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On the limit distribution of lower extreme generalized order statistics

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Abstract. In a wide subclass of generalized order statistics (gOs), which contains most of the known and important models of ordered random variables, weak convergence of lower extremes are developed. A recent result of extreme value theory of $m-$gOs (as well as the classical extreme value theory of ordinary order statistics) yields three types of limit distributions that are possible in case of linear normalization. In this paper a similar classification of limit distributions holds for extreme gOs, where the parameters γ_j, $j = 1, \ldots, n$, are assumed to be pairwise different. Two illustrative examples are given to demonstrate the practical importance for some of the obtained results.

Keywords. Weak convergence; generalized order statistics; extreme value theory; order statistics; progressive Type II censored order statistics.

1. Introduction and auxiliary results

Kamps [6] introduced the concept of generalized order statistics (gOs) as a unification of several models of ascendingly ordered random variables (rv's). The use of such a concept has been steadily growing over the years. This is due to the fact that such concepts include important well-known concepts that have been separately treated in statistical literature.

In testing the strength of materials, reliability analysis, lifetime studies, etc., the realizations of experiments arise in nondecreasing order and therefore we need to consider several models of ascendingly ordered rv's. Theoretically, many of these models are contained in the gOs model, such as ordinary order statistics (oOs), order statistics with non-integral sample size, sequential order statistics (sOs), record values, Pfeifer's record model and progressive type II censored order statistics (pOs). These models can be applied in reliability theory, especially the extreme value ones. For instance, the r-th extreme order statistic represents the lifetime of some $r-$out-of$-n$ system, whereas the sOs model is an extension of the oOs model and serves as a model describing certain dependencies or interactions among the system components caused by failures of components. The pOs model is an important method for obtaining data in lifetime tests. Live units removed early on can be readily used in other tests, thereby saving cost to the experimenter, and a comprise can be achieved between time consumption and observation of some extreme values.

The concept of gOs enables a common approach to structural similarities and analogies. Known results in submodels can be subsumed, generalized and integrated within a general framework. Well-known distributional and inferential properties of oOs and record values turn out to remain valid for gOs. Thus, the concept of gOs provides a large class of models with many interesting and useful properties for both the description and the analysis of practical problems. Due to this reason the question arises whether the general distribution theory of gOs as well as their properties can be obtained by analogy with that for oOs. The present paper proves this coincidence between the limit theory of oOs and an important wide class of gOs including pOs. The limit theory of pOs has not been available so far. Since, the statistical modeling of data that arise in practice mainly depends on the limit theory, the result of this paper will be of considerable practical importance (for more details, see Example 3.2).

Kamps [6] defined gOs by first defining what he called the uniform gOs and then he used the quantile transformation to obtain the general gOs. Namely, uniform gOs $U(r,n,\tilde{m},k),\ r=1,2,\ldots,n,$ are defined by their joint density function

$$f^{(\tilde{m},k)}_{1,2,\ldots,n:n}(u_1,u_2,\ldots,u_n)=f^{(\tilde{m},k)}_{U(1,n,\tilde{m},k),U(2,n,\tilde{m},k),\ldots,U(n,n,\tilde{m},k):n}(u_1,u_2,\ldots,u_n)$$
$$=\left(\prod_{j=1}^{n}\gamma_j\right)\left(\prod_{j=1}^{n-1}(1-u_j)^{\gamma_j-\gamma_{j+1}-1}\right)(1-u_n)^{\gamma_n-1},$$

on the cone $\{(u_1,\ldots,u_n):0\le u_1\le\cdots\le u_n<1\}\subset\mathbb{R}^n$, with parameters $\gamma_1,\ldots,\gamma_n>0$. The parameters $\gamma_1,\ldots,\gamma_n$ are defined by $\gamma_n=k>0$ and $\gamma_r=k+n-r+\sum_{j=r}^{n-1}m_j=k+M'_r,\quad r=1,2,\ldots,n-1,$ where $M'_r=\sum_{j=r}^{n-1}m'_j$, $m'_j=m_j+1,\quad \tilde{m}=(m_1,m_2,\ldots,m_{n-1})$ and $m_1,m_2,\ldots,m_{n-1}\in\mathbb{R}$. Generalized order statistics based on some distribution function (df) F are defined via the quantile transformation $X(r,n,\tilde{m},k)=F^{-1}(U(r,n,\tilde{m},k)),\ r=1,2,\ldots,n$. Particular choices of the parameters $\gamma_1,\ldots,\gamma_n$ lead to different models, e.g., $m-$gOs ($\gamma_n=k,\ \gamma_r=k+(n-r)(m+1),\ r=1,\ldots,n-1$), oOs ($\gamma_n=1,\ \gamma_r=n-r+1,\ r=1,\ldots,n-1$, i.e., $k=1,\ m_i=0,\ i=1,\ldots,n-1$), sOs ($\gamma_n=\alpha_n,\ \gamma_r=(n-r+1)\alpha_r,\ \ \alpha_r>0,\ \ r=1,\ldots,n-1$), pOs with censoring scheme $(R_1,\ldots,R_M)$ ($\gamma_n=R_M+1,\ \gamma_r=n-r+1+\sum_{j=r}^{M}R_j$, if $r\le M-1$ and $\gamma_r=n-r+1+R_M$, if $r\ge M$) and upper records ($\gamma_r=1,\ 1\le r\le n$, i.e., $k=1,\ m_i=-1,i=1,\ldots,n-1$) (see [5–7]). Therefore, all the results obtained in the model of gOs can be applied to the particular models choosing the respective parameters.

In a wide subclass of gOs, where $m_1=m_2=\cdots=m_{r-1}=m$, Kamps [6] derived the marginal df

$$\Phi^{(\tilde{m},k)}_{r:n}(x)=P(X(r,n,\tilde{m},k)\le x)=1-C_{r-1}(1-F(x))^{\gamma_r}\sum_{j=0}^{r-1}\frac{1}{j!C_{r-j-1}}g^j_m(x),$$

where $m\ne-1$, $(m+1)g_m(x)=G_m(x)=1-(1-F(x))^{m+1}$ is a df, while $g_{-1}(x)=-\log(1-F(x))$ (record model), and $C_{r-1}=\prod_{i=1}^{r}\gamma_i,\ r=1,2,\ldots,n$. The possible limit df's of $\Phi^{(\tilde{m},k)}_{n:n}$, i.e., the limit df of the maximum gOs, under the condition $m_1=m_2=\cdots=m_{n-1}=m\ne-1$ (in this case, clearly, the record model is excluded) and their domain of attraction under linear normalization are shown in Nasri-Roudsari

[9]. The analogous results for the power normalization are derived by Nasri-Roudsari [10], by using the transfer theorem of Christoph and Falk [4]. Barakat [3] extended these results to a more general case, where $m_1 = m_2 = \cdots = m_{n-r} = m \neq -1$ and $\check{M} = \frac{1}{r-1}\sum_{j=n-r+1}^{n-1} m_j$ remains fixed, as $n \to \infty$. The main result of Nasri-Roudsari [9] can be summarized in the following theorem.

Theorem 1.1 [9]. *Let* $m_1 = m_2 = \cdots = m_{n-1} = m > -1$ *and* $r \in \{1, 2, \ldots, n\}$. *Then, there exist real normalizing constants* $\tilde{a}_n > 0$ *and* $\tilde{b}_n$ *for which* $\Phi^{(\tilde{m},k)}_{n-r+1:n}(\tilde{a}_n x + \tilde{b}_n) \xrightarrow[n]{w} \tilde{\Phi}^{(\tilde{m},k)}_r(x)$, *where* $\xrightarrow[n]{w}$ *denotes the weak convergence, as* $n \to \infty$, *and* $\tilde{\Phi}^{(\tilde{m},k)}_r(x)$ *is a nondegenerate df, if and only if there exist real normalizing constants* $\tilde{\alpha}_n > 0$ *and* $\tilde{\beta}_n$ *for which*

$$\Phi^{(\tilde{0},1)}_{n-r+1:n}(\tilde{\alpha}_n x + \tilde{\beta}_n) \xrightarrow[n]{w} \tilde{\Phi}^{(\tilde{0},1)}_r(x) = 1 - \Gamma_r(V_{t,\beta}(x)), \quad t \in \{1, 2, 3\},$$

where $\Gamma_r(x)$ *is the incomplete gamma function and*

$$\text{Type I}: V_1(x) = V_{1;\beta}(x) = \mathrm{e}^{-x}, \ \forall\, x;$$

$$\text{Type II}: V_{2;\beta}(x) = \begin{cases} \infty, & x \le 0, \\ x^{-\beta}, & x > 0; \end{cases}$$

$$\text{Type III}: V_{3;\beta}(x) = \begin{cases} (-x)^{\beta}, & x \le 0, \\ 0, & x > 0. \end{cases}$$

In this case $\tilde{\Phi}^{(\tilde{m},k)}_r(x) = 1 - \Gamma_R(V^{m+1}_{t,\beta}(x))$, *where* $R = \frac{k}{m+1} + r - 1$. *Moreover,* $\tilde{a}_n$ *and* $\tilde{b}_n$ *may be chosen such that* $\tilde{a}_n = \tilde{\alpha}_{\tilde{\phi}(n)}$ *and* $\tilde{b}_n = \tilde{\beta}_{\tilde{\phi}(n)}$, *where* $\tilde{\phi}(n) = n^{\frac{1}{m+1}}$.

The following theorem gives the analogous result for the r-th lower gOs.

Theorem 1.2. *Let* $m_1 = m_2 = \cdots = m_{n-1} = m > -1$ *and* $r \in \{1, 2, \ldots, n\}$. *Then, there exist real normalizing constants* $a_n > 0$ *and* b_n *for which* $\Phi^{(\tilde{m},k)}_{r:n}(a_n x + b_n) \xrightarrow[n]{w} \Phi^{(\tilde{m},k)}_r(x)$, *where* $\Phi^{(\tilde{m},k)}_r(x)$ *is a nondegenerate df, if and only if there exist real normalizing constants* $\alpha_n > 0$ *and* β_n *for which*

$$\Phi^{(\tilde{0},1)}_{r:n}(\alpha_n x + \beta_n) \xrightarrow[n]{w} \Phi^{(\tilde{0},1)}_r(x) = \Gamma_r(U_{t,\beta}(x)), \quad t \in \{1, 2, 3\},$$

where

$$\text{Type I}: U_1(x) = U_{1;\beta}(x) = \mathrm{e}^{x}, \quad \forall\, x;$$

$$\text{Type II}: U_{2;\beta}(x) = \begin{cases} (-x)^{-\beta}, & x \le 0, \\ \infty, & x > 0; \end{cases}$$

$$\text{Type III}: U_{3;\beta}(x) = \begin{cases} 0, & x \le 0, \\ x^{\beta}, & x > 0. \end{cases}$$

In this case $\Phi^{(\tilde{m},k)}_r(x) = \Gamma_r(U_{t,\beta}(x)) = 1 - \sum_{j=0}^{r-1} \frac{1}{j!} U^j_{t,\beta}(x) \mathrm{e}^{-U_{t,\beta}(x)}$, *where* a_n *and* b_n *may be chosen such that* $a_n = \alpha_{\phi(n)}$ *and* $b_n = \beta_{\phi(n)}$, *where* $\phi(n) = (m+1)n$.

Theorem 1.2 can be easily proved by considering the representation of Nasri-Roudsari [9], $\Phi_{r:n}^{(\tilde{m},k)}(x) = I_{G_m(x)}(r, N-r)$, where $I_x(a,b) = \frac{1}{B(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}\mathrm{d}t$ denotes the incomplete beta function and $N = n + \frac{k}{m+1}$. On the other hand, by noting that $\Phi_{r:n}^{(\tilde{0},1)}(\alpha_n x + \beta_n) \xrightarrow[n]{w} \tilde{\Phi}_r^{(\tilde{0},1)}(x) = \Gamma_r(U_{t,\beta}(x))$, $t \in \{1,2,3\}$, if and only if $nF(\alpha_n x + \beta_n) \xrightarrow[n]{w} U_{t,\beta}(x)$, which implies that $nG_m(\alpha_n x + \beta_n) = n[1 - \bar{F}^{m+1}(\alpha_n x + \beta_n)] \sim (m+1)nF(\alpha_n x + \beta_n)$, as $n \to \infty$, where $\bar{F} = 1 - F$. Therefore, $nG_m(\alpha_{\phi(n)}x + \beta_{\phi(n)}) \sim \phi(n)F(\alpha_{\phi(n)}x + \beta_{\phi(n)}) \to U_{t,\beta}(x)$, as $n \to \infty$. Thus, we get

$$\begin{aligned}
\Phi_{r:n}^{(\tilde{m},k)}(\alpha_{\phi(n)}x + \beta_{\phi(n)}) &= I_{G_m(\alpha_{\phi(n)}x + \beta_{\phi(n)})}(r, N-r) \\
&= 1 - \sum_{j=0}^{r-1} \frac{\Gamma(N)}{j!\Gamma(N-j)} G_m^j(\alpha_{\phi(n)}x + \beta_{\phi(n)})\bar{G}_m^{N-j-1} \\
&\quad \times (\alpha_{\phi(n)}x + \beta_{\phi(n)}) \xrightarrow[n]{w} \Gamma_r(U_{t,\beta}(x)),
\end{aligned}$$

(see also [8]).

Although, the above two theorems provide a set-up which includes many interesting models such as oOs and pOs, with censoring scheme $(R, \ldots, R) \in \mathbb{N}^M$, a lot of models contained in the family of gOs are excluded in this set-up, eg., pOs with general censoring scheme $(R_1, \ldots, R_M)$. In this paper we extend Theorem 1.2 to a very wide subclass of gOs in which the vector $\tilde{m}$ is arbitrarily chosen such that $m_i' = m_i + 1 > 0$, $i = 1, 2, \ldots, n-1$, and the parameters $\gamma_1, \ldots, \gamma_n$ are pairwise different, i.e., $\gamma_i \neq \gamma_j$, $i \neq j$, for all $i, j \in \{1, \ldots, n\}$. For instance, this assumption is no restriction on pOs with general censoring scheme $(R_1, \ldots, R_M)$. The choice $m_1 = \cdots = m_{n-1} = -1$, which has to be excluded here, corresponds to common record values. However, this excluded case has been extensively studied by various authors (see [1, 11, 12]). The marginal df of the r-th gOs in this case is given in [7] by

$$\Phi_{r:n}^{(\tilde{m},k)}(x) = 1 - C_{r-1}\sum_{i=1}^{r} \frac{a_i(r)}{\gamma_i}(1 - F(x))^{\gamma_i}, \tag{1.1}$$

where $a_i(r) = \prod_{\substack{j=1 \\ j\neq i}}^{r} \frac{1}{\gamma_j - \gamma_i} = \prod_{j=1}^{r}{}^{\langle i \rangle} \frac{1}{\gamma_j - \gamma_i}$. Since the parameters γ_j eventually depend on n we indicate this attribute subsequently by an additional index, i.e., $\gamma_{i,n}$.

Remark 1.1. Although the parameter $a_i(r)$ depends on $\gamma_{i,n}$, the following representation, which can be easily proved

$$a_i(r) = \frac{(-1)^{r-i}}{\left(\prod_{j=1}^{i-1}\sum_{t=j}^{i-1} m_t'\right)\left(\prod_{j=i+1}^{r}\sum_{t=i}^{j-1} m_t'\right)}, \tag{1.2}$$

shows that $a_i(r)$ does not depend on n for all $i = 1, \ldots, r$, whenever r is constant with respect to n. Here we adopt (and in the sequel) the conventions $\prod_{\emptyset} = 1$ and $\sum_{\emptyset} = 0$.

Remark 1.2. For all n and $i = 1, \dots, n-1$, it is easy to prove that

(i) $\gamma_{1,n} > \gamma_{2,n} > \cdots > \gamma_{n-1,n}$;
(ii) $\gamma_{i,n} = \gamma_{1,n} - \sum_{j=1}^{i-1} m'_j$.

Remark 1.3. Since, for all r, n, k and $\tilde{m}$, we have $\Phi_{r:n}^{(\tilde{m},k)}(-\infty) = 0$, then

$$C_{r-1} \sum_{i=1}^{r} \frac{a_i(r)}{\gamma_{i,n}} = \sum_{i=1}^{r} \prod_{j=1}^{r}{}^{\langle i \rangle} \frac{\gamma_{j,n}}{\gamma_{j,n} - \gamma_{i,n}} = 1,$$

for all $r \le n$ and $\gamma_{1,n} > \gamma_{2,n} > \cdots > \gamma_{n-1,n}$.

We end this section with two lemmas which are needed in the next sections.

Lemma 1.1. *For all $\theta \ne \gamma_{i,n}$, $i = m, m+1, \dots, r$, we get*

$$\sum_{i=m}^{r} \frac{a_i^{[m]}(r)}{\theta - \gamma_{i,n}} = \prod_{i=m}^{r} \frac{(-1)^{r-m}}{\theta - \gamma_{i,n}}, \tag{1.3}$$

where

$$a_i^{[m]}(r) = \prod_{j=m}^{r}{}^{\langle i \rangle} \frac{1}{\gamma_{j,n} - \gamma_{i,n}}.$$

Clearly we have $a_i^{[1]}(r) = a_i(r)$.

Proof. We prove the lemma by induction over r. Since

$$\frac{a_i^{[m]}(m)}{\theta - \gamma_{i,n}} = \frac{1}{\theta - \gamma_{i,n}} \prod_{i=m}^{m}{}^{\langle m \rangle} \frac{1}{(\gamma_{i,n} - \gamma_{m,n})} = \frac{1}{\theta - \gamma_{i,n}},$$

the LHS of the relation (1.3) coincides with its RHS at $r = m$. Let us now assume the relation (1.3) to be true for $m \le s < r$. That is

$$\sum_{i=m}^{s} \frac{a_i^{[m]}(s)}{\theta - \gamma_{i,n}} = \prod_{i=m}^{s} \frac{(-1)^{s-m}}{\theta - \gamma_{i,n}}. \tag{1.4}$$

Therefore, for $s+1$, we have

$$\sum_{i=m}^{s+1} \frac{a_i^{[m]}(s+1)}{\theta - \gamma_{i,n}} = \sum_{i=m}^{s} \frac{a_i^{[m]}(s+1)}{\theta - \gamma_{i,n}} + \frac{a_{s+1}^{[m]}(s+1)}{\theta - \gamma_{s+1,n}}.$$

Since $r \le s$ in the first summation on the RHS of the last relation, we have

$$a_i^{[m]}(s+1) = \prod_{j=m}^{s+1}{}^{\langle i \rangle} \frac{1}{(\gamma_{j,n} - \gamma_{i,n})} = \frac{a_i^{[m]}(s)}{\gamma_{s+1,n} - \gamma_{i,n}}.$$

Thus, we get

$$\sum_{i=m}^{s+1} \frac{a_i^{[m]}(s+1)}{\theta - \gamma_{i,n}} = \sum_{i=m}^{s} \frac{a_i^{[m]}(s)}{(\theta - \gamma_{i,n})(\gamma_{s+1,n} - \gamma_{i,n})} + \frac{a_{s+1}^{[m]}(s+1)}{\theta - \gamma_{s+1,n}}.$$

By using the partial fraction and the assumption of induction (1.4), we get

$$\begin{aligned}
&\sum_{i=m}^{s+1} \frac{a_i^{[m]}(s+1)}{\theta - \gamma_{i,n}} \\
&\quad = \sum_{i=m}^{s} \frac{a_i^{[m]}(s)}{(\theta - \gamma_{i,n})(\gamma_{s+1,n} - \theta)} \\
&\qquad + \sum_{i=m}^{s} \frac{a_i^{[m]}(s)}{(\theta - \gamma_{s+1,n})(\gamma_{s+1,n} - \gamma_{i,n})} + \frac{a_{s+1}^{[m]}(s+1)}{\theta - \gamma_{s+1,n}} \\
&\quad = -\frac{1}{\theta - \gamma_{s+1,n}} \prod_{i=m}^{s} \frac{(-1)^{s-m}}{\theta - \gamma_{i,n}} \\
&\qquad + \frac{1}{\theta - \gamma_{s+1,n}} \sum_{i=m}^{s} \frac{a_i^{[m]}(s)}{\gamma_{s+1,n} - \gamma_{i,n}} + \frac{a_{s+1}^{[m]}(s+1)}{\theta - \gamma_{s+1,n}}.
\end{aligned} \tag{1.5}$$

Again, an application of the assumption of induction, with $\theta = \gamma_{s+1,n}$ yields

$$\begin{aligned}
\frac{1}{\theta - \gamma_{s+1,n}} \sum_{i=m}^{s} \frac{a_i^{[m]}(s)}{\gamma_{s+1,n} - \gamma_{i,n}} &= \frac{1}{\theta - \gamma_{s+1,n}} \prod_{i=m}^{s} \frac{(-1)^{s-m}}{\gamma_{s+1,n} - \gamma_{i,n}} \\
&= \frac{1}{\theta - \gamma_{s+1,n}} \prod_{i=m}^{s} \frac{(-1)^{(s-m)+(s-m+1)}}{\gamma_{i,n} - \gamma_{s+1,n}} \\
&= -\frac{1}{\theta - \gamma_{s+1,n}} \prod_{i=m}^{s} \frac{1}{\gamma_{i,n} - \gamma_{s+1,n}} \\
&= -\frac{1}{\theta - \gamma_{s+1,n}} \prod_{i=m}^{s+1} {}^{\langle s+1 \rangle} \frac{1}{\gamma_{i,n} - \gamma_{s+1,n}} \\
&= -\frac{a_{s+1}^{[m]}(s+1)}{\theta - \gamma_{s+1,n}}.
\end{aligned} \tag{1.6}$$

Combining relations (1.5) and (1.6), we can see that relation (1.3) is true for $s+1$, which completes the proof. □

Lemma 1.2. *The df* $\Phi_{r:n}^{(\tilde{m},k)}(x)$ *can be represented as*

$$\Phi_{r:n}^{(\tilde{m},k)}(x) = 1 - \bar{F}^{\gamma_{1,n}}(x) L_{r:n}^{(\tilde{m},k)}(x), \tag{1.7}$$

where

$$L_{r:n}^{(\tilde{m},k)}(x) = \sum_{i=1}^{r} \left(\prod_{j=1}^{r\ \langle i \rangle} \frac{\gamma_{j,n}}{\gamma_{j,n} - \gamma_{i,n}} \right) \bar{F}^{-(\gamma_{1,n} - \gamma_{i,n})}(x)$$

is a polynomial of $\gamma_{1,n}$ *with degree* $r-1$. *Moreover, this polynomial depends on* n *only through the parameter* $\gamma_{1,n}$.

Proof. In view of (1.2) and Remarks 1.2(ii) and 1.3, it is easy to prove the representation

$$L_{r:n}^{(\tilde{m},k)}(x) = \sum_{i=1}^{r} \frac{(-1)^{r-i} \prod_{j=1}^{r\ \langle i \rangle} (\gamma_{1,n} - \sum_{t=1}^{j-1} m'_t)}{\left(\prod_{j=1}^{i-1} \sum_{t=j}^{i-1} m'_t \right) \left(\prod_{j=i+1}^{r} \sum_{t=i}^{j-1} m'_t \right)} \bar{F}^{-\sum_{t=1}^{i-1} m'_t}(x), \tag{1.8}$$

which in turns immediately proves our assertion, and the proof is complete. □

Lemma 1.2 reveals an interesting fact that, given a df F and suitable normalizing constants, the asymptotic behaviour of the df $\Phi_{r:n}^{(\tilde{m},k)}(x)$, as $n \to \infty$ depends solely on $\gamma_{1,n}$ and in fact the parameter $\gamma_{1,n}$ may be thought as the basket, which contains all the information on the asymptotic behaviour of the df $\Phi_{r:n}^{(\tilde{m},k)}(x)$. In §2, we study the weak convergence of the df $\Phi_{r:n}^{(\tilde{m},k)}(x)$, as $\gamma_{1,n} \to \infty$. Section 3 is concerned with the weak convergence of the df $\Phi_{r:n}^{(\tilde{m},k)}(x)$, as $\gamma_{1,n} \to \gamma_1 > 0$ and some illustrative examples.

2. Main result

Throughout this section we assume that there exist normalizing constants $\alpha_n > 0$ and β_n for which

$$\begin{aligned} \Phi_{1:n}^{(\tilde{0},1)}(\alpha_n x + \beta_n) &= 1 - \bar{F}^n(\alpha_n x + \beta_n) \xrightarrow[n]{w} \Phi_1^{(\tilde{0},1)}(x) \\ &= \Gamma_1(U_{t,\beta}(x)) = 1 - \mathrm{e}^{-U_{t,\beta}(x)}, \quad t \in \{1,2,3\}, \end{aligned} \tag{2.1}$$

where the functions $U_{t,\beta}(x), i = 1,2,3$ are defined in Theorem 1. It is known that the necessary and sufficient condition for (2.1) holds if

$$n\,F(\alpha_n x + \beta_n) \longrightarrow U_{t,\beta}(x), \quad t \in \{1,2,3\}, \tag{2.2}$$

which implies that for large n, $F(\alpha_n x + \beta_n) \sim 0$ for all x for which $\Phi_1^{(\tilde{0},1)}(x) < 1$. On the other hand, on account of (1.1), we get $\Phi_{1:n}^{(\tilde{m},k)}(x) = 1 - \bar{F}^{\gamma_{1,n}}(x)$. Therefore

$$\begin{aligned} \Phi_{1:n}^{(\tilde{m},k)}(a_n x + b_n) &= \Phi_{1:n}^{(\tilde{m},k)}(\alpha_{\gamma_{1,n}} x + \beta_{\gamma_{1,n}}) \\ &= 1 - \bar{F}^{\gamma_{1,n}}(\alpha_{\gamma_{1,n}} x + \beta_{\gamma_{1,n}}) \xrightarrow[n]{w} \Phi_1^{(\tilde{m},k)}(x) \\ &= \Gamma_1(U_{t,\beta}(x)), \qquad t \in \{1,2,3\}, \end{aligned} \tag{2.3}$$

if relation (2.2) (as well as relation (2.1)) is satisfied. The following theorem gives the possible limits of the r-th gOs.

Theorem 2.1. *Under the condition* (2.1) (*or equivalently the condition* (2.2)) *and for all* $\tilde{m} \in \mathbb{R}^{n-1}$ *such that* $m'_t > 0,\ t = 1, \ldots, n-1,$ *we get*

$$\Phi_{r:n}^{(\tilde{m},k)}(a_n x + b_n) \xrightarrow[n]{w} \Phi_r^{(\tilde{m},k)}(x) = \Gamma_r(U_{t,\beta}(x))$$
$$= 1 - \sum_{i=0}^{r-1} \frac{U_{t,\beta}^i(x)}{i!} \mathrm{e}^{-U_{t,\beta}(x)}, \quad t \in \{1, 2, 3\}, \tag{2.4}$$

where $a_n = \alpha_{\gamma_{1,n}} > 0$ *and* $b_n = \beta_{\gamma_{1,n}}$.

Proof. Relation (2.3) shows that the theorem is true when $r = 1$. Moreover, when $r = 2$, in view of Remark 1.2(ii) and the relation $F(a_n x + b_n) \to 0$ as $n \to \infty$, Lemma 1.2 yields $\Phi_{2:n}^{(\tilde{m},k)}(a_n x + b_n) = 1 - \bar{F}^{\gamma_{1,n}}(a_n x + b_n) L_{2:n}^{(\tilde{m},k)}(a_n x + b_n)$, where

$$\begin{aligned}
L_{2:n}^{(\tilde{m},k)}(a_n x + b_n) &= \frac{\gamma_{2,n}}{\gamma_{2,n} - \gamma_{1,n}} - \frac{\gamma_{1,n}}{\gamma_{1,n} - \gamma_{2,n}} \bar{F}^{-m'_1}(a_n x + b_n) \\
&= \frac{-(\gamma_{1,n} - m'_1) + \gamma_{1,n} \bar{F}^{-m'_1}(a_n x + b_n)}{m'_1} \\
&= \frac{\gamma_{1,n}(1 + m'_1 F(a_n x + b_n)(1 + \circ(1))) - (\gamma_{1,n} - m'_1)}{m'_1} \\
&= \frac{m'_1 + \gamma_{1,n} m'_1 F(a_n x + b_n)(1 + \circ(1))}{m'_1} \\
&= 1 + \gamma_{1,n} F(a_n x + b_n)(1 + \circ(1)) \to 1 + U_{t,\beta}(x).
\end{aligned}$$

Therefore,

$$\Phi_{2:n}^{(\tilde{m},k)}(a_n x + b_n) \xrightarrow[n]{w} 1 - (1 + U_{t,\beta}(x))\mathrm{e}^{-U_{t,\beta}(x)}$$
$$= \Gamma_2(U_{t,\beta}(x)) = 1 - \sum_{i=0}^{1} \frac{U_{t,\beta}^i(x)}{i!} \mathrm{e}^{-U_{t,\beta}(x)},$$

which means that Theorem 2.1 is also true when $r = 2$.

Now, in order to prove the theorem for all r, we proceed as follows: First, in view of Lemma 1, we note that $L_{r:n}^{(\tilde{m},k)}(a_n x + b_n)$ is a polynomial of $\gamma_{1,n}$ with degree $r - 1$. On the other hand, by expanding $\bar{F}^{-(\gamma_{1,n} - \gamma_{i,n})}(a_n x + b_n) = \bar{F}^{-\sum_{t=1}^{i-1} m'_t}(a_n x + b_n)$ by binomial expansion, $L_{r:n}^{(\tilde{m},k)}$ may be regarded as a polynomial of $\gamma_{1,n}$ with degree $r-1$, as well as an infinite polynomial of $F(a_n x + b_n)$. Since, under the condition of theorem $\gamma_{1,n} F(a_n x + b_n) \to U_{t,\beta}(x)$ and $F(a_n x + b_n) \to 0$ as $n \to \infty$, we can easily see that for large n, all terms of $L_{r:n}^{(\tilde{m},k)}$ as a polynomial of $F(a_n x + b_n)$, starting from $F^r(a_n x + b_n)$ will be vanished. Therefore, in view of (1.7), we have the following representation as $n \to \infty$:

$$\begin{aligned}\Phi_{r:n}^{(\tilde{m},k)}&(a_n x + b_n)\\ &= 1 - \bar{F}^{\gamma_{1,n}}(a_n x + b_n)(A_{0,r} + A_{1,r}F(a_n x + b_n)\\ &\qquad + \cdots + A_{r-1,r}F^{r-1}(a_n x + b_n))(1 + \circ(1)),\end{aligned}$$

where

$$A_{\ell,r} = \frac{1}{\ell!}\sum_{i=1}^{r}\left[\left(\prod_{j=1}^{r}{}^{\langle i\rangle}\frac{\gamma_{j,n}}{\gamma_{j,n} - \gamma_{i,n}}\right)\left(\prod_{j=1}^{\ell}(\gamma_{1,n} - \gamma_{i,n} + j - 1)\right)\right]. \tag{2.5}$$

Moreover, in view of Remark 1.3, we can easily see that $A_{0,r} = 1$ for all r. Our task now is to compute $A_{\ell,r}$ for all $1 \leq \ell \leq r - 1$. To achieve this we first derive a recurrence relation which is satisfied by $A_{\ell,r}$ for all $1 \leq \ell < r - 1, r \geq 3$. In view of (2.5), we get

$$\begin{aligned}(\ell+1)!\,A_{\ell+1,r} &= \sum_{i=1}^{r}\left[\left(\prod_{j=1}^{r}{}^{\langle i\rangle}\frac{\gamma_{j,n}}{\gamma_{j,n} - \gamma_{i,n}}\right)\right.\\ &\qquad \times\left.\left(\prod_{j=2}^{\ell+1}(\gamma_{1,n} - \gamma_{i,n} + j - 1)\right)(\gamma_{1,n} - \gamma_{i,n})\right]\\ &= \sum_{i=2}^{r}\left[\left(\prod_{j=1}^{r}{}^{\langle i\rangle}\frac{\gamma_{j,n}}{\gamma_{j,n} - \gamma_{i,n}}\right)\right.\\ &\qquad \times\left.\left(\prod_{j=1}^{\ell}(\gamma_{1,n} - \gamma_{i,n} + j - 1)\right)(\gamma_{1,n} - \gamma_{i,n} + \ell)\right]\\ &= (\gamma_{1,n} + \ell)\sum_{i=2}^{r}\left[\left(\prod_{j=1}^{r}{}^{\langle i\rangle}\frac{\gamma_{j,n}}{\gamma_{j,n} - \gamma_{i,n}}\right)\right.\\ &\qquad \times\left.\left(\prod_{j=1}^{\ell}(\gamma_{1,n} - \gamma_{i,n} + j - 1)\right)\right]\\ &\quad - \sum_{i=2}^{r}\left[\gamma_{i,n}\left(\prod_{j=1}^{r}{}^{\langle i\rangle}\frac{\gamma_{j,n}}{\gamma_{j,n} - \gamma_{i,n}}\right)\right.\\ &\qquad \times\left.\left(\prod_{j=1}^{\ell}(\gamma_{1,n} - \gamma_{i,n} + j - 1)\right)\right].\end{aligned} \tag{2.6}$$

On the other hand, we can show that

$$\begin{aligned}
\ell!\, A_{\ell,r} &= \sum_{i=1}^{r}\left[\left(\prod_{j=1}^{r}{}^{\langle i\rangle}\frac{\gamma_{j,n}}{\gamma_{j,n}-\gamma_{i,n}}\right)\left(\prod_{j=1}^{\ell}(\gamma_{1,n}-\gamma_{i,n}+j-1)\right)\right]\\
&= \sum_{i=1}^{r}\left[\left(\prod_{j=1}^{r}{}^{\langle i\rangle}\frac{\gamma_{j,n}}{\gamma_{j,n}-\gamma_{i,n}}\right)\left(\prod_{j=2}^{\ell}(\gamma_{1,n}-\gamma_{i,n}+j-1)\right)(\gamma_{1,n}-\gamma_{i,n})\right]\\
&= \sum_{i=2}^{r}\left[\left(\prod_{j=1}^{r}{}^{\langle i\rangle}\frac{\gamma_{j,n}}{\gamma_{j,n}-\gamma_{i,n}}\right)\left(\prod_{j=1}^{\ell}(\gamma_{1,n}-\gamma_{i,n}+j-1)\right)\right]. \qquad (2.7)
\end{aligned}$$

Combining (2.6) and (2.7) we get, for all $1 \leq \ell < r-1,\ r \geq 3$

$$(\ell+1)!A_{\ell+1,r} = \ell!(\gamma_{1,n}+\ell)A_{\ell,r} - \Psi_n(\ell,r),$$

where

$$\begin{aligned}
\Psi_n(\ell,r) &= \sum_{i=2}^{r}\left[\gamma_{i,n}\left(\prod_{j=1}^{r}{}^{\langle i\rangle}\frac{\gamma_{j,n}}{\gamma_{j,n}-\gamma_{i,n}}\right)\left(\prod_{j=1}^{\ell}(\gamma_{1,n}-\gamma_{i,n}+j-1)\right)\right]\\
&= C_{r-1}\sum_{i=2}^{r}\frac{\prod_{j=2}^{\ell}(\gamma_{1,n}-\gamma_{i,n}+j-1)}{\prod_{j=2}^{r}{}^{\langle i\rangle}(\gamma_{j,n}-\gamma_{i,n})}\\
&= C_{r-1}\sum_{i=2}^{r}\left[a_i^{[2]}(r)\prod_{j=2}^{\ell}(\gamma_{1,n}-\gamma_{i,n}+j-1)\right]. \qquad (2.8)
\end{aligned}$$

If one shows that $\Psi_n(\ell,r) = 0$ for all $1 \leq \ell < r-1,\ r \geq 3$, and $A_{1,r} = \gamma_{1,n}$, then by induction over r we can easily see that

$$A_{\ell,r} = \frac{\gamma_{1,n}(\gamma_{1,n}+1)\cdots(\gamma_{1,n}+\ell-1)}{\ell!},$$

which in turn, in view of our conditions $\gamma_{1,n}F(a_nx+b_n) \to U_{t,\beta}(x)$ and $F(a_nx+b_n) \to 0$ as $n \to \infty$, implies that the relation (2.4) and consequently Theorem 2 will be proved (the Mathematica version 6.0 is used to check that $\Psi_{r,\ell} = 0$ for $r = 3, 4, \ldots, 12$ and for all values of $\ell < r$. Although, this will be satisfactory for many practical purposes, it will be better to prove this fact theoretically).

Now, in view of (2.5), we get

$$A_{1,r} = \sum_{i=1}^{r}\left[\left(\prod_{j=1}^{r}{}^{\langle i\rangle}\frac{\gamma_{j,n}}{\gamma_{j,n}-\gamma_{i,n}}\right)(\gamma_{1,n}-\gamma_{i,n})\right]$$

$$= \gamma_{1,n} \sum_{i=2}^{r} \left[\prod_{j=2}^{r \ \langle i \rangle} \frac{\gamma_{j,n}}{\gamma_{j,n} - \gamma_{i,n}} \right]$$

$$= \gamma_{1,n} \sum_{i'=1}^{r-1} \left[\prod_{j'=1}^{r-1 \ \langle i' \rangle} \frac{\gamma_{j'+1,n}}{\gamma_{j'+1,n} - \gamma_{i'+1,n}} \right] = \gamma_{1,n} \sum_{i=1}^{r-1} \left[\prod_{j=1}^{r-1 \ \langle i \rangle} \frac{\gamma^{\star}_{j,n}}{\gamma^{\star}_{j,n} - \gamma^{\star}_{i,n}} \right],$$

where $\gamma^{\star}_{j,n} = \gamma_{j+1,n}$, $j = 1, 2, \ldots, r-1$. In view of the obvious relation $\gamma^{\star}_{1,n} > \cdots > \gamma^{\star}_{r-1,n}$, we can apply Remarks 1.3 with $r-1$ and $\gamma^{\star}_{j,n}$ instead of r and $\gamma_{j,n}$ respectively, to get

$$A_{1,r} = \gamma_{1,n} \sum_{i=1}^{r-1} \left[\prod_{j=1}^{r-1 \ \langle i \rangle} \frac{\gamma^{\star}_{j,n}}{\gamma^{\star}_{j,n} - \gamma^{\star}_{i,n}} \right] = \gamma_{1,n}.$$

Now, let $\bar{\Psi}_n(\ell, r) = \frac{\Psi_n(\ell, r)}{C_{r-1}}$. Therefore, by using (2.8) we get

$$\bar{\Psi}_n(1, r) = \sum_{i=2}^{r} a_i^{[2]}(r) = \sum_{i=2}^{r-1} a_i^{[2]}(r) + a_r^{[2]}(r). \tag{2.9}$$

Since, in the first summation on the RHS of (2.9), $i \leq r-1$, we get

$$a_i^{[2]}(r) = \frac{a_i^{[2]}(r-1)}{\gamma_{r,n} - \gamma_{i,n}}.$$

Thus

$$\bar{\Psi}_n(1, r) = \sum_{i=2}^{r-1} \frac{a_i^{[2]}(r-1)}{\gamma_{r,n} - \gamma_{i,n}} + a_r^{[2]}(r).$$

An application of Lemma 1, with $\theta = \gamma_{r,n}$ thus yields

$$\begin{aligned} \bar{\Psi}_n(1, r) &= \prod_{i=2}^{r-1} \frac{(-1)^{(r-1)-2}}{\gamma_{r,n} - \gamma_{i,n}} + a_r^{[2]}(r) = -\prod_{i=2}^{r-1} \frac{1}{\gamma_{i,n} - \gamma_{r,n}} \\ &= -\prod_{i=2}^{r \ \langle r \rangle} \frac{1}{\gamma_{i,n} - \gamma_{r,n}} + a_r^{[2]}(r) = -a_r^{[2]}(r) + a_r^{[2]}(r) = 0. \end{aligned}$$

Therefore, for all $3 \leq r < n$ and $\gamma_{1,n} > \gamma_{2,n} > \cdots > \gamma_{r,n}$, we get $\Psi_n(1, r) = 0$. We are now going to prove $\bar{\Psi}_n(\ell, r) = 0$ (and consequently $\Psi_n(\ell, r) = 0$). This will be done by induction over ℓ. Assume, for all $3 \leq r < n$ and $\gamma_{1,n} > \gamma_{2,n} > \cdots > \gamma_{r,n}$, we have

$$\bar{\Psi}_n(\xi, r) = 0 \quad \text{for all } \xi < r-2. \tag{2.10}$$

Then, by using (2.8) and the assumption of the induction (2.10), we get

$$
\begin{aligned}
\bar{\Psi}_n(\xi+1,r) &= \sum_{i=2}^{r}\left[a_i^{[2]}(r)\prod_{j=2}^{\xi+1}(\gamma_{1,n}-\gamma_{i,n}+j-1)\right] \\
&= \sum_{i=2}^{r}\left[a_i^{[2]}(r)(\gamma_{1,n}-\gamma_{i,n}+\xi)\prod_{j=2}^{\xi}(\gamma_{1,n}-\gamma_{i,n}+j-1)\right] \\
&= -\sum_{i=2}^{r}\left[a_i^{[2]}(r)\gamma_{i,n}\prod_{j=2}^{\xi}(\gamma_{1,n}-\gamma_{i,n}+j-1)\right] \\
&= \sum_{i=2}^{r}\left[a_i^{[2]}(r)(\gamma_{2,n}-\gamma_{i,n})\prod_{j=2}^{\xi}(\gamma_{1,n}-\gamma_{i,n}+j-1)\right] \\
&\quad -\gamma_{2,n}\bar{\Psi}_n(\xi,r) \\
&= \sum_{i=3}^{r}\left[a_i^{[2]}(r)(\gamma_{2,n}-\gamma_{i,n})\prod_{j=2}^{\xi}(\gamma_{1,n}-\gamma_{i,n}+j-1)\right].
\end{aligned}
\tag{2.11}
$$

Since, in the summation of the RHS of (2.11), $i \geq 3$, we get

$$
a_i^{[2]}(r)=\prod_{j=2}^{r}{}^{\langle i\rangle}\frac{1}{\gamma_{j,n}-\gamma_{i,n}}=\frac{1}{\gamma_{2,n}-\gamma_{i,n}}\prod_{j=3}^{r}{}^{\langle i\rangle}\frac{1}{\gamma_{j,n}-\gamma_{i,n}}=\frac{a_i^{[3]}(r)}{\gamma_{2,n}-\gamma_{i,n}}.
$$

Thus, we get

$$
\begin{aligned}
\bar{\Psi}_n(\xi+1,r) &= \sum_{i=3}^{r}\left[a_i^{[3]}(r)\prod_{j=2}^{\xi}(\gamma_{1,n}-\gamma_{i,n}+j-1)\right] \\
&= \sum_{i'=2}^{r-1}\left[a_{i'+1}^{[3]}(r)\prod_{j=2}^{\xi}(\gamma_{1,n}-\gamma_{i'+1,n}+j-1)\right].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
a_{i'+1}^{[3]}(r) &= \prod_{j=3}^{r}{}^{\langle i'+1\rangle}\frac{1}{\gamma_{j,n}-\gamma_{i'+1,n}}=\prod_{j'=2}^{r-1}{}^{\langle i'\rangle}\frac{1}{\gamma_{j'+1,n}-\gamma_{i'+1,n}} \\
&= \prod_{j'=2}^{r-1}{}^{\langle i'\rangle}\frac{1}{\gamma^{\star}_{j',n}-\gamma^{\star}_{i',n}}=a_{i'}^{\star[2]}(r-1).
\end{aligned}
$$

Therefore,

$$\begin{aligned}\bar{\Psi}_n(\xi+1,r) &= \sum_{i=2}^{r-1}\left[a_i^{\star[2]}(r-1)\prod_{j=2}^{\xi}(\gamma_{1,n}-\gamma^{\star}_{i,n}+j-1)\right]\\ &= \sum_{i=2}^{r-1}\left[a_i^{\star[2]}(r-1)\prod_{j=2}^{\xi}(\gamma_{1,n-1}-\gamma^{\star}_{i,n}+j-1)\right].\end{aligned}$$

Since the assumption of the induction holds for all $3 \le r < n$ and $\gamma_{1,n} > \cdots > \gamma_{r,n}$, we can apply it with $r-1$ and $\gamma_{1,n}, \gamma^{\star}_{2,n}, \dots, \gamma^{\star}_{r-1,n}$ instead of r and $\gamma_{1,n}, \dots, \gamma_{r,n}$ respectively (note that $\gamma_{1,n} > \gamma^{\star}_{2,n} > \dots > \gamma^{\star}_{r-1,n}$) to get $\bar{\Psi}_n(\xi+1,r) = 0$. This completes the proof of the theorem. □

Remark 2.1. Under the condition that $k = 1,\ m'_t = m+1,\ t = 1,\dots,n-1$. Since $\gamma_{1,n-\frac{k}{m+1}} \to \infty$ if $\gamma_{1,n} \to \infty$ and $F(a_n x + b_n) \sim 0$ as $n \to \infty$, we can see that (with $\phi(n) = (m+1)n$)

$$\begin{aligned}\Phi^{(\tilde{m},k)}_{r:n}(\alpha_{\phi(n)}x+\beta_{\phi(n)}) &= \Phi^{(\tilde{m},k)}_{r:n}\left(a_{n-\frac{k}{m+1}}x+b_{n-\frac{k}{m+1}}\right)\\ &\sim \Phi^{(\tilde{m},k)}_{r:n-\frac{k}{m+1}}\left(a_{n-\frac{k}{m+1}}x+b_{n-\frac{k}{m+1}}\right)\\ &\xrightarrow[n]{w} \Gamma_r(U_{t,\beta}(x)) = 1\\ &-\sum_{i=1}^{r-1}\frac{U^i_{t,\beta}(x)}{i!}\,\mathrm{e}^{-U_{t,\beta}(x)}, \quad t \in \{1,2,3\}.\end{aligned}$$

Therefore, Theorem 1 is a simple consequence of Theorem 2 in this case.

3. Further limit theorem and illustrative examples

In this section we drop all the previous conditions and consider only the condition $\gamma_{1,n} \to \gamma_1 > 0$ as $n \to \infty$. The next theorem gives the asymptotic behaviour of the r-th gOs under this condition. It is worth to mention that most of the known models, eg., oOs, pOs are excluded from this situation.

Theorem 3.1. *Under the condition $\gamma_{1,n} \to \gamma_1 > 0$, as $n \to \infty$, and for all $\tilde{m} \in \mathbb{R}^{n-1}$ such that $m'_t > 0,\ t = 1,\cdots,n-1$, we get*

$$\Phi^{(\tilde{m},k)}_{r:n}(a_n x+b_n) \xrightarrow[n]{w} \Phi^{\star(\tilde{m},k)}_r(x) = 1-\sum_{i=1}^{r}\left(\prod_{j=1}^{r}{}^{\langle i\rangle}\frac{\gamma_j}{\gamma_j-\gamma_i}\right)\bar{F}^{\gamma_i}(ax+b), \tag{3.1}$$

where $\gamma_{j,n} \to \gamma_j,\ j = 1,2,\dots,r,\ a_n = a > 0$ and $b_n = b$.

Proof. In view of Remark 1.2(ii) and the two relations $\gamma_{1,n} \to \gamma_1 > 0$ as $n \to \infty$, and $\gamma_{j,n+1} \geq \gamma_{j,n}$ for all n, the parameter $\gamma_{j,n}$ converges to a positive real number γ_j for all $j = 1, \ldots, r$. Therefore, the proof is immediately followed by using (1.7). $\square$

Example 3.1. Let $m'_t = (\frac{1}{2})^t$, $t = 1, \ldots, n-1$ and $k = 0$. Then, $\gamma_{j,n} \to (\frac{1}{2})^{j-1}$, $j = 1, \ldots, r$. In view of Theorem 3, we get

$$\Phi_{r:n}^{(\tilde{m},k)}(x) \xrightarrow[n]{w} \Phi_r^{\star(\tilde{m},k)}(x) = 1 - \sum_{i=1}^{r} \left(\prod_{j=1}^{r \ \langle i \rangle} \frac{1}{1 - 2^{j-1}} \right) \bar{F}^{(2^{1-i})}(ax + b).$$

Example 3.2. (*pOs with general censoring scheme* $(R_1, \ldots, R_M)$). Let $X_1, \ldots, X_n$ be independent lifetimes of n identical units, with X_i having the df F. These units are placed on test at time $t = 0$. At the time of the r-th failure, R_r, $1 \leq r \leq M$, number of surviving units are randomly withdrawn from the experiment. Thus, if M failures are observed, then $R_1 + \cdots + R_M$ number of units are progressively censored and in this case we get $n = M + R_1 + \cdots + R_M$. The r-th failure time $X_{r:n}^{(\tilde{R})}$, where $\tilde{R} = (R_1, \ldots, R_M)$ is called the r-th progressive Type II censored order statistic (e.g., see [2]). Clearly, $X_{r:n}^{(\tilde{R})} = X(r, n, \tilde{m}, k)$, where $k = R_M + 1$, $m_i = R_i$, $i = 1, \ldots, M-1$ and $m_i = 0$, $i = M, \ldots, n-1$. Therefore, $\gamma_{1,n} = n$. Thus, in view of Theorem 2.1, we can conclude that in this important model, the extreme value theory coincides with the classical extreme value theory of oOs, regardless of the value of M. If the failure times are from a continuous population with a df F, it is readily checked that F belongs to the Type I domain, when F is a standard normal. The exponential and log-normal distributions also have Type I limits. The Pareto, Cauchy distributions give Type II limits, whereas the uniform distribution belongs to the Type III domain. For statistical modeling purpose, we first write the full limiting df's model defined in (2.4), by adding location and scale parameters μ and $\sigma > 0$, namely, $\Gamma_r(U_{t,\beta}(\frac{x-\mu}{\sigma}))$, $t = 1, 2, 3$. Then, by taking the reparametrization $\gamma = \frac{1}{\beta}$— due to von Mises (see [13]) one obtains a continuous, unified model

$$\Gamma_r(\tilde{U}_\gamma(x)) = 1 - \sum_{i=0}^{r-1} \frac{\tilde{U}_\gamma^i(x)}{i!} \exp(-\tilde{U}_\gamma(x)), \tag{3.2}$$

where

$$\tilde{U}_\gamma(x) = \exp\left(- \left(1 - \gamma \left(\frac{x - \mu}{\sigma} \right)^{-\frac{1}{\gamma}} \right) \right), \qquad 1 - \gamma \left(\frac{x - \mu}{\sigma} \right) > 0.$$

Apart from a change of origin (the location parameter μ) and a change in the unit on the x-axis (the scale parameter $\sigma > 0$) the df (3.2) yields the three limit types defined in (2.4), according as $\gamma = 0$ ($\gamma \to 0$), $\gamma > 0$ and $\gamma < 0$. In this case, any suitable standard statistical methodology from parametric estimation theory can be utilized in order to derive estimate of the parameters μ, σ and γ. Theorem 2.1 enables us to use the traditional method of analyzing extreme values, which is known as the block method. For example, consider a general Type-II censoring scheme $(R_1, \ldots, R_M)$, in which n randomly selected units were placed on a life test. Suppose, we need to fit the limiting df of the r-th failure time $X_{r:n}^{(\tilde{R})}$. Then we have to repeat this test a suitable number of times, (say K), which is the number of blocks. Since, Theorem 2.1 states that the df (3.2) is the only one which can appear as the limit of linearly normalized $X_{r:n}^{(\tilde{R})}$, we have, in this case, K observed failure times which can be used as a sample of size K from the limiting df (3.2) to get the estimate of the parameters μ, σ and γ. It is worth mentioning that in the test based on oOs we have to wait till nK failures, while in the test based on pOs we have only to wait till rK failures, if we choose $M = r$ (note that M is any integer, for which $r \leq M < n$).

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References

[1] Arnold B C, Balakrishnan N and Nagaraja H N (New York: Records, Wiley) (1998)
[2] Balakrishnan N and Aggarwala R, Progressive censoring. Theory, methods and applications, Statistics for Industry and Technology (Boston: Birkhauser Boston Inc.) (2000)
[3] Barakat H M, Limit theory of generalized order statistics, *J. Statist. Plann. Inference* **137(1)** (2007) 1–11
[4] Christoph G and Falk M, A note on domains of attraction of p-max stable laws, *Stat. Probab. Lett.* **28** (1996) 279–284
[5] Cramer E, Contributions to generalized order statistics, Habililationsschrift, Reprint (University of Oldenburg) (2003)
[6] Kamps U, A Concept of Generalized Order Statistics (Stuttgart: Teubner) (1995)
[7] Kamps U and Cramer E, On distribution of generalized order statistics, *Statistics* **35** (2001) 269–280
[8] Marohn F, On rates of uniform convergence of lower extreme generalized order statistics, *Extremes* **7** (2004) 271–282
[9] Nasri-Roudsari D, Extreme value theory of generalized order statistics, *J. Statist. Plann. Inference* **55** (1996) 281–297
[10] Nasri-Roudsari D, Limit distributions of generalized order statistics under power normalization, *Comm. Statist. Theory Methods* **28** (1999) 1379–1389
[11] Resnick S I, Limit laws for record values, *Stochastic Process. Appl.* **1** (1973a) 67–82
[12] Resnick S I, Record values and maxima, *Ann. Probab.* **1** (1973b) 650–662
[13] Reiss R D and Thomas M, Statistical analysis of extreme values from insurance, finance, hydrology and other fields (Berlin: Birkhäuser Verlag) (2003)