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Discretization of continuous frame

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Abstract. In this paper we consider the notion of continuous frame of subspaces and define a new concept of continuous frame, entitled *continuous atomic resolution of identity*, for arbitrary Hilbert space $\mathcal{H}$ which has a countable reconstruction formula. Among the other results, we characterize the relationship between this new concept and other known continuous frames. Finally, we state and prove the assertions of the stability of perturbation in this concept.

Keywords. Bounded operator; Hilbert space; continuous frame; atomic resolution of identity.

1. Introduction and preliminaries

As we know frames are more flexible tools to convey information than bases, and so they are suitable replacement for bases in a Hilbert space $\mathcal{H}$. Finding a representation of $f \in \mathcal{H}$ as a linear combination of the elements in frames, is the main goal of discrete frame theory. But in continuous frame, which is a natural generalization from discrete, it is not straightforward. However, one of the applications of frames is in wavelet theory. The practical implementation of the wavelet transform in signal processing requires the selection of a discrete set of points in the transformed space. Indeed, all formulas must generally be evaluated numerically, and a computer is an intrinsically discrete object. But this operation must be performed in such a way that no information is lost. So efforts have been done to find methods to discretize classical continuous frames for use in applications like signal processing, numerical solution of PDE, simulation, and modelling; see for example [1, 8]. In particular, the discrete wavelet transform and Gabor frames are prominent examples and have been proven to be a very successful tool for certain applications. Since the problem of discretization is so important it would be nice to have a general method for this purpose. For example, Ali *et al.* [1] asked for conditions which ensure that a certain sampling of a continuous frame $\{\psi_x\}_{x\in X}$ yields a discrete frame $\{\psi_{x_i}\}_{i\in I}$ (see also [9]).

In recent years, there has been considerable interest by harmonic and functional analysts in the frame of subspaces problem of the separable Hilbert space; see [5], [4], [3] and [2] and references therein. Frame of subspaces was first introduced by Casazza and Kutyniok in [5]. They present a reconstruction formula $f = \sum_{i\in I} \nu_i^2 S^{-1}\pi_{W_i}(f)$ for frames of subspaces. Continuous frame of subspace is a natural generalization from discrete frame of subspaces to continuous. As we expect, in discrete frame of subspaces every element in $\mathcal{H}$ has an expansion in terms of frames. But in the continuous case it is with respect to Bochner integral which is not desirable. Therefore, discretization of continuous frame of subspaces is also very important.

Suppose that the measure μ, which appears in the integral of continuous frame, is Radon or discontinuous. (Note that there exist infinite many positive finite discontinuous measure on a locally compact space X which are not counting measure.) Then $\{x \in X : \mu(\{x\} \neq 0\}$ is a nonempty set and we may investigate about some conditions under which every fixed element $f \in \mathcal{H}$ has a countable subfamily J_f of X with frame property for $\mathcal{H}$. This leads us to define *uca-resolution of identity* (Definition 2.1), which is a generalization of the resolution of identity (Definition 3.24 of [5]), and atomic resolution of identity [4], to arbitrary Hilbert space (separable or nonseparable). We then show that in this concept many basic properties of discrete state can be derived within this more general context. In fact uca-resolution identity helps us to investigate continuous frames which have discretization. Because under some extra conditions, every uca-resolution of identity provides a continuous frame of subspace, and conversely. This means that the relationship between uca-resolution of identity and known continuous frames, such as frame of subspaces, is very tight.

Assume $\mathcal{H}$ to be a Hilbert space and X to be a locally compact Hausdorff space endowed with a positive Radon or discontinuous measure μ. Let $\mathcal{W} = \{W_x\}_{x\in X}$ be a family of closed subspaces in $\mathcal{H}$ and let $\omega : X \to [0, \infty)$ be a measurable mapping such that $\omega \neq 0$ almost everywhere (a.e.). We say that $\mathcal{W}_\omega = \{(W_x, \omega(x))\}_{x\in X}$ is a continuous frame of subspaces for $\mathcal{H}$, if:

(a) the mapping $x \mapsto \pi_{W_x}$ is weakly measurable;
(b) there exist constants $0 < A, B < \infty$ such that

$$A\|f\|^2 \leq \int_X \omega(x)^2 \|\pi_{W_x}(f)\|^2 \,\mathrm{d}\mu(x) \leq B\|f\|^2 \tag{1}$$

for all $f \in \mathcal{H}$. The numbers A and B are called the continuous frame of subspaces bounds. If $\mathcal{W}_\omega$ satisfies only the upper inequality in (1), then we say that it is a continuous Bessel frame of subspaces with bound B. Note that if X is a countable set and μ is the counting measure, then we obtain the usual definition of a (discrete) frame of subspaces.

For each continuous Bessel frame of subspaces $\mathcal{W}_\omega = \{(W_x, \omega(x))\}_{x\in X}$, if we define the representation space associated with $\mathcal{W}_\omega$ by $L^2(X, \mathcal{H}, \mathcal{W}_\omega) = \{\varphi : X \to \mathcal{H} |\ \varphi \text{ is measurable}, \varphi(x) \in W_x \text{ and } \int_X \|\varphi(x)\|^2 \mathrm{d}\mu(x) < \infty\}$, then $L^2(X, \mathcal{H}, \mathcal{W}_\omega)$ with the inner product is given by

$$\langle \varphi, \psi \rangle = \int_X \langle \varphi(x), \psi(x) \rangle \mathrm{d}\mu(x), \quad \text{for all } \varphi, \psi \in L^2(X, \mathcal{H}, \mathcal{W}_\omega),$$

is a Hilbert space. Also, the synthesis operator $T_{\mathcal{W}_\omega} : L^2(X, \mathcal{H}, \mathcal{W}_\omega) \to \mathcal{H}$ is defined by

$$\langle T_{\mathcal{W}_\omega}(\varphi), f \rangle = \int_X \omega(x) \langle \varphi(x), f \rangle \mathrm{d}\mu(x),$$

for all $\varphi \in L^2(X, \mathcal{H}, \mathcal{W}_\omega)$ and $f \in \mathcal{H}$. Its adjoint operator is $T^*_{\mathcal{W}_\omega} : \mathcal{H} \to L^2(X, \mathcal{H}, \mathcal{W}_\omega)$; $T^*_{\mathcal{W}_\omega}(f) = \omega \pi_{\mathcal{W}_\omega}(f)$. For more details, see [2].

Now, we give two immediate consequences from the above discussion. As the first, we have the following characterization of continuous Bessel frame of subspaces in term of their synthesis operators as in the discrete frame theory; see [3,6].

Theorem 1.1. *A family $\mathcal{W}_\omega$ is a continuous Bessel frame of subspaces with Bessel fusion bound B for $\mathcal{H}$ if and only if the synthesis operator $T_{\mathcal{W}_\omega}$ is a well-defined bounded operator and $\|T_{\mathcal{W}_\omega}\| \le \sqrt{B}$.*

Also, by an argument similar to the proof of (Theorem 2.6 of [3]), we have a characterization of continuous frame of subspaces as follows:

Theorem 1.2. *The following conditions are equivalent*:

(a) $\mathcal{W}_\omega = (\{W_x\}_{x\in X}, \omega(x))$ *is a continuous frame of subspaces for* $\mathcal{H}$;
(b) *The synthesis operator* $T_{\mathcal{W}_\omega}$ *is a bounded, linear operator from* $L^2(X, \mathcal{H}, \mathcal{W}_\omega)$ *onto* $\mathcal{H}$;
(c) *The analysis operator* $T^*_{\mathcal{W}_\omega}$ *is injective with closed range.*

If $\mathcal{W}_\omega$ is a continuous frame of subspaces for $\mathcal{H}$ with frame bounds A, B, then we define the frame of subspaces operator $S_{\mathcal{W}_\omega}$ for $\mathcal{W}_\omega$ by

$$S_{\mathcal{W}_\omega}(f) = T_{\mathcal{W}_\omega} T^*_{\mathcal{W}_\omega}(f), \quad f \in \mathcal{H},$$

which is a positive, self-adjoint, invertible operator on $\mathcal{H}$ with $A \cdot \mathrm{Id}_{\mathcal{H}} \le S_{\mathcal{W}_\omega} \le B \cdot \mathrm{Id}_{\mathcal{H}}$.

2. Main result

For establishing a relationship between discrete and continuous frame of subspaces, we generalize the concept of continuous frame and resolution of identity to arbitrary Hilbert space $\mathcal{H}$. For this purpose, we introduce the summation to noncountable form. Let $\mathcal{H}$ be a Hilbert space and $\{T_x\}_{x\in X}$ be a family of bounded operators on it. If now, set Γ, the collection of all finite subset of X, then Γ is a directed set ordered under inclusion.

Let f be a fixed element of the Hilbert space $\mathcal{H}$. Define the sum $S(f)$ of the family $\{T_x(f)\}_{x\in X}$ as the limit

$$S(f) = \sum_{x\in X} T_x(f) = \lim \left\{ \sum_{x\in\gamma} T_x(f) : \gamma \in \Gamma \right\}.$$

If this limit exists, we say that the family $\{T_x(f)\}_{x\in X}$ is unconditionally summable. It is easy to see that the family $\{T_x(f)\}_{x\in X}$ is unconditionally summable if and only if for each $\varepsilon > 0$, there exist a finite subset $\gamma_0 \in \Gamma$ such that

$$\left\| \sum_{x\in\gamma_1} T_x(f) - \sum_{x\in\gamma_2} T_x(f) \right\| < \varepsilon,$$

for each $\gamma_1, \gamma_2 > \gamma_0$. Therefore for each $\varepsilon > 0$, there is a finite subset γ_0 of X such that

$$\|T_x(f)\| < \varepsilon$$

for all $x \in X \backslash \gamma_0$. Hence for a fixed element $f \in \mathcal{H}$, if $\{T_x(f)\}_{x\in X}$ is unconditionally summable, then $J_f = \{x \in X : T_x(f) \neq 0\}$ is countable.

DEFINITION 2.1

Let $\mathcal{H}$ be a Hilbert space and let $\omega : X \to [0, \infty)$ be a measurable mapping such that $\omega \neq 0$ almost everywhere. We say that a family of bounded operator $\{T_x\}_{x\in X}$ on $\mathcal{H}$ is an unconditional continuous atomic resolution (uca-resolution) of the identity with respect to ω for $\mathcal{H}$, if there exist positive real numbers C and D such that for all $f \in \mathcal{H}$,

(a) the mapping $x \mapsto T_x$ is weakly measurable;
(b) $C\|f\|^2 \le \int_X \omega(x)^2\|T_x(f)\|^2 \mathrm{d}\mu(x) \le D\|f\|^2$;
(c) $f = \sum_{x\in X} T_x(f)$.

The optimal values of C and D are called the uca-resolution of the identity bounds. It follows from the definition and the uniform boundedness principle that $\sup_{x\in X}\|T_x\|_{x\in X} < \infty$.

Remark 2.2.

(a) If $f \in \mathcal{H}$ satisfies in (c), then as we mention above, there is a countable measurable subset J_f (depends of f) of X such that

$$T_x(f) = 0,$$

for all $x \in X\backslash J_f$. So

$$\int_X \omega(x)^2\|T_x(f)\|^2 \mathrm{d}\mu(x) = \sum_{j\in J_f} \omega(j)^2\|T_j(f)\|^2 \mu(\{j\})$$

and condition (b) transform to

$$C\|f\|^2 \le \sum_{j\in J_f} \omega(j)^2\|T_j(f)\|^2\ \mu(\{j\}) \le D\|f\|^2.$$

(b) If $\mathcal{H}$ is a separable Hilbert space with orthonormal bases $\{e_n\}_{n=1}^{\infty}$, then by condition (c), for each n there exists a countable measurable subset J_n of X such that

$$T_x(e_n) = 0,$$

for all $x \in X\backslash J_n$. So, we can find a countable subset $J = \bigcup_{n=1}^{\infty} J_n$ of X such that

$$T_x(f) = 0,$$

for all $f \in \mathcal{H}$ and $x \in X\backslash J$, and we have

$$\int_X \omega(x)^2\|T_x(f)\|^2 \mathrm{d}\mu(x) = \sum_{j\in J} \omega(j)^2\|T_j(f)\|^2\ \mu(\{j\}).$$

Therefore, if $\mathcal{H}$ is a separable Hilbert space, Definitions 2.1 and 3.1 in [4] coincide.

From now on, $\mathcal{H}$ is a Hilbert space with orthonormal bases $\{e_\lambda\}_{\lambda\in\Lambda}$ and X is a locally compact Hausdorff space endowed with a positive Radon or discontinuous measure μ, and $\omega : X \to [0, \infty)$ is a measurable mapping such that $\omega \neq 0$ almost everywhere. For a fixed element $f \in \mathcal{H}$, by [7] there exists a countable subset J of Λ such that $\langle f, e_\lambda\rangle = 0$ for all $\lambda \in \Lambda\backslash J$.

The following is an important example of uca-resolution compatible with Definition 2.1, and note that this example does not satisfy in the definition of resolution of identity and atomic resolution of identity which is stated in [5] and [4], respectively.

Example 2.3. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_\lambda\}_{\lambda\in\Lambda}$. If, we consider Λ as a locally compact space with discrete topology and measurable space endowed with counting measure, then the family $\{T_\lambda\}_{\lambda\in\Lambda}$ of bounded operators on $\mathcal{H}$ is defined by

$$T_\lambda(f) = \langle e_\lambda, f\rangle e_\lambda, \quad \text{for all } f \in \mathcal{H} \text{ and } \lambda \in \Lambda,$$

is an uca-resolution of identity for $\mathcal{H}$.

In the next theorem we show that every uca-resolution of identity for $\mathcal{H}$, provides a continuous frame of subspace.

Theorem 2.4. *Let $\{T_x\}_{x\in X}$ be a family of bounded operators on $\mathcal{H}$ and for each $x \in X$, set $W_x = \overline{T_x(\mathcal{H})}$. Suppose that there exists $D > 0$ and $R > 0$ such that the following conditions hold*:

(a) $f = \sum_{x\in X} \omega(x)^2 T_x(f)\mu(\{x\})$;
(b) $\int_X \omega(x)^2 \|\pi_{W_x}(f) - T_x(f)\|^2 \mathrm{d}\mu(x) \leq R\|f\|^2$;
(c) $\int_X \omega(x)^2 \|T_x(f)\|^2 \mathrm{d}\mu(x) \leq D\|f\|^2$,

for all $f \in \mathcal{H}$. Then $\{(W_x, \omega(x))\}_{x\in X}$ is a continuous frame of subspaces for $\mathcal{H}$.

Proof. Let f be a fixed element of $\mathcal{H}$. As we mention in Remark 2.2(a), there exists a countable subset J_f of X such that

$$\omega(x)^2 T_x(f)\mu(\{x\}) = 0,$$

for all $x \in X\backslash J_f$, and

$$\int_X \omega(x)^2\|T_x(f)\|^2 \mathrm{d}\mu(x) = \sum_{x\in X} \omega(x)^2\|T_x(f)\|^2\mu(\{x\}).$$

So we can use Cauchy–Schwarz inequality and compute as follows:

$$\begin{aligned}
\|f\|^4 &= \left(\left\langle f, \sum_{x\in X} \omega(x)^2 T_x(f)\mu(\{x\})\right\rangle\right)^2 \\
&= \left(\sum_{x\in X} \omega(x)\langle\sqrt{\mu(\{x\})}\, f, \omega(x)\sqrt{\mu(\{x\})}\, T_x(f)\rangle\right)^2 \\
&= \left(\sum_{x\in X} \omega(x)\langle\sqrt{\mu(\{x\})}\, \pi_{W_x}(f), \omega(x)\sqrt{\mu(\{x\})}\, T_x(f)\rangle\right)^2 \\
&\leq \left(\sum_{x\in X} \omega(x)\|\sqrt{\mu(\{x\})}\, \pi_{W_x}(f)\|\|\omega(x)\sqrt{\mu(\{x\})}\, T_x(f)\|\right)^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\sum_{x\in X}\omega(x)^2\|\pi_{W_x}(f)\|^2\mu(\{x\})\right)\left(\sum_{x\in X}\|\omega(x)\sqrt{\mu(\{x\})}\,T_x(f)\|^2\right)\\
&\leq \left(\int_{x\in X}\omega(x)^2\|\pi_{W_x}(f)\|^2\mathrm{d}\mu(x)\right)\left(\int_X\omega(x)^2\|T_x(f)\|^2\mathrm{d}\mu(x)\right)\\
&\leq D\|f\|^2\left(\int_X\omega(x)^2\|\pi_{W_x}(f)\|^2\mathrm{d}\mu(x)\right).
\end{aligned}
$$

Also, by triangle inequality and hypothesis we have

$$\int_X\omega(x)^2\|\pi_{W_x}(f)\|^2\mathrm{d}\mu(x)\leq D\left(1+\sqrt{\frac{R}{D}}\right)^2\|f\|^2,$$

so the assertion holds. □

Casazza and Kutyniok in [5] introduced an interesting example of atomic resolution of identity. In the next theorem we obtain the uca-resolution of identity form, which is the converse of Theorem 2.4.

Theorem 2.5. *Let $\{(W_x,\omega(x))\}_{x\in X}$ be a continuous Bessel frame of subspaces for $\mathcal{H}$ with Bessel bound D, and for each $x\in X$. Let $T_x:\mathcal{H}\to W_x$ be a bounded operator such that $T_x\pi_{W_x}=T_x$. Also assume that for each $f\in\mathcal{H}$,*

$$f=\sum_{x\in X}\omega(x)^2T_x(f)\mu(\{x\}).$$

Then for all $f\in\mathcal{H}$ we have

$$\frac{1}{D}\|f\|^2\leq\int_X\omega(x)^2\|T_x(f)\|^2\mathrm{d}\mu(x)\leq DE\|f\|^2,$$

where $E=\sup_{x\in X}\|T_x\|_{x\in X}$.

Proof. By a similar proof of Theorem 2.4, we obtain

$$\frac{1}{D}\|f\|^2\leq\int_X\omega(x)^2\|T_x(f)\|^2\mathrm{d}\mu(x).$$

Also we have

$$
\begin{aligned}
\frac{1}{D}\|f\|^2 &\leq \int_X\omega(x)^2\|T_x(f)\|^2\mathrm{d}\mu(x)\\
&= \int_X\omega(x)^2\|T_x\pi_{W_x}(f)\|^2\mathrm{d}\mu(x)\\
&\leq \int_X\omega(x)^2\|T_x\|^2\|\pi_{W_x}(f)\|^2\mathrm{d}\mu(x)\\
&\leq E\int_X\omega(x)^2\|\pi_{W_x}(f)\|^2\mathrm{d}\mu(x)\leq DE\|f\|^2.
\end{aligned}
$$

Whence, for each $f\in\mathcal{H}$,

$$\frac{1}{D}\|f\|^2\leq\int_X\omega(x)^2\|T_x(f)\|^2\mathrm{d}\mu(x)\leq DE\|f\|^2.$$

as we required. □

PROPOSITION 2.6

Let $\{W_x\}_{x\in X}$ be a family of closed subspaces of Hilbert space $\mathcal{H}$ such that the mapping $x \to \pi_{W_x}$ is weakly measurable. Also suppose ω is a bounded map and the following conditions hold for all $f \in \mathcal{H}$:

(a) *There exists $C > 0$ such that*

$$\int_X \| \pi_{W_x}(f) \|^2 \mathrm{d}\mu(x) \leq \frac{1}{C} \| f \|^2,$$

(b) $f = \sum_{x\in X} \omega(x)\pi_{W_x}(f)\mu(\{x\})$.

Then $\{(W_x, \omega(x))\}_{x\in X}$ is a continuous frame of subspaces for $\mathcal{H}$.

Proof. By condition (a) we see that

$$\int_X \omega(x)^2 \|\pi_{W_x}(f)\|^2 \mathrm{d}\mu(x) \leq \frac{\sup_{x\in X}\omega(x)}{C} \|f\|^2, \quad (f \in \mathcal{H}).$$

Condition (b) implies that for a fixed element f of $\mathcal{H}$,

$$\int_X \omega(x)^2 \|\pi_{W_x}(f)\|^2 \mathrm{d}\mu(x) = \sum_{x\in X} \omega(x)^2 \|\pi_{W_x}(f)\|^2 \mu(\{x\})$$

and

$$\int_X \|\pi_{W_x}(f)\|^2 \mathrm{d}\mu(x) = \sum_{x\in X} \|\pi_{W_x}(f)\|^2 \mu(\{x\}).$$

Now, since the family $\{\omega(x)\mu(\{x\})T_x\}$ is unconditionally summable, we can use Cauchy–Schwarz inequality and compute as follows:

$$\begin{aligned}
\|f\|^4 &= \left(\left\langle \sum_{x\in X} \omega(x)\mu(\{x\})\pi_{W_x}(f), f \right\rangle\right)^2 \\
&= \left(\sum_{x\in X} \omega(x)\mu(\{x\})\|\pi_{W_x}(f)\|^2\right)^2 \\
&\leq \left(\sum_{x\in X} \omega(x)^2\mu(\{x\})\|\pi_{W_x}(f)\|^2\right)\left(\sum_{x\in X} \|\pi_{W_x}(f)\|^2\mu(\{x\})\right) \\
&\leq \frac{1}{C}\|f\|^2 \left(\sum_{x\in X} \omega(x)^2 \|\pi_{W_x}(f)\|^2 \mu(\{x\})\right).
\end{aligned}$$

Thus

$$C\|f\|^2 \leq \sum_{x\in X} \omega(x)^2 \|\pi_{W_x}(f)\|^2 \mu(\{x\}) = \int_X \omega(x)^2 \|\pi_{W_x}(f)\|^2 \mathrm{d}\mu(x)$$

for all $f \in \mathcal{H}$, and this completes the proof. □

In the following proposition we give a reconstruction formula for continuous frame of subspaces in the special case.

PROPOSITION 2.7

Let $\{W_x\}_{x\in X}$ be a family of orthogonal closed subspaces of Hilbert space $\mathcal{H}$. If $\{(W_x, \omega(x)\}_{x\in X}$ is a continuous frame of subspaces for $\mathcal{H}$ with bounds C, D, then for each $f \in \mathcal{H}$,

$$f = \sum_{x\in X} \pi_{W_x}(f).$$

The converse is true if ω is bounded and there exists $C > 0$ such that

$$\int_X \|\pi_{W_x}(f)\|^2 \mathrm{d}\mu(x) \le \frac{1}{C}\|f\|^2,$$

for all $f \in \mathcal{H}$.

Proof. Let $\{(W_x\}_{x\in X}$ be a continuous frame of subspaces. First, we should note that for each $f \in \mathcal{H}$, by the Hahn-Banach theorem and orthogonality of the family $\{W_x\}_{x\in X}$, there exists a sequence $\{f_n\}$ in $\mathcal{H}$ such that $f_n \longrightarrow f$ and for each n we have the following equality

$$f_n = \sum_{x\in X} \pi_{W_x}(f_n).$$

Now we define $S_\gamma(f) = \sum_{x\in\gamma} \pi_{W_x}(f)$, where γ is an arbitrary finite subset of X and $f \in \mathcal{H}$. Therefore

$$\begin{aligned} C\|S_\gamma(f) - f\|^2 &\le \int_X \omega(x)^2 \|\pi_{W_x}(S_\gamma(f) - f)\|^2 \mathrm{d}\mu(x) \\ &\le \int_X \omega(x)^2 \|\pi_{W_x}(f)\|^2 \mathrm{d}\mu(x) \\ &\le D\|f\|^2. \end{aligned}$$

By replacing f with $f_n - f$ we obtain

$$\|S_\gamma(f_n - f) - (f_n - f)\| \le \sqrt{\frac{D}{C}}\|f_n - f\|.$$

The converse holds by Proposition 2.6. □

Now we want to show that, by a given uca-resolution of identity, each $f \in \mathcal{H}$ has a new countable reconstruction formula. First we need the following lemma:

Lemma 2.8. *Let $\{T_x\}_{x\in X}$ be an uca-resolution of the identity with respect to weight ω for $\mathcal{H}$ with bounds C and D, and let $\{f_i\}_{i\in I}$ be a frame sequence. Then there exists a countable subset J of X, such that $\{\omega(j)\sqrt{\mu(\{j\})}T_j^*(f_i)\}_{i\in I, j\in J}$ is a frame for $\overline{\mathrm{span}}\{f_i\}_{i\in I}$.*

Proof. If we set $J_i = \{x \in X : T_x(f_i) \neq 0\}$, then by definition of uca-resolution of identity, J_i is a countable and measurable subset of X. Now, set $J = \bigcup_{i\in I} J_i$. So J is a countable and measurable subset of X, and for each $f \in \overline{\mathrm{span}}\{f_i\}_{i\in I}$ and $x \in X\backslash J$ we have

$$T_x(f) = 0.$$

Hence we see that for each $f \in \overline{\operatorname{span}}\{f_i\}_{i\in I}$,

$$C\|f\|^2 \le \sum_{j\in J} \omega^2(j)\mu(\{j\})\|T_j(f)\|^2 \le D\|f\|^2$$

and

$$f = \sum_{j\in J} T_j(f),$$

and these series converge unconditionally.

Now, suppose that A and B are frame bounds of $\{f_i\}_{i\in I}$. For each $f \in \overline{\operatorname{span}}\{f_i\}_{i\in I}$ we have

$$\begin{aligned} A\sum_{j\in J} \omega^2(j)\mu(\{j\})\|T_j(f)\|^2 &\le \sum_{j\in J}\sum_{i\in I} |\langle \omega^2(j)\mu(\{j\})T_j(f), f_i\rangle|^2 \\ &\le B\sum_{j\in J} \omega^2(j)\mu(\{j\})\|T_j(f)\|^2, \end{aligned}$$

and therefore

$$\begin{aligned} AC\|f\|^2 &\le A\sum_{j\in J} \omega^2(j)\mu(\{j\})\|T_j(f)\|^2 \\ &\le \sum_{j\in J}\sum_{i\in I} |\langle f, \omega^2(j)\mu(\{j\})T_j^*(f_i)\rangle|^2 \\ &\le B\sum_{j\in J} \omega^2(j)\mu(\{j\})\|T_j(f)\|^2 \le BD\|f\|^2 \end{aligned}$$

and this complete the proof. □

Theorem 2.9. *Let $\{T_x\}_{x\in X}$ be an uca-resolution of the identity with respect to weight ω for $\mathcal{H}$ with bounds C and D. Then for each $f \in \mathcal{H}$, there exists a countable subset I (dependents on f) of X, such that we have the following reconstruction formula*

$$f = \sum_{i\in I} \omega^2(i)\mu(\{i\})S^{-1}T_i^*T_i(f) = \sum_{i\in I} \omega^2(i)\mu(\{i\})T_i^*T_iS^{-1}(f),$$

where S is a frame operator of a frame sequence.

Proof. Let f be a fixed element of Hilbert space $\mathcal{H}$. Set

$$\mathcal{H}_f = \overline{\operatorname{span}}\{e_j\}_{j\in J},$$

where $J = \{j \in \Lambda : \langle e_j, f\rangle \neq 0\}$ is a countable subset of Λ. Then, by Lemma 2.8, there is a countable subset I of X such that the sequence $\{\omega(i)\sqrt{\mu(\{i\})}T_i^*(e_j)\}_{i\in I, j\in J}$ is a frame for $\mathcal{H}_f$.

If now, $S \in B(\mathcal{H})$ is the frame operator of $\{\omega(i)\sqrt{\mu(\{i\})}T_i^*(e_j)\}_{i\in I, j\in J}$, then we have

$$\begin{aligned} S(f) &= \sum_{i\in I}\sum_{j\in J} \left\langle f, \omega(i)\sqrt{\mu(\{i\})}T_i^*(e_j)\right\rangle \omega(i)\sqrt{\mu(\{i\})}T_i^*(e_j) \\ &= \sum_{i\in I} \omega^2(i)\mu(\{i\})T_i^*\left(\sum_{j\in J}\langle T_i(f), e_j\rangle e_j\right) \\ &= \sum_{i\in I} \omega^2(i)\mu(\{i\})T_i^*T_i(f). \end{aligned}$$

Hence, the reconstruction formula follows immediately from the invertibility of the operator S. $\square$

In the rest of paper we consider the stability of perturbation in uca-resolution of identity. First, let us state and prove the following useful lemma.

Lemma 2.10. *Let* $\{T_x\}_{x\in X}$ *and* $\{S_x\}_{x\in X}$ *be two families of bounded operators on* $\mathcal{H}$ *and there exists* $0 < \lambda < 1$ *such that for all finite subset* I *of* X,

$$\left\| \sum_{i\in I} (T_i - S_i)(f) \right\| \leq \lambda \left\| \sum_{i\in I} T_i(f) \right\|, \quad f \in \mathcal{H}. \tag{2}$$

If $\{(T_x, \omega(x)\}_{x\in X}$ *is an uca-resolution of identity then we have the following reconstruction formula*

$$f = \sum_{x\in X} S_x S^{-1}(f), \quad f \in \mathcal{H}$$

where S *is an invertible operator on* $\mathcal{H}$.

Proof. Let $f \in \mathcal{H}$ and let I be a finite subset of X. Since

$$\left\| f - \sum_{i\in I} S_i(f) \right\| \leq \left\| f - \sum_{i\in I} T_i(f) \right\| + \left\| \sum_{i\in I} T_i(f) - \sum_{i\in I} S_i(f) \right\|,$$

therefore by inequality (2) we have

$$\left\| f - \sum_{i\in I} S_i(f) \right\| \leq \left\| f - \sum_{i\in I} T_i(f) \right\| + \lambda \left\| \sum_{i\in I} T_i(f) \right\|. \tag{3}$$

Hence, the family $\{S_x(f)\}_{x\in X}$ is unconditionally summable. Now, we define $S : \mathcal{H} \to \mathcal{H}$ by $S(f) = \sum_{x\in X} S_x(f)$. By inequality (3) and using that $\{(T_x, \omega(x)\}$ is assumed to be uca-resolution of identity, S is well defined and we have

$$\|f - S(f)\| \leq \lambda \|f\|,$$

for all $f \in \mathcal{H}$. So $\|\mathrm{id}_{\mathcal{H}} - S\| \leq \lambda < 1$, and therefore S is an invertible operator on $\mathcal{H}$. Hence for all $f \in \mathcal{H}$ we have

$$\sum_{x\in X} S_x S^{-1}(f) = SS^{-1}(f) = f,$$

and this complete the proof. $\square$

DEFINITION 2.11

Let $\{T_x\}_{x\in X}$ and $\{S_x\}_{x\in X}$ be two families of bounded operators on $\mathcal{H}$, and let $\omega : X \to [0, \infty)$ be a measurable map such that $\omega(x) \neq 0$ almost everywhere. Suppose that $0 \leq \lambda_1, \lambda_2 < 1$, and $\varphi : X \to [0, \infty)$ is an arbitrary positive map such that $\int_X \varphi(x)^2 \mathrm{d}\mu(x) < \infty$. If

$$\|\omega(x)(T_x - S_x)(f)\| \leq \lambda_1 \|\omega(x) T_x(f)\| + \lambda_2 \|\omega(x) S_x(f)\| + \varphi(x)\|f\|$$

for all $f \in \mathcal{H}$ and $x \in X$, then we say that $\{(S_x, \omega(x))\}_{x\in X}$ is a $(\lambda_1, \lambda_2, \varphi)$-perturbation of $\{(T_x, \omega(x))\}_{x\in X}$.

From now on, let $\{S_x\}_{x\in X}$ be a family of bounded operators on $\mathcal{H}$ such that the mapping $x \mapsto S_x(f)$ is weakly measurable. Then for each bounded operator $S : \mathcal{H} \to \mathcal{H}$, the map $x \mapsto S_x S(f)$ is weakly measurable. Hence by Lemma 2.9, we have the following theorem.

Theorem 2.12. *Let $\{(T_x, \omega(x))\}_{x\in X}$ be an uca-resolution of identity for $\mathcal{H}$ with bounds C and D, and let $\{(S_x, \omega(x))\}_{x\in X}$ be a $(\lambda_1, \lambda_2, \varphi)$-perturbation of $\{(T_x, \omega(x))\}_{x\in X}$ for some $0 \le \lambda_1, \lambda_2 < 1$. Moreover assume that $(1 - \lambda_1)\sqrt{C} - (\int_X \varphi(x)^2 \mathrm{d}\mu(x))^{\frac{1}{2}} > 0$ and for some $0 \le \lambda < 1$,*

$$\|\sum_{i\in I}(T_i - S_i)(f)\| \le \lambda \|\sum_{i\in I} T_i(f)\|, \quad f \in \mathcal{H},$$

for all finite subset I of X. Then there exists an invertible operator S on $\mathcal{H}$ such that $\{(S_x S^{-1}, \omega(x)\}_{x\in X}$ is a uca-resolution of the identity on $\mathcal{H}$.

Proof. First it should be noted that by Lemma 2.10, there exists an invertible operator S on $\mathcal{H}$, such that the family $\{S_x S^{-1}\}_{x\in X}$ satisfies in Definition 2.1(c). Also by open mapping theorem and closed graph theorem, there exist $A > 0$ and $B > 0$ such that

$$A\|f\| \le \|S^{-1}(f)\| \le B\|f\|$$

for all $f \in \mathcal{H}$.

Now, for $f \in \mathcal{H}$ we obtain

$$\begin{aligned}
&\left(\int_X \omega(x)^2 \|S_x(f)\|^2 \mathrm{d}\mu(x)\right)^{\frac{1}{2}} \\
&\le \left(\int_X \omega(x)^2 (\|T_x(f)\| + \|(T_x - S_x)(f)\|)^2 \mathrm{d}\mu(x)\right)^{\frac{1}{2}} \\
&\le \left(\int_X ((\omega(x)^2 \, (\|T_x(f)\| + \lambda_1 \|T_x(f)\|\| + \lambda_2 \|S_x(f)\|)) \right. \\
&\qquad \left. + \ \varphi(x)\|f\|)^2 \, \mathrm{d}\mu(x)\right)^{\frac{1}{2}} \\
&\le (1 + \lambda_1) \left(\int_X \omega(x)^2 \|T_x(f)\|^2 \mathrm{d}\mu(x)\right)^{\frac{1}{2}} \\
&+\lambda_2 \left(\int_X \omega(x)^2 \|S_x(f)\|^2 \mathrm{d}\mu(x)\right)^{\frac{1}{2}} + \|f\| \left(\int_X \varphi(x)^2 \mathrm{d}\mu(x)\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_X \omega(x)^2 \|S_x S^{-1}(f)\|^2 \mathrm{d}\mu(x) \\
&\le \left(\frac{(1 + \lambda_1)\sqrt{D} + (\int_x \varphi(x)^2 \mathrm{d}\mu(x))^{\frac{1}{2}}}{1 - \lambda_2}\right)^2 B^2 \|f\|^2.
\end{aligned}$$

To prove the lower bound, first we observe that

$$\|f\|^2 \leq \frac{1}{C}\int_X \omega(x)^2\|T_x(f)\|^2 \mathrm{d}\mu(x),$$

for all $f \in \mathcal{H}$. Therefore, by triangle inequality we have

$$\begin{aligned}
&\left(\int_X \omega(x)^2\|T_x(f)\|^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}} - \left(\int_X \omega(x)^2\|S_x(f)\|^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}}\\
&\quad\leq \left(\int_X \|\omega(x)(T_x - S_x)(f)\|^2\right)^{\frac{1}{2}}\\
&\quad\leq \lambda_1\left(\int_X \omega(x)^2\|T_x(f)\|^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}} + \lambda_2\left(\int_X \omega(x)^2\|S_x(f)\|^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}}\\
&\quad+\frac{1}{\sqrt{C}}\left(\int_X \varphi(x)^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}}\left(\int_X \omega(x)^2\|T_x(f)\|^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left(\frac{1-\lambda_1 - \frac{1}{\sqrt{C}}(\int_x \varphi(x)^2\mathrm{d}\mu(x))^{\frac{1}{2}}}{1+\lambda_2}\right)\left(\int_X \omega(x)^2\|T_x(f)\|^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}}\\
&\quad\leq \left(\int_X \omega(x)^2\|S_x(f)\|^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}}.
\end{aligned}$$

So

$$\begin{aligned}
&\left(\frac{(1-\lambda_1)\sqrt{C} - (\int_x \varphi(x)^2\mathrm{d}\mu(x))^{\frac{1}{2}}}{1+\lambda_2}\right)^2 A^2\|f\|^2\\
&\quad\leq \int_X \omega(x)^2\|S_x S^{-1}(f)\|^2\mathrm{d}\mu(x),
\end{aligned}$$

as we required. □

Remarks 2.13. Suppose $\{T_x\}_{x\in X}$ and $\{S_x\}_{x\in X}$ are two families of bounded operators on $\mathcal{H}$. If $\{(T_x, \omega(x))\}_{x\in X}$ is a uca-resolution of identity, then by Cauchy–Schwarz inequality we have

$$\begin{aligned}
|\langle T_x S_x(f), g\rangle| &= |\langle S_x(f), T_x^*(g)\rangle|\\
&\leq \|S_x(f)\|\|T_x^*\|\|g\|\\
&\leq \|S_x(f)\|\|g\|\sup_{x\in X}\|T_x\|,
\end{aligned}$$

for all $f, g \in \mathcal{H}$ and $x \in X$. Hence, for each $f \in \mathcal{H}$ and $x \in X$,

$$\|T_x S_x(f)\| \leq \|S_x(f)\|E,$$

where $E = \sup_{x\in X}\|T_x\|$.

Theorem 2.14. *Let $\{(T_x, \omega(x))\}_{x\in X}$ be an uca-resolution of identity for $\mathcal{H}$ with bounds C and D, and let $\{S_x\}_{x\in X}$ be a family of bounded operators on $\mathcal{H}$ such that for some K,*

$$\int_X \omega(x)^2 \|S_x(f)\|^2 \mathrm{d}\mu(x) \leq K\|f\|^2,$$

for all $f \in \mathcal{H}$. Suppose that $\varphi : X \to [0, \infty)$ is a positive map, and there exist $0 < \lambda_1, \lambda_2 < 1$ such that

$$\|\omega(x)f - \omega(x)T_xS_x(f)\| \leq \lambda_1\|\omega(x)T_x(f)\| + \lambda_2\|\omega(x)T_xS_x(f)\| + \varphi(x)\|f\|.$$

Also

$$\|\sum_{i\in I}(T_i - S_i)(f)\| \leq \lambda\|\sum_{i\in I} T_i(f)\|$$

for all finite subset I of X and for all $f \in \mathcal{H}$, where $0 < \lambda < 1$. If $\int_X \varphi(x)\mathrm{d}\mu(x) < \infty$ and $0 < (\int_X \omega(x)^2\mathrm{d}\mu(x))^{\frac{1}{2}} - \lambda_1\sqrt{D} - (\int_X \varphi(x)^2\mathrm{d}\mu(x)) < \infty$, then there exists an invertible operator S on $\mathcal{H}$ such that $\{(S_xS^{-1}, \omega(x))\}_{x\in X}$ is an uca-resolution of the identity on $\mathcal{H}$.

Proof. For $f \in \mathcal{H}$ we have

$$\begin{aligned}
&\|f\|\left(\int_X \omega(x)^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}} \\
&\leq \left(\int_X (\|\omega(x)f - \omega(x)T_xS_x(f)\| + \|\omega(x)T_xS_x(f)\|)^2\,\mathrm{d}\mu(x)\right)^{\frac{1}{2}} \\
&\leq \left(\int_X \|\omega(x)f - \omega(x)T_xS_x(f)\|^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}} \\
&+ \left(\int_X \|\omega(x)T_xS_x(f)\|^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}} \\
&\leq \left(\int_X (\lambda_1\|\omega(x)T_x(f)\| + \lambda_2\|\omega(x)^2T_xS_x(f)\| + \varphi(x)\|f\|)^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}} \\
&+ \left(\int_X \|\omega(x)T_xS_x(f)\|^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}} \\
&\leq \lambda_1\sqrt{D}\|f\| + (1+\lambda_2)\left(\int_X \omega(x)^2\|T_xS_x(f)\|^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}} \\
&+\|f\|\left(\int_X \varphi(x)^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}} \\
&\leq \lambda_1\sqrt{D}\|f\| + (1+\lambda_2)E\left(\int_X \omega(x)^2\|S_x(f)\|^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}} \\
&+\|f\|\left(\int_X \varphi(x)^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}}
\end{aligned}$$

where $E = \sup_{x\in X} \|T_x\|$. Therefore

$$\|f\| \frac{(\int_X \omega(x)^2 \mathrm{d}\mu(x))^{\frac{1}{2}} - \lambda_1\sqrt{D} - (\int_X \varphi(x)^2 \mathrm{d}\mu(x))^{\frac{1}{2}}}{E(1+\sqrt{\lambda_2})}$$

$$\leq \left(\int_X \omega(x)^2 \|S_x(f)\|^2 \mathrm{d}\mu(x)\right)^{\frac{1}{2}}.$$

Now by Lemma 2.10, and similar to the proof of of Theorem 2.12, the assertion holds. □

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