Proc. Indian Acad. Sci. (Math. Sci.) Vol. 122, No. 2, May 2012, pp. 283–295.


# Asymptotic behavior of stochastic two-dimensional Navier–Stokes equations with delays

HUABIN CHEN

Department of Mathematics, School of Science, Nanchang University,
Nanchang 330031, People's Republic of China
Emai: chb_00721@126.com

MS received 14 January 2011; revised 14 November 2011

**Abstract.** The paper proves the $L^2$-exponential stability of weak solutions of two-dimensional stochastic Navier–Stokes equations in the presence of delays. The results extend some of the existing results.

**Keywords.** Exponential stability; weak solution; almost surely exponential stability; stochastic Navier–Stokes equations; delays.

## 1. Introduction

The long-time behavior of flows has been regarded as an interesting and important problem in the theory of fluid dynamics, and has been receiving much attention for many years; see, for example [1–3, 5–8, 13] and the references therein. One of the most studied models is the Navier–Stokes model (and its variants) since it can provide a suitable model which covers several important fluids (see [12, 13] and the references therein).

However, most of the studies above are in connection with the deterministic case and no heredity. As a general rule, another interesting question is to analyze the effects produced on a deterministic system by stochastic or random disturbances and heredity. In this paper, we consider the following stochastic incompressible two-dimensional Navier–Stokes equations with delays:

$$\begin{cases} \mathrm{d}u = [\nu\Delta u - (u,\nabla)u - \nabla p + f(t) + G(u(t-\rho(t)))]\mathrm{d}t \\ \qquad +\Lambda(u(t-\rho(t)))\mathrm{d}w(t),\ (0,+\infty)\times D, \\ \operatorname{div} u = 0,\quad (0,+\infty)\times D, \\ u(t,x) = 0,\quad (t,x)\in(0,+\infty)\times\partial D, \\ u(0) = u_0(x),\quad x\in D, \\ u(t,x) = \varphi(t,x),\quad (t,x)\in(-\tau,0)\times D, \end{cases}$$

where $D$ is a regular open bounded domain of $R^2$ with boundary $\Gamma$, $u$ is the velocity field of the fluid, $p$ the pressure, $\nu > 0$ the kinematic viscosity, $\varphi$ the initial velocity field, $f$ the external force field without delay, $G(u(t-\rho(t)))$ the external force field with delay, and $\Lambda(u(t-\rho(t)))\mathrm{d}w(t)$ the random force field with delay where $w(t)$ is an infinite dimensional Wiener process.

Recently, stochastic Navier–Stokes equations have been studied by some authors and many valuable results on the existence, uniqueness and the asymptotic behavior of the weak solution and the strong solution for such equations have been established. For example, Caraballo *et al* in [4], have discussed the exponential behavior and stabilizability of stochastic 2D Navier–Stokes equations. The existence and uniqueness of solutions to the backward 2D stochastic Navier–Stokes equations was obtained in [10]. In [6], Caraballo and Real have considered the asymptotic behavior of two-dimensional Navier–Stokes equations with delays with the help of the Lyapunov function and the Razuminkhin theorem, respectively. Wei and Zhang, in [14], have investigated the exponential stability and almost surely exponential stability of the weak solution for stochastic 2D Navier–Stokes equations with variable delays by using the approach proposed in [6]. However, although the desired results can be given in [14], the differentiability of variable delays must be imposed. So, in this paper, in order to remove this restrictive condition about variable delays, we proceed to study the exponential stability and almost surely exponential stability of the weak solution for stochastic two dimensional Navier–Stokes equations with delays by establishing an integral inequality.

The format of this work is organized as follows. In §2, some necessary definitions, notations and lemmas used in this paper will be introduced. By establishing *an integral inequality*, some sufficient conditions about the exponential stability in mean square and almost surely exponential stability for the weak solution of stochastic two dimensional Navier–Stokes equations with delays are given in §3.

## 2. Preliminaries

Let $H$ be the closure of the set $\{u \in C_0^\infty(D, R^2) : \text{div } u = 0\}$ in the space $L^2(D, R^2)$ with the norm $|u| = (u, u)^{\frac{1}{2}}$, where for any $u,\ v \in L^2(D, R^2)$,

$$(u, v) = \sum_{j=1}^{2} \int_D u^j(x)v^j(x)\mathrm{d}x.$$

Let $V$ denote the closure of the set $\{u \in C_0^\infty(D, R^2) : \text{div } u = 0\}$ in the space $H_0^1(D, R^2)$ with the norm $\|u\| = ((u, v))^{\frac{1}{2}}$, where for any $u,\ v \in H_0^1(D, R^2)$,

$$((u, v)) = \sum_{i,j=1}^{2} \int_D \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\mathrm{d}x.$$

Thus, it is obviously seen that $H$ and $V$ are two separable Hilbert spaces with associated inner $(\cdot, \cdot)$ and $((\cdot, \cdot))$, and the following expression is satisfied, i.e.

$$V \subset H \equiv H' \subset V',$$

where injections are dense, continuous and compact ($H'$ and $V'$ are the dual spaces of $H$ and $V$, respectively). $\lambda_1$ denotes the first eigenvalue of $A$, and we remark that $\|v\|^2 \geq \lambda_1|v|^2,\ \forall v \in V$. Now, we can set $A = -P\Delta$, where $P$ is the orthogonal projector from $L^2(D, R^2)$ onto $H$, and define the trilinear form $b$ by

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_D u^i(x)\frac{\partial v^j(x)}{\partial x_i}w^j(x)\mathrm{d}x.$$

As we shall need some properties on this trilinear form $b$, we list here the ones we will use later on (see [13]),

$$\begin{aligned}
&|b(u, v, w)| \leq c_1 |u|^{\frac{1}{2}} \|u\|^{\frac{1}{2}} \|v\| |w|^{\frac{1}{2}} \|w\|^{\frac{1}{2}}, \quad \forall u, v, w \in V,\\
&b(u, v, v) = 0, \quad \forall u, v \in V,\\
&b(u, u, v-u) - b(v, v, v-u) = -b(v-u, u, v-u), \qquad \forall u, v \in V,
\end{aligned} \tag{2.1}$$

where $c_1 > 0$ is an appropriate constant which depends on the regular open domain $D$ (see p. 50 of [7]). Furthermore, we can define the operator $B: \ V \times V \to V'$ by

$$\langle B(u, v), w \rangle = b(u, v, w), \qquad \forall u, v, w \in V,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality $\langle V', V \rangle$. And, we also set

$$B(u) = B(u, u), \qquad \forall u \in V.$$

Let $(\Omega, \Im, P)$ be a complete probability space equipped with some filtration $\Im_t$ $(t \geq 0)$ satisfying the usual conditions, i.e., the filtration is right continuous and $\Im_0$ contains all $P$-null sets. And let $\beta_n(t)$ $(n = 1, 2, \ldots)$ denote a sequence of real valued one-dimensional standard Brownian motions mutually independent on $(\Omega, \Im, P)$. Setting

$$w(t) = \sum_{n=1}^{+\infty} \sqrt{\lambda'_n} \beta_n(t) e_n, \qquad t \geq 0,$$

where $\lambda'_n \geq 0$ $(n = 1, 2, \ldots)$ are some nonnegative real numbers such that $\sum_{n=1}^{+\infty} \lambda'_n < +\infty$, and $\{e_n\}$ $(n = 1, 2, \ldots)$ is a complete orthonormal basis in a real and separable Hilbert space $K$. Let $Q \in L(K, K)$ be the operator defined by $Qe_n = \lambda'_n e_n$ $(n = 1, 2, \ldots)$. The above $K$-valued stochastic process $w(t)$ is called a $Q$-Wiener process. $L(K, H)$ denotes the space of bounded linear operators from $K$ to $H$.

DEFINITION 2.1 [9]

Let $\sigma \in L(K, H)$ and define

$$\|\sigma\|^2_{L_2^0} := \mathrm{tr}(\sigma Q \sigma^*) = \left\{ \sum_{n=1}^{+\infty} \|\sqrt{\lambda'_n} \sigma e_n\|^2 \right\}.$$

If $\|\sigma\|^2_{L_2^0} < +\infty$, then $\sigma$ is called a $Q$-Hilbert–Schmidt operator and let $L_2^0(K, H)$ denote the space of all $Q$-Hilbert–Schmidt operators $\sigma: \ K \to H$.

Now, for the definition of a $H$-valued stochastic integral of a $L_2^0(K, H)$-valued and $\Im_0$-adapted predictable process $\Phi(t)$ with respect to the $Q$-Wiener process $w(t)$, the readers can refer to [9].

Thus, the stochastic two-dimensional incompressible Navier–Stokes equations with delays can be written as follows:

$$\begin{cases}
\mathrm{d}u(t) = [-\nu A u(t) - B(u(t)) + f(t) + G(u(t-\rho(t)))]\mathrm{d}t\\
\qquad\qquad + \Lambda(u(t-\rho(t)))\mathrm{d}w(t), \quad t \geq 0,\\
u_0(\theta) = \varphi \in L^2(\Omega, C([-\tau, 0], H)), \quad \theta \in [-\tau, 0],
\end{cases} \tag{2.2}$$

where $L^2(\Omega, C([-\tau, 0], H))$ denotes the family of all almost surely bounded $\Im_t$ $(t \geq 0)$-measurable and $C([-\tau, 0], H)$-valued stochastic process and as usual, equipped with the supremum norm $\|\varphi\|_0 = E \sup_{\theta\in[-\tau,0]} |\varphi(\theta)|^2$; the function $\rho : [0, +\infty) \to [0, \tau]$ $(\tau > 0)$ is bounded and measurable; $f : [0, +\infty) \to V'$, $G : V \to V'$ and $\Lambda : V \to L(K, H)$ are three appropriate Borel measurable functions.

The corresponding deterministic system of system (2.2) can be represented as follows:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}u(t) = -\nu Au(t) - B(u(t)) + f(t) + G(u(t - \rho(t))), \quad t \geq 0, \\ u_0(\theta) = \varphi \in C([-\tau, 0], H), \quad \theta \in [-\tau, 0]. \end{cases} \tag{2.3}$$

DEFINITION 2.2 [9]

A stochastic process $u(t)$ $(t \geq -\tau)$ is said to be a weak solution of system (2.2) if

(i) $u(t)$ is $\Im_t$-adapted;
(ii) $u(t) \in L^\infty(-\tau, T; H) \cap L^2(-\tau, T; V)$ almost surely for all $T > 0$;
(iii) the following equation holds as an identity in $V'$ almost surely, for all $t \in [0, +\infty)$,

$$u(t) = u(0) + \int_0^t [-\nu Au(s) - B(u(s)) + f(s) + G(u(s - \rho(s)))]\mathrm{d}s + \int_0^t \Lambda(u(s - \rho(s)))\mathrm{d}w(s).$$

Let $C^{(1,2)}([0, +\infty) \times H, R^+)$ denote the space of all $R^+$-valued functions $\Phi$ defined on $[0, +\infty) \times H$ with the following assumptions:

(1) $\Phi(t, u)$ is differentiable in $t \in [0, +\infty)$ and twice Fréchet differentiable in $u$ with $\Phi_t(t, \cdot)$, $\Phi_u(t, \cdot)$ and $\Phi_{uu}(t, \cdot)$ locally bounded on $H$;
(2) $\Phi(t, \cdot)$, $\Phi_t(t, \cdot)$ and $\Phi_u(t, \cdot)$ are continuous on $H$;
(3) for all trace class operators $Z$, $\mathrm{tr}(\Phi_{uu}(t, \cdot)Z)$ is continuous from $H$ into $R$;
(4) if $v \in V$, then $\Phi_u(t, v) \in V$, and $x \to \langle \Phi_u(t, x), v^*\rangle$ is continuous for each $v^* \in V'$;
(5) $\|\Phi_u(t, x)\| \leq C_0(t)(1 + \|x\|)$, $C_0(t) > 0$, for all $x \in V$.

*Lemma* 2.3 (*Ito's formula*) [9]. *Let* $\Phi \in C^{(1,2)}([0, +\infty) \times H, R^+)$. *If the stochastic process* $u(t)$ *is a weak solution of system* (2.2), *then it holds that*

$$\begin{aligned} \Phi(t, u(t)) = {} & \Phi(0, u(0)) + \int_0^t L\Phi(s, u(s))\mathrm{d}s \\ & + \int_0^t (\Phi_u(s, u(s)), \Lambda(u(s - \rho(s)))\mathrm{d}w(s)), \end{aligned}$$

*where*

$$\begin{aligned} L\Phi(t, u(t)) = {} & \Phi_t(t, u(t)) + \langle -\nu Au(t) - B(u(t)) + f(t) \\ & + G(u(t - \rho(t))), \Phi_x(t, x(t))\rangle \\ & + \frac{1}{2}\mathrm{tr}(\Phi_{uu}(t, u(t))\Lambda(u(t - \rho(t)))Q\Lambda(u(t - \rho(t)))^*). \end{aligned}$$

DEFINITION 2.4 [4]

A weak solution $u(t)$ of system (2.2) converges to $u_\infty \in H$ exponentially stable in mean square if there exist two positive numbers $a > 0$ and $M_0 > 0$, such that

$$E|u(t) - u_\infty|^2 \le M_0 \mathrm{e}^{-at}, \qquad t \ge 0.$$

In particular, if $u_\infty$ is a stationary solution of system (2.2), then $u_\infty$ is called exponentially stable in the mean square provided that any weak solution to (2.2) converges *in* $L^2$ to $u_\infty$ at the same exponential rate $a > 0$.

DEFINITION 2.5 [4]

A weak solution $u(t)$ of system (2.2) converges to $u_\infty \in H$ almost surely exponentially stable if there exists $\gamma > 0$ such that

$$\lim_{t\to+\infty} \frac{1}{t} \log |u(t) - u_\infty| \le -\gamma, \quad \text{a.s.}$$

In particular, if $u_\infty$ is a stationary solution of system (2.2), then $u_\infty$ is called almost surely exponentially stable provided that any weak solution to (2.2) converges in $L^2$ to $u_\infty$ with the same constant $\gamma > 0$.

## 3. Main results

In order to discuss the exponential stability in mean square and almost surely exponential stability of the weak solution to system (2.2), we need the following assumptions:

($\mathrm{H}_1$) There exists a positive number $\beta_1 > 0$ such that

$$\|G(u) - G(v)\|_{V'} \le \beta_1 |u - v|,$$

for any $u,\ v \in H$ and $G(0) = 0$.

($\mathrm{H}_2$) There exists a positive number $\beta_2 > 0$ such that

$$\|\Lambda(u) - \Lambda(v)\|_{L_2^0} \le \beta_2 |u - v|,$$

for any $u,\ v \in H$ and $\Lambda(0) = 0$.

*Remark* 1. For $f \in L^2([0, +\infty), V')$ and $\nu > 0$, under the conditions ($\mathrm{H}_1$)–($\mathrm{H}_2$), we can show the existence and uniqueness of the weak solution to system (2.2). The proof is extremely similar to that provided in [12]. Here, we omit it.

Firstly, assume that $f$ is independent of $t$. We consider the existence of the stationary solution to the equation

$$\nu A u + B u = f + G(u). \tag{3.1}$$

*Lemma* 3.1 [6]. *Supposed that condition* ($\mathrm{H}_1$) *holds and* $f \in V'$. *Then*

(i) *if* $\nu > \lambda_1^{-1}$, *there exists a stationary solution* $u_\infty \in V$ *to system* (3.1);

(ii) *furthermore, if* $(\nu - \lambda_1 \beta_1)^2 > C(D)\|f\|_{V'}$, *then the stationary solution* $u_\infty$ *to system* (3.1) *is unique, where* $C(D)$ *is a positive constant.*

*Lemma* 3.2. *For $\gamma > 0$, there exist two positive constants $\lambda' > 0$, $\lambda'' > 0$ and a function $y : [-\tau, +\infty) \to [0, +\infty)$. If $\lambda'' < \gamma$, the following inequality*

$$y(t) \leq \begin{cases} \lambda' \mathrm{e}^{-\gamma t} + \lambda'' \int_0^t \mathrm{e}^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} y(s+\theta)\mathrm{d}s, & t \geq 0, \\ \lambda' \mathrm{e}^{-\gamma t}, & t \in [-\tau, 0] \end{cases} \tag{3.2}$$

*holds. Then, we have $y(t) \leq M_1 \mathrm{e}^{-\mu t}$ $(t \geq -\tau)$, where $\mu \in (0, \gamma)$ such that $\frac{\lambda''}{\gamma-\mu}\mathrm{e}^{\mu\tau} = 1$ and $M_1 = \lambda'$.*

*Proof.* Letting $F(\mu) = \frac{\lambda''}{\gamma-\mu}\mathrm{e}^{\mu\tau} - 1$, we have $F(0)F(\gamma-) < 0$. That is, there exists a positive constant $\mu \in (0, \gamma)$ such that $F(\mu) = 0$.

For any $\varepsilon > 0$ and letting $C_\varepsilon = \lambda' + \varepsilon$, we only claim that (3.2) implies

$$y(t) \leq C_\varepsilon \mathrm{e}^{-\mu t}, \quad t \geq -\tau. \tag{3.3}$$

It is easily seen that (3.3) holds for any $t \in [-\tau, 0]$. Assume, for the sake of contradiction, that there exists a $t_1^* > 0$ such that

$$y(t) < C_\varepsilon \mathrm{e}^{-\mu t}, \quad t \in [-\tau, t_1^*), \quad y(t_1^*) = C_\varepsilon \mathrm{e}^{-\mu t_1^*}. \tag{3.4}$$

From (3.2), it follows that

$$\begin{aligned} y(t_1^*) &\leq \lambda' \mathrm{e}^{-\gamma t_1^*} + \lambda'' C_\varepsilon \int_0^{t_1^*} \mathrm{e}^{-\gamma(t_1^*-s)} \sup_{\theta\in[-\tau,0]} \mathrm{e}^{-\mu(s+\theta)}\mathrm{d}s \\ &< \lambda' \mathrm{e}^{-\gamma t_1^*} + \lambda'' C_\varepsilon \mathrm{e}^{-\gamma t_1^*} \int_0^{t_1^*} \mathrm{e}^{(\gamma-\mu)s}\mathrm{d}s\mathrm{e}^{\mu\tau} \\ &\leq \lambda' \mathrm{e}^{-\gamma t_1^*} - \frac{\lambda'' C_\varepsilon \mathrm{e}^{\mu\tau}}{\gamma-\mu}\mathrm{e}^{-\gamma t_1^*} + C_\varepsilon \frac{\lambda'' \mathrm{e}^{\mu\tau}}{\gamma-\mu}\mathrm{e}^{-\mu t_1^*}. \end{aligned} \tag{3.5}$$

From the definitions of $\mu$ and $C_\varepsilon$, we have

$$\frac{\lambda'' \mathrm{e}^{\mu\tau}}{\gamma-\mu} = 1$$

and

$$\begin{aligned} \lambda' \mathrm{e}^{-\gamma t_1^*} - \frac{\lambda'' C_\varepsilon \mathrm{e}^{\mu\tau}}{\gamma-\mu}\mathrm{e}^{-\gamma t_1^*} &\leq \lambda' \mathrm{e}^{-\gamma t_1^*} - \frac{\lambda'' \mathrm{e}^{\mu\tau}}{\gamma-\mu}\mathrm{e}^{-\gamma t_1^*}(\varepsilon+\lambda')\frac{(\gamma-\mu)}{\lambda'' \mathrm{e}^{\mu\tau}} \\ &< 0. \end{aligned}$$

Thus, (3.5) yields

$$y(t_1^*) < C_\varepsilon \mathrm{e}^{-\mu t_1^*},$$

which contradicts (3.4), that is, (3.3) holds.

As $\varepsilon > 0$ is arbitrarily small, in view of (3.3), it follows that

$$y(t) \leq M_1 \mathrm{e}^{-\mu t}, \quad t \geq -\tau,$$

where $M_1 = \lambda' > 0$. The proof of this lemma is completed. □

**Theorem 3.3.** *Let $u_\infty$ be the unique stationary solution of* (3.1) *and $\Lambda(u_\infty) = 0$. Suppose that the conditions* ($H_1$)–($H_2$) *are satisfied, then the weak solution $u(t)$ of system* (2.2) *converges to the stationary solution $u_\infty$ of system* (3.1) *exponentially stable in the mean square provided that the following inequality*

$$2\nu > 2\frac{c_1}{\sqrt{\lambda_1}}\|u_\infty\| + \frac{2\beta_1 + \beta_2^2}{\lambda_1}, \tag{3.6}$$

*holds.*

*Proof.* From (3.6), we can choose a positive constant $\lambda > 0$ such that

$$\lambda - \lambda_1\left(2\nu - 2\frac{c_1}{\sqrt{\lambda_1}}\|u_\infty\| - \frac{2\beta_1 + \beta_2^2}{\lambda_1}\right) \geq 0. \tag{3.7}$$

Then, by using Lemma 2.3 to the function $\mathrm{e}^{\lambda t}|u(t) - u_\infty|^2$, we have

$$\begin{aligned}
&\mathrm{e}^{\lambda t}|u(t) - u_\infty|^2 \\
&\quad = |u(0) - u_\infty|^2 + \lambda \int_0^t \mathrm{e}^{\lambda s}|u(s) - u_\infty|^2 \mathrm{d}s \\
&\qquad -2\int_0^t \mathrm{e}^{\lambda s}\langle \nu A u(s), u(s) - u_\infty\rangle \mathrm{d}s \\
&\qquad -2\int_0^t \mathrm{e}^{\lambda s}\langle B(u(s)), u(s) - u_\infty\rangle \mathrm{d}s + 2\int_0^t \mathrm{e}^{\lambda s}\langle f, u(s) - u_\infty\rangle \mathrm{d}s \\
&\qquad +2\int_0^t \mathrm{e}^{\lambda s}\langle G(u(s-\rho(s))), u(s) - u_\infty\rangle \mathrm{d}s \\
&\qquad -2\int_0^t \mathrm{e}^{\lambda s}\langle \Lambda(u(s-\rho(s))), u(s) - u_\infty\rangle \mathrm{d}w(s) \\
&\qquad + \int_0^t \mathrm{e}^{\lambda s}\|\Lambda(u(s-\rho(s)))\|_{L_2^0}^2 \mathrm{d}s.
\end{aligned} \tag{3.8}$$

Since $u_\infty$ is the stationary solution to (3.1),

$$\begin{aligned}
&\int_0^t \mathrm{e}^{\lambda s}\langle \nu A u_\infty, u(s) - u_\infty\rangle \mathrm{d}s + \int_0^t \mathrm{e}^{\lambda s}\langle B(u_\infty), u(s) - u_\infty\rangle \mathrm{d}s \\
&\quad = \int_0^t \mathrm{e}^{\lambda s}\langle f, u(s) - u_\infty\rangle \mathrm{d}s + \int_0^t \mathrm{e}^{\lambda s}\langle G(u_\infty), u(s) - u_\infty\rangle \mathrm{d}s,
\end{aligned} \tag{3.9}$$

and, noting the next identity:

$$\langle B(u(t)) - B(u_\infty), u(t) - u_\infty\rangle = b(u(t) - u_\infty, u_\infty, u(t) - u_\infty). \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), we can obtain

$$\begin{aligned}
&\mathrm{e}^{\lambda t}|u(t)-u_\infty|^2\\
&\quad=|u(0)-u_\infty|^2+\lambda\int_0^t \mathrm{e}^{\lambda s}|u(s)-u_\infty|^2\mathrm{d}s\\
&\qquad-2\int_0^t \mathrm{e}^{\lambda s}\langle \nu A(u(s)-u_\infty),u(s)-u_\infty\rangle \mathrm{d}s\\
&\qquad-2\int_0^t \mathrm{e}^{\lambda s}b(u(s)-u_\infty,u_\infty,u(s)-u_\infty)\mathrm{d}s\\
&\qquad+2\int_0^t \mathrm{e}^{\lambda s}\langle G(u(s-\rho(s)))-G(u_\infty),u(s)-u_\infty\rangle \mathrm{d}s\\
&\qquad-2\int_0^t \mathrm{e}^{\lambda s}\langle \Lambda(u(s-\rho(s)))-\Lambda(u_\infty),u(s)-u_\infty\rangle \mathrm{d}w(s)\\
&\qquad+\int_0^t \mathrm{e}^{\lambda s}\|\Lambda(u(s-\rho(s)))-\Lambda(u_\infty)\|^2_{L_2^0}\mathrm{d}s. \qquad (3.11)
\end{aligned}$$

Since $\int_0^t \mathrm{e}^{\lambda s}\langle \Lambda(u(s-\rho(s))),u(s)-u_\infty\rangle \mathrm{d}w(s)$ is a martingale [11], we have

$$E\int_0^t \mathrm{e}^{\lambda s}\langle \Lambda(u(s-\rho(s)))-\Lambda(u_\infty),u(s)-u_\infty\rangle \mathrm{d}w(s)=0.$$

So, from (3.11) it follows that

$$\begin{aligned}
&\mathrm{e}^{\lambda t}E|u(t)-u_\infty|^2\\
&\quad=E|u(0)-u_\infty|^2+\lambda\int_0^t \mathrm{e}^{\lambda t}E|u(s)-u_\infty|^2\mathrm{d}s\\
&\quad-2\int_0^t \mathrm{e}^{\lambda s}E\langle \nu A(u(s)-u_\infty),u(s)-u_\infty\rangle \mathrm{d}s\\
&\quad-2\int_0^t \mathrm{e}^{\lambda s}Eb(u(s)-u_\infty,u_\infty,u(s)-u_\infty)\mathrm{d}s\\
&\quad+2\int_0^t \mathrm{e}^{\lambda s}E\langle G(u(s-\rho(s))-G(u_\infty),u(s)-u_\infty\rangle \mathrm{d}s\\
&\quad+\int_0^t \mathrm{e}^{\lambda s}E\|\Lambda(u(s-\rho(s)))-\Lambda(u_\infty)\|^2_{L_2^0}\mathrm{d}s. \qquad (3.12)
\end{aligned}$$

From the properties of trilinear form $b$, we have

$$\begin{aligned}
&|b(u(s)-u_\infty,u_\infty,u(s)-u_\infty)|\\
&\quad\le c_1|u(t)-u_\infty|\;\|u(t)-u_\infty\|\;\|u_\infty\|\\
&\quad\le \frac{c_1}{\sqrt{\lambda_1}}\|u_\infty\|\;\|u(t)-u_\infty\|^2. \qquad (3.13)
\end{aligned}$$

In view of (3.7), (3.13) and the conditions ($H_1$) and ($H_2$), we obtain

$$
\begin{aligned}
& e^{\lambda t} E|u(t)-u_\infty|^2 \\
&\quad = E|u(0)-u_\infty|^2 + \lambda \int_0^t e^{\lambda t} E|u(s)-u_\infty|^2 ds \\
&\qquad -2\int_0^t e^{\lambda s} E\langle \nu A(u(s)-u_\infty), u(s)-u_\infty\rangle ds \\
&\qquad -2\int_0^t e^{\lambda s} Eb(u(s)-u_\infty, u_\infty, u(s)-u_\infty) ds \\
&\qquad +\beta_1 \int_0^t e^{\lambda s} E\|u(s)-u_\infty\|^2 ds \\
&\qquad +(\beta_1+\beta_2^2)\int_0^t e^{\lambda s} E|u(s-\rho(s))-u_\infty|^2 ds \\
&\quad \le E|u(0)-u_\infty|^2 + \lambda \int_0^t e^{\lambda t} E|u(s)-u_\infty|^2 ds \\
&\qquad -(2\nu - 2\frac{c_1}{\sqrt{\lambda_1}}\|u_\infty\| - \beta_1)\int_0^t e^{\lambda s} E\|u(s)-u_\infty\|^2 ds \\
&\qquad +(\beta_1+\beta_2^2)\int_0^t e^{\lambda s} E|u(s-\rho(s))-u_\infty|^2 ds \\
&\quad \le E|u(0)-u_\infty|^2 + \left[\lambda - \lambda_1(2\nu - 2\frac{c_1}{\sqrt{\lambda_1}}\|u_\infty\| - \beta_1)\right] \\
&\qquad \int_0^t e^{\lambda s} E|u(s)-u_\infty|^2 ds \\
&\qquad +(\beta_1+\beta_2^2)\int_0^t e^{\lambda s} E|u(s-\rho(s))-u_\infty|^2 ds \\
&\quad \le E|u(0)-u_\infty|^2 + \left[\lambda - \lambda_1(2\nu - 2\frac{c_1}{\sqrt{\lambda_1}}\|u_\infty\| - \frac{2\beta_1+\beta_2^2}{\lambda_1})\right] \\
&\qquad \int_0^t e^{\lambda s} \sup_{\theta\in[-\tau,0]} E|u(s+\theta)-u_\infty|^2 ds.
\end{aligned} \tag{3.14}
$$

Consequently, from (3.14), we have

$$
\begin{aligned}
E|u(t)-u_\infty|^2 \;\le\; & \sup_{\theta\in[-\tau,0]} E|u(\theta)-u_\infty|^2 e^{-\lambda t} \\
& +\left[\lambda - \lambda_1(2\nu - 2\frac{c_1}{\sqrt{\lambda_1}}\|u_\infty\| - \frac{2\beta_1+\beta_2^2}{\lambda_1})\right] \\
& \times \int_0^t e^{-\lambda(t-s)} \sup_{\theta\in[-\tau,0]} E|u(s+\theta)-u_\infty|^2 ds,
\end{aligned}
$$

for all $t \ge 0$. Obviously, it is easily derived that $E|u(t)-u_\infty|^2 \le \sup_{\theta\in[-\tau,0]} E|u(\theta)-u_\infty|^2 e^{-\lambda t}$, $t \in [-\tau, 0]$.

*Case* 1. If $\lambda = \lambda_1\left(2\nu - 2\frac{c_1}{\sqrt{\lambda_1}}\|u_\infty\| - \frac{2\beta_1+\beta_2^2}{\lambda_1}\right)$, the desired result is obviously obtained.

*Case* 2. If $\lambda > \lambda_1\left(2\nu - 2\frac{c_1}{\sqrt{\lambda_1}}\|u_\infty\| - \frac{2\beta_1+\beta_2^2}{\lambda_1}\right)$, from Lemma 3.2, it follows

$$E|u(t)-u_\infty|^2 \leq M_2 \mathrm{e}^{-\alpha t}, \quad t \geq 0,$$

where $\alpha \in (0, \lambda)$ and

$$\begin{aligned} M_2 &= \max\Bigg\{ \sup_{\theta\in[-\tau,0]} E|u(\theta)-u_\infty|^2(\lambda-\alpha) \\ &\quad \times\left(\left[\lambda - \lambda_1(2\nu - 2\frac{c_1}{\sqrt{\lambda_1}}\|u_\infty\| - \frac{2\beta_1+\beta_2^2}{\lambda_1})\right]\mathrm{e}^{\alpha\tau}\right)^{-1}, \\ &\quad \sup_{\theta\in[-\tau,0]} E|u(\theta)-u_\infty|^2 \Bigg\} > 0. \end{aligned}$$

The proof of this theorem is completed. □

**Theorem 3.4.** *Supposed that all conditions of Theorem* 3.3 *are satisfied, then the weak solution of system* (2.2) *converges to the stationary solution* $u_\infty$ *of system* (3.1) *almost surely exponentially stable.*

*Proof.* Let $n = 1, 2, \ldots$, from Itô formula, for $t \geq n\tau$,

$$\begin{aligned} |u(t)-u_\infty|^2 &= |u(nh)-u_\infty|^2 - 2\int_{nh}^t \langle \nu A(u(s)-u_\infty), u(s)-u_\infty\rangle \mathrm{d}s \\ &\quad -2\int_{nh}^t \langle B(u(s)) - B(u_\infty), u(s)-u_\infty\rangle \mathrm{d}s \\ &\quad +2\int_{nh}^t \langle G(u(s-\rho(s))) - G(u_\infty), u(s)-u_\infty\rangle \mathrm{d}s \\ &\quad +\int_{nh}^t \|\Lambda(u(s-\rho(s)))\|^2_{L_2^0}\mathrm{d}s \\ &\quad +2\int_{nh}^t (u(s)-u_\infty, \Lambda(u(s-\rho(s))))\mathrm{d}w(s). \end{aligned}$$

In view of Burkholder–Davis–Gundy formula,

$$\begin{aligned} &E\left(\sup_{nh\leq t\leq(n+1)h}\int_{nh}^t (u(s)-u_\infty, \Lambda(u(s-\rho(s))))\mathrm{d}w(s)\right) \\ &\leq CE\left(\int_{nh}^{(n+1)h} |u(s)-u_\infty|^2\ \|\Lambda(u(s-\rho(s)))\|^2_{L_2^0}\mathrm{d}s\right)^{\frac{1}{2}} \\ &\leq CE(\sup_{nh\leq s\leq(n+1)h} |u(s)-u_\infty|^2 \int_{nh}^{(n+1)h} \|\Lambda(u(s-\rho(s)))\|^2_{L_2^0}\mathrm{d}s)^{\frac{1}{2}} \\ &\leq \frac{1}{4}E(\sup_{nh\leq s\leq(n+1)h} |u(s)-u_\infty|^2) + C'\int_{nh}^{(n+1)h} E\|\Lambda(u(s-\rho(s)))\|^2_{L_2^0}\mathrm{d}s, \end{aligned}$$

where $C$, $C'$ denote two positive constants. From Theorem 3.3, we have

$$\int_{nh}^{(n+1)h} E|u(s-\rho(s)) - u_\infty|^2 \mathrm{d}s \le M_2 \mathrm{e}^{\alpha\tau} \mathrm{e}^{-\alpha nh}.$$

Thus, it follows that

$$\begin{aligned}
& E\left(\sup_{nh\le t\le(n+1)h} |u(t)-u_\infty|^2\right) \\
& \le E|u(nh)-u_\infty|^2 - 2\nu \int_{nh}^{(n+1)h} E\|u(s)-u_\infty)\|^2 \mathrm{d}s \\
& \quad + 2\frac{c_1}{\sqrt{\lambda_1}}\|u_\infty\| \int_{nh}^{(n+1)h} \|u(s)-u_\infty\|^2 \mathrm{d}s \\
& \quad + \frac{\beta_1}{\lambda_1} \int_{nh}^{(n+1)h} E\|u(s)-u_\infty\|^2 \mathrm{d}s \\
& \quad + (\beta_1 + (1+2C')\beta_2^2) \int_{nh}^{(n+1)h} E|u(s-\rho(s)) - u_\infty|^2 \mathrm{d}s \\
& \quad + \frac{1}{2} E\left(\sup_{nh\le t\le(n+1)h} |u(t)-u_\infty|^2\right) \\
& \le E|u(nh)-u_\infty|^2 - \frac{1}{\lambda_1}\left(2\nu - 2\frac{c_1}{\sqrt{\lambda_1}} - \frac{\beta_1}{\lambda_1}\right) \int_{nh}^{(n+1)h} E|u(s)-u_\infty|^2 \mathrm{d}s \\
& \quad + (\beta_1 + (1+2C')\beta_2^2) M_2 \mathrm{e}^{\alpha\tau} \mathrm{e}^{-\alpha nh} + \frac{1}{2} E\left(\sup_{nh\le t\le(n+1)h} |u(t)-u_\infty|^2\right).
\end{aligned}$$

From (3.6), it implies that

$$E(\sup_{nh\le t\le(n+1)h} |u(t)-u_\infty|^2) \le M\mathrm{e}^{-\alpha nh},$$

where $\alpha$ is given in Theorem 3.3 and $M > 0$. Based on the Chebychev inequality,

$$P\left\{\omega : \sup_{nh\le t\le(n+1)h} |u(t)-u_\infty| > \mathrm{e}^{-\frac{(\alpha-\varepsilon)nh}{2}}\right\} \le M'\mathrm{e}^{-\varepsilon nh},$$

where $M'$ is a positive constant and $\varepsilon \in (0, \alpha)$.

From the Borel–Cantelli lemma, there is a finite integer $n_0(\omega)$ such that

$$\sup_{nh\le t\le(n+1)h} |u(t)-u_\infty| \le \mathrm{e}^{-\frac{(\alpha-\varepsilon)nh}{2}}, \quad \text{a.s.}$$

for all $n \ge n_0$. The proof of this theorem is completed. □

*Remark* 2. Recently, in [14], Wei and Zhang have obtained the exponential stability in mean square and almost surely exponential stability for the weak solution to system (2.2) by constructing the Lyapunov functional proposed in [6]. But, an additional condition:

$\rho_* = \sup_{t\geq 0} \rho'(t) < 1$ must be imposed in [14]. However, this restrictive condition about the time-varying delay can be removed in this paper. So, our result can improve the one given in [14].

COROLLARY 3.5

*Assuming that condition* $(\mathrm{H}_1)$ *holds, the weak solution* $u(t)$ *of system* (2.3) *converges to its stationary solution* $u_\infty$ *exponentially stable provided the following inequality*

$$\nu > \frac{c_1}{\sqrt{\lambda_1}}\|x_\infty\| + \frac{\beta_1}{\lambda_1},$$

*holds.*

*Remark* 4. In [6], Caraballo and Real have studied the asymptotic behavior of the weak solution to system (2.3) by utilizing the direct method. However, a strong restrictive condition of the delay are also imposed in [6], i.e. $\rho'(t) < 1$, for $\forall t \geq 0$. In this paper, we can remove this one. Thus, we can generalize and improve the result in [6].

**Acknowledgement**

The author is indebted the anonymous referee for their valuable comments and suggestions and for carefully reading the paper. This work is supported by the National Natural Science Foundation of China under Grant No. 11126278, and Grant No. 11101202, the Natural Science Foundation of Jiangxi Province under Grant No. 20114BAB211001 and Grant No. 2009GQS0018, and the Youth Foundation of Jiangxi Provincial Educations of China under Grant No. GJJ11045 and Grant No. GJJ10051.

## References

[1] Barbu V and Sritharan S, Navier–Stokes equation with hereditary viscosity, *Z. Angew. Math. Phys.* **54** (2003) 449–461

[2] Bensoussan A, Stochastic Navier–Stokes equations, *Acta Appl. Math.* **38** (1995) 267–304

[3] Capinski M and Gatarek D, Stochastic equations in Hilbert spaces with applications to Navier–Stokes equations in any dimension, *J. Funct. Anal.* **126** (1994) 26–35

[4] Caraballo T, Langa J and Taniguchi T, The exponential behavior and stabilizability of stochastic 2D-Navier–Stokes equations, *J. Diff. Equations* **179** (2002) 714–737

[5] Caraballo T and Real J, Navier–Stokes equations with delays, *R. Soc. London* **A457** (2001) 2441–2453

[6] Caraballo T and Real J, Asymptotic behavior of Navier–Stokes equations with delays, *R. Soc. London* **A459** (2003) 3181–3194

[7] Constantin P and Foias C, Navier–Stokes Equations (Chicago/London: Univ. of Chicago Press) (1988)

[8] Planas G and Hernandez E, Asymptotic behavior of two-dimensional time-delayed Navier–Stokes equations, *Discrete Contin. Dynam. Syst.* **31** (2008) 1245–1258

[9] Prato G Da and Zabczyk J, Stochastic Equations in Infinite Dimensions (Cambridge University Press) (1992)

[10] Sundar P and Yin H, Existence and uniqueness of solutions to the backward 2D stochastic Navier–Stokes equations, *Stoch. Proc. Appl.* **119** (2009) 1216–1234

[11] Sritharan S S and Sundar P, Large derivations for the two-dimensional Navier–Stokes equations with multiplicative noise, *Stoch. Proc. Appl.* **116** (2006) 1636–1659
[12] Temam R, Navier–Stokes Equations, Theory and Numerical Analysis (Amsterdam: North-Holland) (1979)
[13] Temam R, Navier–Stokes Equations and Nonlinear Functional Analysis, 2nd edition (Philadelphia: SIAM) (1995)
[14] Wei M and Zhang T, Exponential stability for stochastic 2D-Navier–Stokes equations with time delay, *Appl. Math. J. Chinese Univ.* **24** (2009) 493–500 (in Chinese)