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Rees valuations

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Abstract. We study dicritical divisors and Rees valuations.

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1. Introduction

In Example (8.3) of our recent paper [17], we considered the local ring R of the point $(0, 0, 0)$ on the irreducible surface

$$Z^m - F_1(X, Y) \dots F_h(X, Y) = 0,$$

where $m > h > 0$ are integers such that m is nondivisible by the characteristic of the ground field K, and $F_1(X, Y), \dots, F_h(X, Y)$ are pairwise coprime homogeneous linear polynomials. Now geometrically speaking, a section of the tangent cone of the surface at $(0, 0, 0)$ consists of the h lines $F_1(X, Y) = 0, \dots, F_h(X, Y) = 0$ in the (X, Y)-plane passing through $(0, 0)$, and they 'give rise to' all the distinct h elements of $\mathfrak{D}(R, M)$, where M is the maximal ideal $M(R)$ of R and $\mathfrak{D}(R, M)$ is the set of all dicritical divisors of M in R. See §2 for the definition of dicritical divisors and their relation to Rees valuations. The analytical (topological) theory of dicritical divisors was developed in [19–21, 23, 24]. It was algebracized in [9] to [14] and [16–18].

In (3.2) and (3.3) of §3 we shall show that M is a normal ideal in R, i.e., for every positive integer p, the ideal M^p is a complete ideal in the normal local domain R and, if $F_0(X, Y)$ is a homogeneous linear polynomial which is coprime to $F_1(X, Y), \dots, F_h(X, Y)$ then the ideal generated by $F_0(X, Y)^p$ and Z^p is a reduction of M^p; as a special example we can take $F_1(X, Y) \dots F_h(X, Y) = X^h - Y^h$ and $F_0(X, Y)^p = Y^p$. Actually, in (3.2) and (3.3), we shall prove a generalized higher dimensional version of this result by two different methods. In (3.4) to (3.7) we shall make some relevant observations. In (4.9) to (4.11) we shall raise some questions.

In the above phrase 'give rise to', we are trying to mimic Max Noether's concept of 'infinitely near in the first neighborhood'. Thus, making a QDT (quadratic transformation) centered at the point $(0, 0, 0)$ of the above displayed surface, i.e., substituting $(X, Y) = (ZX', ZY')$ and factoring out the 'exceptional locus' Z^h we get the transformed surface

$$Z^{m-h} - F_1(X', Y') \dots F_h(X', Y') = 0.$$

Let V_j be the local ring of the line $Z = F_j(X', Y') = 0$ on the transformed surface. Then $V_1, \ldots, V_h$ are exactly all the distinct members of $\mathfrak{D}(R, M)$.

In §4 we shall elucidate the contact number $c(R, V, W)$ which appears in Proposition (3.5) of [12]. Here V and W are prime divisors of a two dimensional regular local domain R. We shall prove some commutativity properties of the contact number and relate them to calculations of the local intersection multiplicity of two curves at a simple point of an algebraic or arithmetical surface.

2. Terminology and preliminaries

We shall use the notation and terminology introduced in §2 to §9 of [16] and §2 to §9 of [17] which themselves were based on [8] to [14]. Relevant background material can be found in [1] to [7] and [25–27].

Referring to pages 145–161 of [8], for the foundations of models, recall that $\mathfrak{V}(A) = \{A_P : P \in \mathrm{spec}(A)\}$ = the modelic spec of a domain A. $S^{\mathfrak{N}}$ = the set of all members of $\mathfrak{V}(\bar{S})$ which dominate S where $\bar{S}$ is the integral closure of a quasilocal domain S in its quotient field QF(S). $U^{\mathfrak{N}} = \cup_{B \in U} B^{\mathfrak{N}}$ for any set U of quasilocal domains. $\mathfrak{W}(A, J) = \cup_{0 \neq x \in J} \mathfrak{V}(A[Jx^{-1}])$ = the modelic blowup of A at a nonzero ideal J in a domain A; note that $Jx^{-1} = \{y/x : y \in J\}$; see pages 152–160 of [8]. $\bar{D}(L/A)$ = the set of all valuation rings V with quotient field L such that $A \subset V$, where A is a subring of L. For this and other related definitions, see §2 of [11]. In the cited reference we had defined the completion of an ideal J in a domain A with quotient field L only when A is normal. i.e., integrally closed in L; now we extend this without assuming A to be normal by saying that the *completion* $\bar{J}$ of J is always obtained by putting $\bar{J} = \cap_{V \in \bar{D}(L/A)}(JV)$; note that $\bar{J}$ is clearly an ideal in the integral closure of A in L.

As usual $\mathbb{N}$ (resp: $\mathbb{N}_+$) denotes the set of all nonnegative (resp positive) integers. The set of all nonzero elements in a ring A is denoted by $A^{\times}$.

For any set U of quasilocal domains and any $i \in \mathbb{N}$, U_i denotes the set of all i-dimensional members of U. $\mathfrak{W}(S, J)_i^{\Delta}$ = the set of all i-dimensional members of $\mathfrak{W}(S, J)$ which dominates S where J is a nonzero ideal in a quasilocal domain S and $i \in \mathbb{N}$. $\mathfrak{D}(S, J) = (\mathfrak{W}(S, J)_1^{\Delta})^{\mathfrak{N}}$ = the *dicritical set* of a nonzero ideal J in a quasilocal domain S; members of this set are called *dicritical divisors* of J in S.

More generally, $\mathfrak{W}(S, J, M)_i^{\Delta}$ = the set of all i-dimensional members T of $\mathfrak{W}(S, J)$ such that $M \subset M(T)$ where $J \subset M$ are nonzero ideals in a domain S and $i \in \mathbb{N}$. $\mathfrak{W}^*(S, J, M)_i^{\Delta}$ = the set of all i-dimensional members T of $\mathfrak{W}(S, J)^{\mathfrak{N}}$ such that $M \subset M(T)$ where $J \subset M$ are nonzero ideals in a domain S and $i \in \mathbb{N}$. $\mathfrak{D}(S, J, M) = (\mathfrak{W}(S, J, M)_1^{\Delta})^{\mathfrak{N}}$ = the *dicritical set* of a pair of nonzero ideals (J, M) in a domain S with $J \subset M$; members of this set are again called the *dicritical divisors* of (J, M) in S. $\mathfrak{D}^*(S, J, M) = (\mathfrak{W}(S, J, M)^{\mathfrak{N}})_1^{\Delta}$ = the *starred dicritical set* of a pair of nonzero ideals (J, M) in a domain S with $J \subset M$; members of this set are called the *starred dicritical divisors* of (J, M) in S.

We make the following observations concerning dicritical divisors.

(I) If $J \subset M$ are nonzero ideals in a domain S and N is an ideal in S with $J \subset N$ such that $\mathrm{rad}_S N = \mathrm{rad}_S M$, then for all $i \in \mathbb{N}$ we have $\mathfrak{W}(S, J, N)_i^{\Delta} = \mathfrak{W}(S, J, M)_i^{\Delta}$ with $\mathfrak{W}^*(S, J, N)_i^{\Delta} = \mathfrak{W}^*(S, J, M)_i^{\Delta}$, and we have $\mathfrak{D}(S, J, N) = \mathfrak{D}(S, J, M)$ with $\mathfrak{D}^*(S, J, N) = \mathfrak{D}^*(S, J, M)$. If $M = S$, then for all $i \in \mathbb{N}$ we have $\mathfrak{W}(S, J, M)_i^{\Delta} = \emptyset$ with $\mathfrak{W}^*(S, J, M)_i^{\Delta} = \emptyset$, and we have $\mathfrak{D}(S, J, M) = \emptyset$ with $\mathfrak{D}^*(S, J, M) = \emptyset$.

(II) If $J \subset M$ are nonzero ideals in a quasilocal domain S with $M = M(S)$, then $\mathfrak{W}(S, J, M)_i^{\Delta} = \mathfrak{W}(S, J)_i^{\Delta}$ for all $i \in \mathbb{N}$, and we have $\mathfrak{D}(S, J, M) = \mathfrak{D}(S, J)$.

(III) If R is a noetherian domain and $I \subset M$ are nonzero nonunit ideals in R, then $\mathfrak{D}(R, I, M)$ is a finite set of DVRs. As in (5.6)(†*) of [8], this follows from (33.2) on page 115 of [25] or (33.10) on page 118 of [25]; here (33.2) is called the Krull–Akizuki theorem and, according to page 218 of [25], (33.10) may be called the Mori–Nagata theorem. Note that if $M \subset \mathrm{rad}_R I$, then $\mathfrak{D}(R, I, M) \neq \emptyset$.

(IV) Let us note that, for a nonzero ideal J in a noetherian domain R, members of $\mathfrak{D}^*(R, J, J)$ are called *Rees valuation rings* of J in R, and their order functions are called *Rees valuations* of J in R. Note that, by the Mori–Nagata theorem ((33.10) on page 118 of [25]) Rees valuation rings are DVRs.

3. Normality of the maximal ideal

In (3.1) and (3.2) we use the set-up of a QDT; for details see pages 534–577 of [8] which are based on pages 7–45, 155–192, 262–283 of [3]. Let us start by discussing the following.

3.1 *Tangent cones and quadratic transformations*

Let d be a positive integer and let $X_1, \ldots, X_{d+1}$ be indeterminates over a field K. Let $m \geq h$ be positive integers. Let

$$G = G(X_1, \ldots, X_{d+1}) = \sum_{h \leq i \leq m} G_i \in C = K[X_1, \ldots, X_{d+1}]$$

be irreducible with

$$G_i = G_i(X_1, \ldots, X_{d+1}) \in C$$

such that depending on i, $G_i = 0$ or G_i is a homogeneous polynomial of degree i, and we have $G_h \neq 0 \neq G_m$. Note that then $G_h = 0$ is the tangent cone of the hypersurface $G = 0$ at the origin $(0, \ldots, 0)$. Let R be the local ring of the origin on the hypersurface and let M be the maximal ideal $M(R)$ in R. Let B be the affine coordinate ring of the hypersurface. Then $B = C/(GC) = K[x_1, \ldots, x_{d+1}]$, where $x_1, \ldots, x_{d+1}$ are the images of $X_1, \ldots, X_{d+1}$ in $C/(GC)$ under the residue class epimorphism $\Phi : C \to C/(GC)$ and we have identified K with its image in $C/(GC)$. The d-dimentional local domain R is the localization of B at the maximal ideal $(x_1, \ldots, x_{d+1})B$ and we have $M = (x_1, \ldots, x_{d+1})R$. The proof of the following *relation* between the tangent cone and the modelic blowup $\mathfrak{W}(R, M)$ is essentially contained in pages 534–577 of [8].

(I) *Relation*. Assume that the hyperplane $X_{d+1} = 0$ is not a component of the tangent cone, i.e., $G_h(X_1, \ldots, X_d, 0) \neq 0$, and let $A_R = R[x_1/x_{d+1}, \ldots, x_d/x_{d+1}]$. Then we have $\mathfrak{W}(R, M)^{\Delta} \subset \mathfrak{V}(A_R)$. Hence in particular $\mathfrak{D}(R, M) = (\mathfrak{V}(A_R)_1)^{\mathfrak{N}}$.

Now let us prove the following *reduction* property of the tangent cone.

(II) *Reduction*. Let $H_1, \ldots, H_d$ be homogeneous linear members of C which are linearly independent over K and which do not divide G_h. Then the ideal in R generated by $\Phi(H_1), \ldots, \Phi(H_d)$ is a reduction of M.

Proof. Without loss of generality, we may assume that $H_1 = X_1, \ldots, H_d = X_d$. Now the assumption that $H_1, \ldots, H_d$ do not divide G_h is equivalent to saying that $G_h(0, \ldots, 0, X_{d+1}) \neq 0$. The form ring $F_{(R,M)}(M)$ is, in a natural manner, isomorphic to the associated graded ring $\mathrm{grad}(R, M)$ as discussed on pages 272–277 of [8] and hence, in view of the material on these pages, by (6.1) of [16] we see that the ideal $(x_1, \ldots, x_d)R$ is a reduction of M. □

3.2 *First (high-school) method*

Let d, m, h be positive integers with $m > h$. Let $X_1, \ldots, X_d, Z$ be indeterminates over a field K, and let

$$G = G(X_1, \ldots, X_d, Z) = Z^m + \sum_{h \leq i \leq m-1} G_i \in C = K[X_1, \ldots, X_d, Z]$$

be irreducible with

$$G_i = G_i(X_1, \ldots, X_d, Z) \in C$$

such that depending on i, $G_i = 0$ or G_i is a homogeneous polynomial of degree i. Let R be the local ring of the origin $(0, \ldots, 0)$ on the hypersurface $G = 0$, and let M be the maximal ideal $M(R)$ in R. Assume (♯) and (♯♯) sandwiched between items (17) and (18) of the following detailed description (DD). In DD, we shall show that $\mathfrak{D}(R, M)$ consists of h distinct DVRs $V_1, \ldots, V_h$ and upon letting (32) of DD we have (33), (45), and (46) of DD.

Detailed description. Consider the polynomial rings

$$C' = K[X_1, \ldots, X_d] \subset K[X_1, \ldots, X_d, Z] = C, \tag{1}$$

where d is a positive integer and $X_1, \ldots, X_d, Z$ are indeterminates over a field K. Recalling that the mspec of a ring is the set of all maximal ideals in it, let

$$Q_{C'} = (X_1, \ldots, X_d)C' \in \mathrm{mspec}(C')$$

with

$$R_{C'} = C'_{Q_{C'}} \quad \text{and} \quad L_{C'} = \mathrm{QF}(C') = K(X_1, \ldots, X_d)$$

and

$$Q_C = (X_1, \ldots, X_d, Z)C \in \mathrm{mspec}(C)$$

with

$$R_C = C_{Q_C} \quad \text{and} \quad L_C = \mathrm{QF}(C) = K(X_1, \ldots, X_d, Z).$$

Note that then R_C is a $(d+1)$-dimensional regular local domain which dominates the d-dimensional regular local domain $R_{C'}$. Let

$$x_1 = X_1/Z, \dots, x_d = X_d/Z \tag{2}$$

and note that then $x_1, \dots, x_d, Z$ may be regarded as indeterminates over K. Let

$$A_{C'} = K[x_1, \dots, x_d] \subset K[x_1, \dots, x_d, Z] = A_C. \tag{3}$$

Then

$$\begin{cases} C \subset A_C \text{ are } (d+1)\text{-variable polynomial rings} \\ \text{with a common quotient field } L_C \\ \text{while } C' \not\subset A_{C'} \not\subset C' \text{ are } d\text{-variable polynomial rings} \\ \text{which are subrings of the field } L_C. \end{cases}$$

Let

$$m = t + h, \text{ where } t \text{ and } h \text{ are positive integers.} \tag{4}$$

Let

$$G = G(X_1, \dots, X_d, Z) = \sum_{h \le i \le m} G_i \in C \setminus C' \tag{5}$$

with

$$G_i = G_i(X_1, \dots, X_d, Z) \in C \tag{6}$$

such that

$$\text{depending on } i,\ G_i = 0 \text{ or } G_i \text{ is homogeneous of degree } i \tag{7}$$

and we have

$$G_h \neq 0 \neq G_m. \tag{8}$$

Since $G \in C \setminus C'$, we get $C' \cap (GC) = 0$. Therefore we can construct an overring B of C' together with

$$\begin{cases} C'\text{-epimorphism } \Phi : C \to B = K[X_1, \dots, X_d, z] \\ \text{with } \Phi(Z) = z \text{ and } \ker(\Phi) = GC \end{cases} \tag{9}$$

which may be depicted by the following commutative diagram:

$$\begin{array}{ccccc} GC & \xrightarrow[\text{inj}]{\ker(\Phi)=GC} & C = K[X_1, \dots, X_d, Z] & \xrightarrow[\text{sur}]{\Phi} & K[X_1, \dots, X_d, z] = B \\ & & \text{inj}\big\uparrow & & \big\uparrow = \\ & & C' = K[X_1, \dots, X_d] & \xrightarrow{\text{inj}} & K[X_1, \dots, X_d, z] = B, \end{array}$$

where ‘inj’ and ‘sur’ indicate injective and surjective maps respectively.

Henceforth

(†) assume that G is irreducible

i.e., equivalently, assume that B is a domain. Let

$$Q = (X_1, \dots, X_d, z)B \in \operatorname{mspec}(B) \tag{10}$$

with

$$R = B_Q \quad \text{and} \quad L = \operatorname{QF}(B) = K(X_1, \dots, X_d, z). \tag{11}$$

Note that then R is a d-dimensional local domain which dominates $R_{C'}$ and Φ extends to a unique $(R_{C'})$-epimorphism

$$\Phi^* : R_C \to R \quad \text{with} \quad \ker(\Phi^*) = GR_C.$$

Henceforth

(‡) assume that $G_m(X_1, \dots, X_d, Z) = Z^m$

and let

$$g = g(x_1, \dots, x_d, Z) = \frac{G(Zx_1, \dots, Zx_d, Z)}{Z^h}$$

and

$$x'_1 = X_1/z, \dots, x'_d = X_d/z, \tag{12}$$

$$A' = K[x'_1, \dots, x'_d] \quad \text{with} \quad L' = K\left(x'_1, \dots, x'_d\right), \tag{13}$$

$$A = K[x'_1, \dots, x'_d, z] \quad \text{with} \quad L_A = K\left(x'_1, \dots, x'_d, z\right). \tag{14}$$

Then clearly

$$g = g(x_1, \dots, x_d, Z) = Z^t + \sum_{1 \le i \le t} g_i Z^{t-i} \in A_C, \tag{15}$$

where

$$g_i = g_i(x_1, \dots, x_d) = \frac{G_{m-i}(Zx_1, \dots, Zx_d, Z)}{Z^{m-i}} \in A_{C'} \tag{16}$$

with

$$g_t \neq 0$$

and

$$\begin{cases} A' \text{ is a } d\text{-variable polynomial ring with } \operatorname{QF}(A') = L', \\ A \text{ is a } d\text{-dimentional noetherian domain with } \operatorname{QF}(A) = L = L_A, \\ \text{and we have the subring inclusions } A' \subset B \text{ and } B \subset A \end{cases}$$

and Φ can be uniquely extended to

$$\begin{cases} K\text{-epimorphism } \phi : A_C \to A \\ \text{with } \phi(x_1, \ldots, x_d, Z) = x'_1, \ldots, x'_d, z \text{ and } \ker(\phi) = gA_C \end{cases} \tag{17}$$

which may be depicted by the following commutative diagram

$$\begin{tikzcd}
 & C = K[X_1, \ldots, X_d, Z] \arrow[d, "\text{inj}"'] & K\big[x'_1, \ldots, x'_d\big] = A' \arrow[d, "\text{inj}"] \\
gA_C \arrow[r, "\ker(\phi)=gA_C", "\text{inj}"'] & A_C = K\big[x_1, \ldots, x_d, Z\big] \arrow[r, "\phi", "\text{sur}"'] & K\big[x'_1, \ldots, x'_d, z\big] = A \\
 & A_{C'} = K[x_1, \ldots, x_d] \arrow[u, "\text{inj}"] \arrow[r, "\text{inj}"] & K\big[x'_1, \ldots, x'_d, z\big] = A \arrow[u, "="'] \\
 & & K[X_1, \ldots, X_d, z] = B, \arrow[u, "X_i = zx'_i", "\text{inj}"']
\end{tikzcd}$$

where 'inj' and 'sur' indicate injective and surjective maps respectively.

For a moment suppress assumption (†) but henceforth assume that

(♯) $$\sum_{h \le i < m} G_i \in FC \text{ with } F = F(X_1, \ldots, X_d) = \prod_{1 \le j \le h} F_j,$$

where

(♯♯) $$\begin{cases} F_j = F_j(X_1, \ldots, X_d) \in C' = K[X_1, \ldots, X_d] \\ \text{and } F_1, \ldots, F_h \text{ are nonzero homogeneous linear polynomials} \\ \text{which are coprime;} \\ \text{here coprime means the ideals } F_1C', \ldots, F_hC' \text{ are distinct.} \end{cases}$$

By Eisenstein's criterion we see that

$$(\sharp) + (\sharp\sharp) \Rightarrow (\dagger). \tag{18}$$

Upon letting

$$f = f(x_1, \ldots, x_d) = \prod_{1 \le j \le h} f_j$$

with

$$f_j = f_j(x_1, \ldots, x_d) = \frac{F_j(Zx_1, \ldots, Zx_d)}{Z},$$

we see that

$$f_j = f_j(x_1, \ldots, x_d) \in A_{C'}$$

are such that

$$\begin{cases} f_1, \ldots, f_h \text{ are coprime nonzero homogeneous linear polynomials} \\ \text{where coprime means the ideals } f_1 A_{C'}, \ldots, f_h A_{C'} \text{ are distinct,} \end{cases}$$

and we have

$$g_t A_{C'} = f A_{C'} \quad \text{and} \quad \sum_{1 \le i \le t} g_i Z^{t-i} \in f A_C.$$

Now upon letting

$$f' = f'(x'_1, \ldots, x'_d) = f(x'_1, \ldots, x'_d) \in A' \tag{19}$$

and

$$f'_j = f'_j(x'_1, \ldots, x'_d) = f_j(x'_1, \ldots, x'_d) \in A' \quad \text{for } 1 \le j \le h, \tag{20}$$

we see that

$$f' = f'(x'_1, \ldots, x'_d) = \prod_{1 \le j \le h} f'_j, \tag{21}$$

$$\begin{cases} f'_1, \ldots, f'_h \text{ are coprime nonzero homogeneous linear polynomials} \\ \text{where coprime means the ideals } f'_1 A', \ldots, f'_h A' \text{ are distinct} \end{cases} \tag{22}$$

and

$$\phi(g_t) A' = f' A' \quad \text{and} \quad \phi(g_i) \in f' A' \quad \text{for } 1 \le i \le t. \tag{23}$$

Upon letting

$$V'_j = \text{the localization } A'_{f'_j A'} \quad \text{for } 1 \le j \le h, \tag{24}$$

we see that

$$V'_1, \ldots, V'_h \text{ are distinct DVRs with quotient field } L'. \tag{25}$$

Now $L = L'(z)$ and z is a root of

$$Z^t + \sum_{1 \le i \le t} \phi(g_i) Z^{t-i}$$

and hence upon letting

$$V_j = \text{the integral closure of } V'_j \text{ in } L \quad \text{for } 1 \le j \le h \tag{26}$$

by (12) to (25) we see that

$$V_1, \ldots, V_h \text{ are distinct DVRs with quotient field } L \tag{27}$$

and for $1 \le j \le h$ we have

$$V_j(y) = \begin{cases} t+1, & \text{if } y = F_j, \\ 1, & \text{if } y = z, \\ 1, & \text{if } y = F_i \text{ with } i \in \{1, \dots, h\} \setminus \{j\}, \\ e, & \text{if } y \in C' \setminus (F_j C') \text{ is homogeneous of degree } e \in \mathbb{N}, \end{cases} \tag{28}$$

where the third line is a special case of the fourth line. Upon letting

$$M = M(R) \tag{29}$$

by (10) and (11) we see that

$$M = (X_1, \dots, X_d, z)R. \tag{30}$$

In view of 3.1(I), by (12) to (30) we see that

$$V_1, \dots, V_h \text{ are exactly all the distinct members of } \mathfrak{D}(R, M). \tag{31}$$

Upon letting

$$I_p = (M(V_1)^p \cap R) \cap \cdots \cap (M(V_h)^p) \cap R) \quad \text{for all } \ p \in \mathbb{N}, \tag{32}$$

we claim that

$$I_p = M^p \quad \text{for all } \ p \in \mathbb{N}. \tag{33}$$

Namely, by (28) to (32) we get $M^p \subset I_p$. Given any $y \in I_p$ we shall show that $y \in M^p$ and this will prove (33). Suppose if possible that $y \notin M^p$. Then, for some $0 \le q < p$, we must have $y \in M^q$ with $y \notin M^{q+1}$. Now clearly

$$y = y' + \sum_{0 \le i \le q} a_i(X_1, \dots, X_d) z^{q-1} \quad \text{with } \ y' \in M^{q+1}, \tag{34}$$

where

$$a_i(X_1, \dots, X_d) \in C' \text{ is either 0 or homogeneous of degree } i. \tag{35}$$

Let

$$\Lambda = \{i : 0 \le i \le q \qquad \text{and} \qquad a_i(X_1, \dots, X_d) \notin FC'\}. \tag{36}$$

By (4)–(9), (‡), (♯), (♯♯) we see that $0 \ne F \in C'$ is homogeneous of degree h with $F \in M^{h+1}$; consequently by (28) and (34) we see that if $i \in \{0, \dots, q\} \setminus \Lambda$, then $a_i(X_1, \dots, X_d) z^{q-i} \in M^{q+1}$; therefore upon letting

$$y'' = \sum_{i \in \Lambda} a_i(X_1, \dots, X_d) z^{q-i} \tag{37}$$

by (34) we see that

$$y - y'' \in M^{q+1}. \tag{38}$$

Since $y \notin M^{q+1}$, by (37) and (38) we get

$$\Lambda \neq \emptyset. \tag{39}$$

Upon letting

$$y^* = y''/z^q, \tag{40}$$

by (12), (35), and (37) we see that

$$y^* = \sum_{i \in \Lambda} a_i\big(x'_1, \ldots, x'_d\big), \tag{41}$$

where, for each $i \in \Lambda$,

$$a_i\big(x'_1, \ldots, x'_d\big) \in A' \setminus (f'A') \text{ is homogeneous of degree } i \tag{42}$$

and hence by (21) and (22) we see that for some $j \in \{1, \ldots, h\}$ we have

$$a_i\big(x'_1, \ldots, x'_d\big) \in A' \setminus \big(f'_j A'\big) \quad \text{for all } i \in \Lambda \tag{43}$$

and now by (41) to (43) we conclude that

$$y^* \in A' \setminus \big(f'_j A'\big)$$

and therefore by (26), (28), (38), (39) and (40) we get

$$V_j(y) = q$$

which contradicts the assumption $y \in I_p$. Therefore we must have $y \in M^p$. This completes the proof of (33).

> *Observe that* item (28) can be sharpened by saying that, for $1 \leq j \leq h$, the DVR $V_j \cap L_{C'}$ can be described thus: (44)

We can write C' = the polynomial ring $K[F_j, Y_{2j}, \ldots, Y_{dj}]$, where $Y_{2j}, \ldots, Y_{dj}$ are suitable homogeneous linear members of C'. Taking a $(t+1)$-th root Y_{1j} of F_j we may regard C' as a subring of the polynomial ring $C_j = K[Y_{1j}, \ldots, Y_{dj}]$. We get a DVR $\bar{V}_j$ with quotient field $K(Y_{1j}, \ldots, Y_{dj})$ such that for every $0 \neq y \in C_j$ we have $\bar{V}_j(y) = \mathrm{ord}(y)$ where $\mathrm{ord}(y)$ = the degree of the smallest degree term when we express y as a polynomial in $Y_{1j}, \ldots, Y_{dj}$. Now clearly $V_j \cap L_{C'} = \bar{V}_j \cap L_{C'}$.

Taking $H_d = Z$ in 3.1(II) and invoking 3.4(II) below we see that

$$\begin{cases} \text{if there exist homogeneous linear members } H_1, \ldots, H_{d-1} \text{ of } C' \\ \text{which are linearly independent over } K \\ \text{and which do not divide } F_1 \ldots F_h \text{ in } C' \text{ then, for every } p \in \mathbb{N}, \\ \text{the ideal } (H_1^p, \ldots, H_{d-1}^p, z^p)R \text{ is a reduction of } M^p. \end{cases} \tag{45}$$

Finally, by standard arguments we see that

$$\begin{cases} \text{if } G_i = 0 \text{ for } h < i < m \\ \text{and } m \text{ is nondivisible by the characteristic of } K \\ \text{then } R \text{ is normal.} \end{cases} \tag{46}$$

3.3 *Second* (*Rees-ring*) *method*

Let I be an ideal in a nonnull ring R. Referring to pages 272–277 of [8] for definitions, for the associated graded ring

$$\operatorname{grad}(R, I) = \bigoplus_{n\in\mathbb{N}} (I^n/I^{n+1}),$$

we have the following.

Lemma 3.3.1. *Assume that the ring grad*(R, I) *is reduced, i.e., has no nonzero nilpotent element. Then for every positive integer* c, *the ideal* I^c *coincides with its integral closure in* R.

Proof. Let, if possible, c be a positive integer such that I^c does not coincide with its integral closure in R. Then there exists $x \in R \setminus I^c$ such that x is integral over I^c. Let

$$x^n + \sum_{1\le i\le n} a_i x^{n-i} = 0 \quad \text{with } a_i \in I^{ci}$$

be an equation of integral dependence. Clearly there is a unique nonegative integer $r < c$ such that $x \in I^r \setminus I^{r+1}$. Let

$$\bar{x} = \operatorname{lefo}_{(R,I)}(x) = \text{the leading form of } x \text{ relative to } (R, I).$$

Then $0 \neq \bar{x} \in \operatorname{grad}(R, I)$. Now for all i we have $ci + (n-i)r \geq nr + 1$ and hence $a_i x^{n-i} \in I^{nr+1}$. Therefore

$$x^n = -\sum_{1\le i\le n} a_i x^{n-i} \in I^{nr+1}$$

and hence $\bar{x}^n = 0$. Thus $\bar{x}$ is a nonzero nilpotent element in grad(R, I). This is a contradiction. Therefore for every positive integer c, the ideal I^c coincides with its integral closure in R. □

Lemma 3.3.2. *Assume that* R *is a normal noetherian domain and* I *is a nonzero nonunit normal ideal in* R. *Then the height zero primes of* grad(R, I) *correspond in a one-to-one manner with the Rees valuations of* I *as follows. Upon letting* L *be the quotient field of* R *and* $E(I)$ *be the Rees ring of* I *relative to* R *with variable* Z, $P \mapsto E(I)_P \cap L$ *gives a bijection of the set of all minimal primes of* $IE(I)$ *onto the set of all Rees valuation rings of* I.

Observe that the height zero primes of grad(R, I) form a nonempty finite set. Also observe that under the natural epimorphism

$$E(I) \to \frac{E(I)}{IE(I)} \to \operatorname{grad}(R, I),$$

the set of minimal primes of $IE(I)$ bijectively map onto the height zero primes of grad(R, I). Finally observe that, by (8.1)(VI) of [16], $E(I)$ is a normal noetherian domain and $\mathfrak{W}(R, I)^{\mathfrak{N}} = \mathfrak{W}(R, I)$, and hence the Rees valuation rings $\mathfrak{D}^*(R, I, I)$ coincide with the dicritical divisors $\mathfrak{D}(R, I, I)$.

Proof. Follows from (6.11) of [17].

Note 3.4. Let $J = (y_1, \ldots, y_b)R \subset (x_1, \ldots, x_a)R = I$ be any finitely generated ideals in a domain R with quotient field L. Then we have the following:

(I) J is a reduction of $I \Leftrightarrow JV = IV$ for all $V \in \bar{D}(L/R)$.
(II) If J is a reduction of I then, for every $p \in \mathbb{N}$, the ideal $(y_1^p, \ldots, y_b^p)R$ is a reduction of I^p.

Proof of (*I*). First suppose that J is a reduction of I. Then $JI^n = I^{n+1}$ for some $n \in \mathbb{N}$. Let there be given any $V \in \bar{D}(L/R)$. Since J and I are finitely generated, they extend to principal ideals $JV = \beta V$ and $IV = \alpha V$ with $\beta \in J$ and $\alpha \in I$. Therefore, since $JI^n = I^{n+1}$, we get $\alpha\beta^n V = \beta^{n+1} V$. Consequently $\alpha V = \beta V$. Hence $JV = IV$.

Next suppose that $JV = IV$ for all $V \in \bar{D}(L/R)$. Then I is the integral over J by (3.6) below. Therefore, by (4.4) of [16], for every $i \in \{1, \ldots, a\}$ we can find $n_i \in \mathbb{N}_+$ such that $x_i^{n_i} \in JI^{n_i - 1}$. Let $n = n_1 + \cdots + n_a$. We claim that $I^{n+1} \subset JI^n$. From this it will follow that $JI^n = I^{n+1}$ and hence J is a reduction of I. To prove the claim it suffices to show that $x_1^{m_1} \ldots x_a^{m_a} \in JI^n$ whenever $m_1 + \cdots + m_a = n + 1$. But the last equation implies $m_i \geq n_i$ for some i and hence $x_i^{m_i} \in JI^{m_i - 1}$. Therefore $x_1^{m_1} \ldots x_a^{m_a} \in JI^n$. $\square$

Proof of (*II*). Assume that J is a reduction of I. Then for all $V \in \bar{D}(L/R)$, by (I) we have $JV = IV$ and hence $J^p V = I^p V$ for every $p \in \mathbb{N}$. But clearly $J^p V = (y_1^p, \ldots, y_b^p)V$ and hence $(y_1^p, \ldots, y_b^p)V = I^p V$. Therefore again by (I) we see that $(y_1^p, \ldots, y_b^p)R$ is a reduction of I^p. $\square$

Zariski's theorem 3.5 (*see Theorem* 1 *on page* 350, *Volume II of* [27] *and the beginning of Section* 8 *of* [16]). *Let $\bar{R}$ be the integral closure of a domain R in its quotient field L. Then the completion of any ideal J in R coincides with the integral closure of J in $\bar{R}$.*

ZARISKI'S COROLLARY 3.6

In the situation of (3.3), *let $J \subset I$ be any ideals in R such that $JV = IV$ for all $V \in \bar{D}(L/R)$. Then I is integral over J.*

Proof. By the definition of completion, the completion of I coincides with the completion of J. Therefore by (3.3) the integral closure of I in $\bar{R}$ coincides with the integral closure of J in $\bar{R}$. Therefore I is integral over J. $\square$

Remark 3.7. Comparing the concrete (3.2) to the abstract (3.3), we note that in (3.3) we are saying that the dicriticals are in a bijective correspondence with the height zero primes of $\mathrm{grad}(R, I)$ under the assumption that R is a normal noetherian domain and $\mathrm{grad}(R, I)$ is reduced, without explicitly describing how to find these height zero primes or how many such height zero primes there are. In (3.2) we are taking a concrete hypersurface case and giving an explicit description of the dictitical divisors of the maximal ideal by using affine QDTs.

4. Contact numbers

Given any prime divisors V, W of a two dimensional regular local domain R, i.e., elements V, W of $D(R)^{\Delta}$, in Proposition (3.5) of [11], the *contact number* $c(R, V, W)$ of V with W at R was defined by putting

$$c(R, V, W) = V(\zeta_R(W)),$$

where we recall that for any $a \in \mathrm{QF}(V)$ we have put

$$V(a) = \mathrm{ord}_V(a)$$

and for any ideal I in a noetherian subring S of V we have put

$$V(I) = \min\{V(a) : a \in I\}$$

with the understanding that $V(I) = \infty$ if $I \subset \{0\}$.

Our goal in this section is to prove the following commutativity Theorem 4.6 about contact numbers. Our main tool will be Lemma (6.11) of our previous paper [17]. First we shall introduce some terminology and prove a string of lemmas.

Recall that $H_T : T \to H(T) = T/M(T)$ denotes the residue class epimorphism of any quasilocal ring T, and for any subring S of T let us put

$$\chi(S, T) = [K' : K],$$

where K' is the algebraic closure of $K = \mathrm{QF}(H_T(S))$ in $H(T)$.

For any subring S of a ring T we put

$$\bar{S}^T = \text{the integral closure of } S \text{ in } T$$

and we note that $\bar{S}^T$ is a subring of T, and for any ideal J in S we put

$$J^{-T} = \text{the integral closure of } J \text{ in } T$$

and we note that J^{-T} is an ideal in $\bar{S}^T$.

Recall that the *length* of a module M over a ring D is denoted by

$$\ell_D M.$$

Note that M is a *finite D-module* means M is finitely generated as a module.

By $\iota(\mathfrak{a}, \mathfrak{a}'; R)$ we denote the *intersection multiplicity* of any nonzero principal ideals $\mathfrak{a}, \mathfrak{a}'$ in a two dimensional regular local domain R; for respective generators a, a' of $\mathfrak{a}, \mathfrak{a}'$, we may write $\iota(a, a'; R)$ instead of $\iota(aR, a'R; R)$; recall that

$$\iota(a, a'; R) = \ell_R(R/(a, a')R)$$

and note that this is zero or a positive integer or infinity according as the ideal $(a, a')R$ is the unit ideal or an $M(R)$-primary ideal or is contained in a nonzero nonunit principal ideal.

Let t, t^* be independent indeterminates over a field L. Referring to the beginning of §2 of [13] for the definition of the t-extension R^t of any subring R of L, we define the

(t, t^*)-*extension* R^{t,t^*} of R by putting $R^{t,t^*} = (R^t)^{t^*}$. The ring R^{t,t^*} can be directly defined as the localization of $R[t, t^*]$ at the multiplicative set of all polynomials $f(t, t^*) \in R[t, t^*]$ whose coefficients generate the unit ideal in R. Note that if R is a regular local domain of dimension d with quotient field L then R^t and R^{t,t^*} are regular local domains of dimension d with quotient fields $L^t = L(t)$ and $L^{t,t^*} = L(t, t^*)$ respectively.

Note 4.1. Let J be an $M(R)$-primary ideal in a two dimensional regular local domain, and let $I = J^{-R}$. Then by (4.5) and (4.6) of [16] we know that I is an $M(R)$-primary ideal in R and, by (3.3), I coincides with the completion of J. By (8.1)(VI) of [16] and Zariski's theorem stated as ZQT in §2 of [16], I is a normal ideal in R, $\mathfrak{D}(R, J) = \mathfrak{D}(R, I, I) =$ a set consisting of a finite number of distinct DVRs with $h \in \mathbb{N}_+$, and we have

$$I = \prod_{1 \le i \le h} \zeta_R(V_i)^{n(i)} \quad \text{with } n(i) \in \mathbb{N}_+.$$

DEFINITION 4.2

In the above set-up of (4.1), let $V_1^*, \ldots, V_{h^*}^*$ be a finite number of distinct DVRs with $h^* \in \mathbb{N}_+$ and let

$$I^* = \prod_{1 \le i \le h^*} \zeta_R(V_i^*)^{n^*(i)} \quad \text{with } n^*(i) \in \mathbb{N}_+.$$

Let us define the *contact number* $c(R, I, I^*)$ of I with I^* at R by putting

$$c(R, I, I^*) = \sum_{1 \le i \le h, 1 \le j \le h^*} n(i)n^*(j)\chi(R, V_i)c(R, V_i, V_j^*).$$

Note that this is always a positive integer. In (4.6) we shall prove a commutativity property of $c(R, I, I^*)$ and, as a special case, it will imply a commutativity property of $c(R, V, W)$.

Lemma 4.3. *Let D be a one-dimensional local domain with quotient field L and let $B = \bar{D}^L$. Assume that B is finite D-module. Then B is a Dedekind domain having only a finite number of distinct nonzero prime ideals $P_1, \ldots, P_h$ with $h \in \mathbb{N}_+$, and upon letting $V_i = B_{P_i}$ we have that $V_1, \ldots, V_h$ are DVRs with $B = V_1 \cap \cdots \cap V_h$ and $M(V_i) \cap D = M(D)$ for $1 \le i \le h$. Moreover, for any $0 \ne z \in M(D)$ we have*

$$\ell_D(D/zD) = \sum_{1 \le i \le h} e_i f_i, \quad \text{where } e_i = V_i(z) \text{ and } f_i = \chi(D, V_i).$$

Proof. Everything except the 'moreover' part is well-known. Now for any $0 \ne z \in D$, we have

$$zB = \prod_{1 \le i \le h} P_i^{e_i} \quad \text{with } e_i = V_i(z).$$

Considering the natural surjective maps

$$\phi : B \to B/zB \quad \text{and} \quad \phi_i : B \to B/P_i^{e_i}$$

and

$$\psi_i : B/P_1^{e_1} \oplus \cdots \oplus B/P_h^{e_h} \to B/P_i^{e_i}$$

of B-modules, by Chinese remaindering we get a unique isomorphism

$$\psi : B/zB \to B/P_1^{e_1} \oplus \cdots \oplus B/P_h^{e_h}$$

of B-modules such that for all $x \in B$ we have $\psi_i(\psi(\phi(x)) = \phi_i(x)$ for $1 \le i \le h$. Clearly ψ is also an isomorphism of D-modules, and hence

$$\ell_D(D/zD) = \sum_{1\le i\le h} \ell_D\big(B/P_i^{e_i}\big).$$

We shall show that for each i we have $\ell_D(B/P_i^{e_i}) = e_i f_i$ and this will complete the proof. Again by Chinese remaindering we see that each P_i is generated by a single nonzero element y_i. We have a sequence

$$P_i^{e_i} \subset P_i^{e_i-1} \subset P_i = P_i^0 = B$$

of B-modules and multiplication by y_i^j gives a B-module isomorphism $B/P_i \to P_i^j/P_i^{j+1}$ and hence a D-module isomorphism $B/P_i \to P_i^j/P_i^{j+1}$. Therefore for $0 \le j \le e_i - 1$, we have $\ell_D(B/P_i) = \ell_D(P_i^j/P_i^{j+1})$. Consequently

$$\ell_D\big(D/P_i^{e_i}\big) = e_i \ell_D(B/P_i).$$

Now $V_i/M(V_i)$ and B/P_i are isomorphic as B-modules, and hence also as D-modules. Therefore it suffices to show that $\ell_D H(V_i) = f_i$. Clearly $H(V_i)$ is annihilated by $M(D)$ and hence $H(V_i)$ may be regarded as a $D/M(D)$-module and we have $\ell_D H(V_i) = \ell_{D/M(D)} H(V_i)$. Obviously $\ell_{D/M(D)} H(V_i) = f_i$. □

Lemma 4.4. *Let R be a normal noetherian domain with quotient field L. Let I be a nonzero nonunit normal ideal in R. Let J be an ideal in R such that $I = J^{-R}$. Let $0 \ne x \in I$ be such that for every $V \in \mathfrak{D}(R, I, I)$ we have $V(x) = V(I)$. Let $C = R[Jx^{-1}]$ and $A = R[Ix^{-1}]$. Then A is a finite C-module with $A = \bar{C}^L$.*

Proof. By (6.11) of [17] we know that $\mathfrak{W}(R, I)^{\mathfrak{N}} = \mathfrak{W}(R, I)$ and, because $0 \ne x \in I$, clearly $\mathfrak{V}(A)$ is an affine piece of $\mathfrak{W}(R, I)$, i.e.,

$$A = \bigcap_{S\in\mathfrak{V}(A)} S \quad \text{with } \mathfrak{V}(A) \subset \mathfrak{W}(R, I)$$

and hence $\bar{A}^L = A$. Since I is integral over J, any $y \in I$ satisfies an equation of the form

$$y^n + y_1 y^{n-1} + \cdots + y_i y^{n-i} y_n = 0, \quad \text{where } y_i \in J^i \text{ for } 1 \le i \le n$$

with $n \in \mathbb{N}_+$. We can write y_i as a finite sum $\sum y_{i1} \dots y_{ii}$ with $y_{i1}, \dots, y_{ii}$ in J. Dividing the displayed equation by x^n we get

$$\left(\frac{y}{x}\right)^n + \sum_{1\le i\le n} \left(\sum \left(\frac{y_{i1}}{x}\right) \cdots \left(\frac{y_{ii}}{x}\right)\right) \left(\frac{y}{x}\right)^{n-i} = 0$$

with $\frac{y_{ij}}{x} \in Jx^{-1}$ for all i, j. Since I is a finitely generated ideal, it follows that A is a finitely generated ring extension of C and every element of A is integral over C. Therefore by (E2) on page 161 of [8] we conclude that A is a finite C-module with $A = \bar{C}^L$. □

Note 4.5. Let R be a normal quasilocal domain with quotient field L, and let $C = R[F/G]$, where F, G are nonzero elements in R such that $F/G \notin R$ and $G/F \notin R$. Let $Q = M(R)C$ and let Z be an indeterminate.

(1) By the bracketed remark on pages 75–76 of [2], there exists a unique ring epimorphism $\phi : C \to H(R)[Z]$ with $\phi(F/G) = Z$ such that for all $a \in R$ we have $\phi(a) = H_R(a)$. By the said bracketed remark we also see that $\ker(\phi) = Q$ and hence Q is a nonzero depth-one prime ideal in C.

(2) Let us observe that if R is a two-dimensional normal local domain then C is a two-dimensional noetherian domain and Q is a height-one prime ideal in C. To see this note that, by Lemma (T30) on page 235 of [8], $R[t]$ is a three-dimensional noetherian domain, and by sending t to $-F/G$ we get an R-epimorphism (= a ring epimorphism which keeps R elementwise fixed) $\mu : R[t] \to C$ with $\mu(\Phi) = 0$ where

$$\Phi = F + tG$$

and hence C is a two dimensional noetherian domain and Q is a height-one prime ideal in C.

(3) Henceforth assume that R is a two-dimensional regular local domain and dividing F and G by their GCD (which is determined up to a unit in R) let us arrange that the ideal $J = (F, G)R$ is $M(R)$-primary. Also let $I = J^{-R}$ and $A = R[I/G]$. By (2) we see that Q is a height-one prime ideal in the two-dimensional noetherian domain C, and hence upon letting $D = C_Q$ we see that D is a one-dimensional local domain and μ uniquely extends to an R-epimorphism $\nu : R^t \to D$. Note that now

$$\ker(\mu) = \Phi R[t] \quad \text{and} \quad \ker(\nu) = \Phi R^t.$$

By (4.1) we know that I is a normal ideal in R and hence by (4.4) we see that A is a finite C-module with $A = \bar{C}^L$. Consequently, upon letting $B = A_{C\setminus Q}$ we conclude that B is a finite D-module with $B = \bar{D}^L$ and B is a Dedekind domain having only a finite number of distinct nonzero prime ideals $P_1, \ldots, P_h$ with $h \in \mathbb{N}_+$, and upon letting $V_i = B_{P_i}$ for $1 \le i \le h$ we have that $V_1, \ldots, V_h$ are DVRs with $B = V_1 \cap \cdots \cap V_h$. In view of Lemma 8.3 of [16], by (4.1) it follows that $V_1, \ldots, V_h$ are all the distinct members of $\mathfrak{D}(R, J)$ and we have

$$I = \prod_{1 \le i \le h} \zeta_R(V_i)^{n(i)} \quad \text{with } n(i) \in \mathbb{N}_+.$$

By (4.3) we see that for any $0 \neq z \in M(D)$, we have

$$\ell_D(D/zD) = \sum_{1 \le i \le h} \chi(D, V_i)\, V_i(z).$$

By the description given in Lemma 8.3(II) of [16] we see that for $1 \le i \le h$ we have

$$\chi(D, V_i) = n(i)\chi(R, V_i).$$

Combining the above two displays we obtain

$$\ell_D(D/zD) = \sum_{1\le i\le h} n(i)\chi(R, V_i)\, V_i(z).$$

(4) In the set-up of (3), let

$$I^* = \prod_{1\le i\le h^*} \zeta_R\big(V_i^*\big)^{n^*(i)} \quad \text{with } n^*(i) \in \mathbb{N}_+$$

where $V_1^*, \ldots, V_{h^*}^*$ are pairwise distinct members of $D(R)^\Delta$ with $h^* \in \mathbb{N}_+$. Also let there be given any $\Phi^* \in I^*R[t]$ such that $(\Phi, \Phi^*)R^t$ is $M(R^t)$-primary and

$$(\sharp) \qquad (V_i)^t(\Phi^*) = V_i(I^*) = V_i(\mu(\Phi^*)) \quad \text{for} \quad 1 \le i \le h.$$

We *claim* that then

$$\iota(\Phi, \Phi^*; R^t) = c(R, I, I^*).$$

Proof of the claim. Upon letting $z = \mu(\Phi^*)$ we get a nonzero element z in $M(D)$ such that $R^t/(\Phi, \Phi^*)R^t$ and D/zD are isomorphic R^t-modules and we have

$$\ell_{R^t}(R^t/(\Phi, \Phi^*)R^t) = \ell_D(D/zD).$$

By definition the LHS equals $\iota(\Phi, \Phi^*; R^t)$ and by (3) the RHS equals

$$\sum_{1\le i\le h} n(i)\chi(R, V_i)\, V_i(z)$$

and hence we get

$$\iota(\Phi, \Phi^*; R^t) = \sum_{1\le i\le h} n(i)\chi(R, V_i)\, V_i(z).$$

Therefore by $(\sharp)$ we obtain

$$\iota(\Phi, \Phi^*; R^t) = \sum_{1\le i\le h} n(i)\chi(R, V_i)\, V_i(I^*).$$

But for $1 \le i \le h$ we have

$$V_i(I^*) = \sum_{1\le j\le h^*} n^*(j) V_i(\zeta_R(V_j^*)).$$

By the above two equations we get

$$\iota(\Phi, \Phi^*; R^t) = \sum_{1\le i\le h, 1\le j\le h^*} n(i)n^*(j)\chi(R, V_i)\, V_i(\zeta_R(V_j^*)).$$

Therefore by the definition of $c(R, I, I^*)$ we conclude that

$$\iota(\Phi, \Phi^*; R^t) = c(R, I, I^*).$$

□

Theorem 4.6 (*On commutativity of contact numbers*). *Let R be a two-dimensional regular local domain with quotient field L. Let*

$$I = \prod_{1 \le i \le h} \zeta_R(V_i)^{n(i)},$$

where $h, n(1), \ldots, n(h)$ are positive integers, and $V_1, \ldots, V_h$ are pairwise distinct members of $D(R)^\Delta$. Let

$$I^* = \prod_{1 \le i \le h^*} \zeta_R\big(V_i^*\big)^{n^*(i)},$$

where $h^*, n^*(1), \ldots, n^*(h^*)$ are positive integers, and $V_1^*, \ldots, V_{h^*}^*$ are pairwise distinct members of $D(R)^\Delta$. Let F, G, F^*, G^* be nonzero members of $M(R)$ such that

$$((F, G)R)^{-R} = I \ \text{ and } \ ((F^*, G^*)R)^{-R} = I^*.$$

Then we have the following:
(4.6.1) *Upon letting*

$$\Phi = F + tG \ \text{ and } \ \Phi^* = F^* + tG^*$$

and, *assuming* (•) $(\Phi, \Phi^*)R^t$ *to be* $M(R^t)$*-primary*, *we have*

$$c(R, I, I^*) = \iota(\Phi, \Phi^*; R^t) = c(R, I^*, I).$$

(4.6.2) *We always have*

$$c(R, I, I^*) = c(R, I^*, I).$$

(4.6.3) *Upon letting*

$$\Phi = F + tG \ \text{ and } \ \Phi^* = F^* + t^*G^*,$$

we have

$$c(R, I, I^*) = \iota(\Phi, \Phi^*; R^{t,t^*}) = c(R, I^*, I).$$

Proof of 4.6.1. By symmetry, it is enough to prove that

$$\iota(\Phi, \Phi^*; R^t) = c(R, I, I^*).$$

Therefore, in view of Claim (4.5)(4), we only need to show that for $1 \le i \le h$, we have

$$(V_i)^t(\Phi^*) = V_i(I^*) = V_i(\mu(\Phi^*)).$$

Clearly

$$(V_i)^t(\Phi^*) = \min(V_i(F^*), V_i(G^*)) = V_i(I^*)$$

and hence $(V_i)^t(\Phi^*) = V_i(I^*)$. Now

$$\mu(\Phi^*) = F^* - (F/G)G^*$$

and by §2 of [11] we know that F/G is residually transcendental over R at V_i. Therefore $V_i(\mu(\Phi^*)) = \min(V_i(F^*), V_i(G^*))$. $\square$

Proof of 4.6.2. Let $\Phi = F + tG$. Let

$$\Phi^* = \begin{cases} F^* + tG^*, & \text{if } (\Phi, F^* + tG^*)R^t \text{ is } M(R^t)\text{-primary} \\ G^* + tF^*, & \text{otherwise.} \end{cases}$$

Then $(\Phi, \Phi^*)R^t$ is $M(R^t)$-primary, and hence we are done by (4.5.1). $\square$

Proof of 4.6.3 (see (4.3) on page 334 of [22]). By symmetry, it is enough to prove that

$$(^*) \qquad \iota\big(\Phi, \Phi^*; R^{t,t^*}\big) = c(R, I, I^*).$$

Let $(\bar{R}, \bar{L}) = (R^{t^*}, L^{t^*})$ and $(\bar{I}, \bar{I}^*) = (I\bar{R}, I^*\bar{R})$ and

$$\big(\bar{V}_1, \dots, \bar{V}_h, \bar{V}_1^*, \dots, \bar{V}_{h^*}^*\big) = ((V_1)^{t^*}, \dots, (V_h)^{t^*}, (V_1^*)^{t^*}, \dots, (V_{h^*}^*)^{t^*}).$$

Then $\bar{V}_1, \dots, \bar{V}_h$ are pairwise distinct members of $D(\bar{R})^\Delta$ and F, G are nonzero members of $M(\bar{R})$ such that

$$((F, G)\bar{R})^{-\bar{R}} = \bar{I} = \prod_{1 \le i \le h} \zeta_{\bar{R}}(\bar{V}_i)^{n(i)}.$$

Likewise, $\bar{V}_1^*, \dots, \bar{V}_h^*$ are pairwise distinct members of $D(\bar{R})^\Delta$ and F^*, G^* are nonzero members of $M(\bar{R})$ such that

$$((F^*, G^*)\bar{R})^{-\bar{R}} = \bar{I}^* = \prod_{1 \le i \le h^*} \zeta_{\bar{R}}(\bar{V}_i^*)^{n^*(i)}.$$

Clearly the ideal $(\Phi, \Phi^*)\bar{R}^t$ is $M(\bar{R}^t)$-primary and for $1 \le i \le h$ we have

$$(\bar{V}_i)^t(\Phi^*) = \min(\bar{V}_i(F^*), \bar{V}_i(G^*)) = \bar{V}_i(I^*)$$

and hence $(\bar{V}_i)^t(\Phi^*) = \bar{V}_i(I^*)$. Also

$$\mu(\Phi^*) = F^* + t^*G^*,$$

where $\mu : \bar{R}[t] \to C = \bar{R}[F/G]$ is the unique $\bar{R}$-epimorphism such that $\mu(t) = -F/G$ and $\ker(\mu) = \Phi\bar{R}[t]$; therefore $\bar{V}_i(\mu(\Phi^*)) = \min(\bar{V}_i(F^*), \bar{V}_i(G^*))$. It follows that

$$(\sharp) \qquad (\bar{V}_i)^t(\Phi^*) = \bar{V}_i(\bar{I}^*) = \bar{V}_i(\mu(\Phi^*)) \quad \text{for } 1 \le i \le h.$$

Hence, by putting a bar on the relevant quantities in the Claim of (4.5)(4), we conclude that

$$\iota(\Phi, \Phi^*; \bar{R}^t) = c(\bar{R}, \bar{I}, \bar{I}^*).$$

Clearly the above LHS equals the LHS of (*), and the above RHS equals the RHS of (*). This establishes (*) and concludes the proof. □

Note 4.7. As noted in the proof of (4.6.2), the condition (•) of (4.6.1) is satisfied by at least one of the two pairs $(\Phi, F^* + tG^*)$ or $(\Phi, G^* + tF^*)$. It is also satisfied by letting $\Phi = F + tG$ and $\Phi^* = F + \tau F$ where $\tau =$ any unit in R excluding at most one value.

Note 4.8. By a *testing curve* for $V \in D(R)^\Delta$, where R is a two dimensional regular local domain, we mean an element $\delta_V \in R$ which does 'something' useful for V at R. For instance we ask whether, for every $V \in D(R)^\Delta$, there is a testing curve δ_V such that $W(\delta_V) = W(\zeta_R(V))$ for all $W \in D(R)^\Delta$. To answer this negatively, let

$$V = o(R)$$

and note that

$$\zeta_R(V) = M \quad \text{with } V(\zeta_R(V)) = 1$$

and for any $\theta \in R$ we have

$$V(\theta) = V(\zeta_R(V)) \Leftrightarrow \theta \in M \setminus M^2. \tag{47}$$

Now let (x, y) be generators of M, let K be a coefficient set of R and, for every $t \in K \cup \{\infty\}$, let

$$(p_t, q_t) = \begin{cases} (y + tx, x), & \text{if } t \in K, \\ (x, y), & \text{if } t = \infty, \end{cases}$$

$$\Gamma_t = \{\theta \in M \setminus M^2 : (\theta R) + M^2 = (p_t R) + M^2\},$$

$$S_t = R[p_t/q_t]_{(p_t/q_t, q_t)R[p_t/q_t]} \in Q_1(R)$$

and

$$W_t = o(S_t) \in D(R)^\Delta.$$

Note that then

$$\begin{cases} t \mapsto S_t \text{ gives a bijection of } K \cup \{\infty\} \\ \text{onto the set of all those members of } Q_1(R) \\ \text{which are residually rational over } R \end{cases}$$

and

$$M \setminus M^2 = \coprod_{t \in K \cup \{\infty\}} \Gamma_t = \text{a partition into disjoint nonempty subsets.} \tag{48}$$

Also note that for each $t \in K \cup \{\infty\}$ we have

$$(p_t, q_t)R = M$$

with

$$W_t(\theta) = W_t(\zeta_R(V)) = 1 \quad \text{for all } \theta \in (M/\setminus M^2) \setminus \Gamma_t \tag{49}$$

and

$$W_t(\theta) = 2 \quad \text{for all } \theta \in \Gamma_t. \tag{50}$$

By (47)–(50) we see that

$$\begin{cases} \text{there is no } \delta_V \in R \\ \text{such that } V(\delta_V) = V(\zeta_R(V)) \\ \text{and } W_t(\delta_V) = W_t(\zeta_R(V)) \text{ for all } t \in K \cup \{\infty\}, \end{cases} \tag{51}$$

but

$$\begin{cases} \text{for every } \tau \in K \cup \{\infty\} \text{ and every } \delta_V \in \Gamma_\tau \\ \text{we have } V(\delta_V) = V(\zeta_R(V)) \\ \text{and } W_t(\delta_V) = W_t(\zeta_R(V)) \text{ for all } t \in (K \cup \{\infty\}) \setminus \{\tau\}. \end{cases} \tag{52}$$

In [15] it will be shown how the difficulty in defining a good testing curve is removed by introducing the concept of 'free points' of V.

Note 4.9. Here is a *question* related to the definitions of dicritical divisors and Rees valuation rings introduced in §2. Let J be a nonzero nonunit ideal in a normal noetherian domain R with quotient field L, and for $0 \neq G \in J$, let $C = R[J/G]$ and $A = \bar{C}^L$. Then does every height-one prime of GA contract to a height-one prime of GC?

Note 4.10. Here is another *question*. In 4.5(4), we found a sufficient condition for the intersection formula to work. That condition assumes an indeterminate inside Φ but not inside Φ^*. The question is whether we can find a condition in which both Φ and Φ^* stay inside R. In other words, can we find a sufficient (and necessary?) condition on $\Phi \in I$ and $\Phi^* \in I^*$ which would imply $\iota(\Phi, \Phi^*; R) = c(R, I, I^*)$?

Note 4.11. Here is yet another *question*. In (46) of §3, can you prove the normality of R under more general conditions?

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