Proc. Indian Acad. Sci. (Math. Sci.) Vol. 122, No. 3, August 2012, pp. 335–337.

Finite groups with three conjugacy class sizes of some elements

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MS received 24 March 2011; revised 13 July 2011

Abstract. Let G be a finite group. We prove as follows: Let G be a p-solvable group for a fixed prime p. If the conjugacy class sizes of all elements of primary and biprimary orders of G are $\{1, p^a, n\}$ with a and n two positive integers and $(p, n) = 1$, then G is p-nilpotent or G has abelian Sylow p-subgroups.

Keywords. Conjugacy class sizes; p-nilpotent groups; finite groups.

1. Introduction

All groups considered in this paper are finite. If G is a group, then x^G denotes the conjugacy class containing x and $|x^G|$ the size of x^G. Following [1], we call $\mathrm{Ind}_G(x) = |x^G| = |G : C_G(x)|$, the index of x in G. The rest of our notation and terminology are standard. The reader may refer to [10].

It is well-known that there is a strong relation between the structure of a group and the sizes of its conjugacy classes and there exist several results studying the structure of a group under some arithmetical conditions on its conjugacy class sizes. For example, Itô in [6] showed that if the sizes of the conjugacy classes of a group G are $\{1, m\}$, then G is nilpotent, $m = p^a$ for some prime p and $G = P \times A$, with P a Sylow p-subgroup of G and $A \subseteq Z(G)$. Then he showed in [7] that G is solvable if the conjugacy class sizes of G are $\{1, n, m\}$. Other authors replace conditions for all conjugacy classes by conditions referring to only some conjugacy classes to investigate the structure of a finite group. For instance, Baer in [1] proved that a group G is solvable if its elements of prime power order have also prime power index. In [8], Li proved that G is solvable if the finite group G has exactly two conjugacy class sizes of elements of prime power order. Recently in [2], Berkovich and Kazarin described groups in which the indices of all elements of primary or biprimay orders, are prime powers. In this paper, we show how imposing some arithmetical conditions on conjugacy class sizes of all elements of primary or biprimary orders of G yields restrictions on the structure of G. Stimulated by papers [8] and [2], we consider that the finite group G has three conjugacy class sizes of elements of primary and biprimay orders of G and get the following main result.

Theorem. *Let G be a p-solvable group for a fixed prime p. If the conjugacy class sizes of all elements of primary and biprimary orders of G are $\{1, p^a, n\}$ with a and n two positive integers and $(p, n) = 1$, then G is p-nilpotent or G has abelian Sylow p-subgroups.*

2. Basic definitions and preliminary results

In this section, we state some lemmas which are useful for our main results.

Lemma 2.1 (*Lemma* 6 *of* [1]). *$O_p(G)$ contains every element in G whose order and index are powers of p.*

Lemma 2.2 (*Chap.* 5, *Theorem* 3.4 *of* [4]). *Let $A \times B$ be a group of automorphisms of the p-group P with A a p'-group and B a p-group. If A acts trivially on $C_P(B)$, then $A = 1$.*

Lemma 2.3 (*Theorem* 5 *of* [9]). *Let G be a finite group and p a prime divisor of $|G|$. Then there is in G no p'-element of prime power order whose index is divisible by p if and only if $G = P \times H$, where P is a Sylow p-subgroup of G and H has order prime to p.*

By Lemma 2.3 we can easily get the following result.

Lemma 2.4. *Let G be a group. A prime p does not divide any conjugacy class length of any element of prime power order of G if and only if G has a central Sylow p-subgroup.*

Lemma 2.4 can be seen as a generalization of Theorem 33.4 of [5]: Let G be a group. A prime p does not divide any conjugacy class length of G if and only if G has a central Sylow p-subgroup.

Lemma 2.5 (*Lemma* 1.1 *of* [3]). *Let $N \trianglelefteq G$, $x \in N$, and $y \in G$. Then*

(i) $|x^N| \mid |x^G|$;
(ii) $|(yN)^{G/N}| \mid |y^G|$.

3. The proof of the main result

Proof. Let G be a p-solvable group such that the conjugacy class size of any p-element of G is coprime with p. Then the Sylow p-subgroup of G is abelian. Indeed, we may assume without loss of generality, by Lemma 2.5, that $O_{p'}(G) = 1$. Since for every p-element $x \in G$ the index $[G : C_G(x)]$ is coprime with p, it follows that x centralizes $O_p(G)$. By Theorem 6.3.2 of [4], $x \in O_p(G)$. This implies that G is p-closed and $x \in Z(O_p(G))$. Hence the Sylow p-subgroup of G is abelian as required.

Now by the hypothesis we suppose that x is a p-element of G such that $[G : C_G(x)] = p^a$. By Lemma 2.1, we know that the normal closure of x will be a p-group, say H. Let $Z = C_G(H)$. Now $[G : C_G(x)] = p^a$, and so if $y \in C_G(x)$ and y has prime power order prime to p, $[C_G(x) : C_G(xy)]$ is prime to p. Otherwise p^{a+1} would divide the index of xy, contrary to the hypothesis. However, $C_G(xy) = C_G(x) \bigcap C_G(y)$, as x and y has coprime order and $[x, y] = 1$. As $[C_G(x) : C_G(xy)]$ is prime to p, we can assume that $C_G(xy)$ contains a Sylow p-subgroup of $C_G(x)$, say P. Obviously $H \bigcap C_G(x) \leq H$. Since H is normal in G, we have that $H \bigcap C_G(x) \leq O_p(C_G(x)) \leq P$. Thus $H \bigcap C_G(x) \leq H \bigcap P$. As $C_G(xy) = C_G(x) \bigcap C_G(y) \subseteq C_G(y)$, we get that $P \leq C_G(y)$, and it follows that $H \bigcap C_G(x) \leq H \bigcap C_G(y)$ or $C_H(x) \leq C_H(y)$. We can now use Lemma 2.2 to deduce that $C_H(y) = H$.

As $[G : C_G(x)] = p^a$, we can deduce that $[G : Z]$ is a power of p. Now let w be any p'-element of prime power order in Z. By the previous argument, $[C_G(x) : C_G(w) \bigcap C_G(x)]$ is prime to p, but, as Z is a normal subgroup of $C_G(x)$, we have that

$[Z : C_Z(w)]$ is prime to p. Thus every p'-element of prime power order in Z has index in Z prime to p and so, by Lemma 2.3 we have that $Z = K \times P_1$, where K has order prime to p and P_1 is the Sylow p-subgroup of Z. As $[G : Z]$ is a power of p, K is a normal p-complement of G. Hence G is p-nilpotent.

The proof of the theorem is now complete. □

Acknowledgements

The author is very grateful to the referee for reading the manuscript carefully and providing a lot of valuable suggestions and useful comments. It should be mentioned that the final version of this paper could not have been refined so well without his or her outstanding efforts. This research is supported by the National Natural Science Foundation of China (10771132), SGRC (GZ310), the Research Grant of Shanghai University and Shanghai Leading Academic Discipline Project (J50101).

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