left(\int_{H^n} |(z,t)|^{2a}|W_\phi f(z,t,\rho)|^2 \mathrm{d}z\mathrm{d}t\right)^{\frac{b}{a+b}} \\ & \times\left(\int_{H^n} |\mathscr{L}^{b/2} W_\phi f(z,t,\rho)|^2 \mathrm{d}z\mathrm{d}t\right)^{\frac{a}{a+b}}. \end{aligned}$$

Integrating with respect to $d\rho/\rho^{n+2}$ we obtain

$$\int_0^\infty\int_{H^n}|W_\phi f(z,t,\rho)|^2\frac{dzdtd\rho}{\rho^{n+2}}\leq C\int_0^\infty\left(\int_{H^n}|(z,t)|^{2a}|W_\phi f(z,t,\rho)|^2dzdt\right)^{\frac{b}{a+b}} \times\left(\int_{H^n}|\mathscr{L}^{b/2}W_\phi f(z,t,\rho)|^2dzdt\right)^{\frac{a}{a+b}}\frac{d\rho}{\rho^{n+2}}.$$

Then by Hölder's inequality and (2.3) we have

$$\|f\|_2^2\leq C\left(\int_0^\infty\int_{H^n}|(z,t)|^{2a}|W_\phi f(z,t,\rho)|^2\frac{dzdtd\rho}{\rho^{n+2}}\right)^{\frac{b}{a+b}} \times\left(\int_{H^n}|\mathscr{L}^{b/2}f(z,t)|^2dzdt\right)^{\frac{a}{a+b}}.$$

This completes the proof of this theorem. □

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