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Split Malcev algebras

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Abstract. We study the structure of split Malcev algebras of arbitrary dimension over an algebraically closed field of characteristic zero. We show that any such algebras M is of the form $M = \mathcal{U} + \sum_j I_j$ with $\mathcal{U}$ a subspace of the abelian Malcev subalgebra H and any I_j a well described ideal of M satisfying $[I_j, I_k] = 0$ if $j \neq k$. Under certain conditions, the simplicity of M is characterized and it is shown that M is the direct sum of a semisimple split Lie algebra and a direct sum of simple non-Lie Malcev algebras.

Keywords. Malcev algebras; structure theory; roots; root spaces.

1. Introduction and preliminaries

The class of Malcev algebras contains one of the Lie algebras and so a question arises whether some known results on Lie algebras can be extended to the framework of Malcev algebras (see [4, 7, 9, 10]). In the present paper, we are interested in studying the structure of arbitrary Malcev algebras by focussing on the split ones. After introducing the concept of split Malcev algebra as the natural extension of one of the split Lie algebra (see [1, 6]), we improve in §2 the techniques of connections of roots introduced for split Lie algebras and split Lie triple systems in [1, 2], so as to develop a theory of connections of roots for split Malcev algebras M which let us prove the first decompositions of M. Finally, in §3 and under certain conditions, the simplicity of M is characterized and it is shown that M is the direct sum of a semisimple split Lie algebra and a direct sum of simple non-Lie Malcev algebras. Throughout this paper, $\mathbb{K}$ denotes an algebraically closed field of characteristic zero.

DEFINITION 1.1

Denote by H a maximal abelian subalgebra (MASA) of a Malcev algebra $(M, [\cdot, \cdot])$. For a linear functional $\alpha : H \longrightarrow \mathbb{K}$, we define the root space of M associated to α as the subspace $M_\alpha = \{v_\alpha \in M : [h, v_\alpha] = \alpha(h)v_\alpha \text{ for any } h \in H\}$. The elements $\alpha \in H^*$ satisfying $M_\alpha \neq 0$ are called roots of M and we denote by $\Lambda := \{\alpha \in H^* \setminus \{0\} : M_\alpha \neq 0\}$ the root system of M. We say that M is a split Malcev algebra if $M = H \oplus (\bigoplus_{\alpha \in \Lambda} M_\alpha)$.

We also say that a root system Λ is *symmetric* if it satisfies that $\alpha \in \Lambda$ implies $-\alpha \in \Lambda$. It is clear that the root space associated to the zero root satisfies $M_0 = H$, and that split Lie algebras are examples of split Malcev algebras. As in the finite dimensional case

[3, 5], we can show that if $[M_\alpha, M_\beta] \neq 0$ with $\beta \neq \alpha$ then $\alpha + \beta \in \Lambda \cup \{0\}$ and $[M_\alpha, M_\beta] \subseteq M_{\alpha+\beta}$; and that if $[M_\alpha, M_\alpha] \neq 0$ then $[M_\alpha, M_\alpha] \subseteq M_{2\alpha} + M_{-\alpha}$.

2. Connections of roots: Decompositions

In the following, M denotes a split Malcev algebra with a symmetric root system Λ. Let us denote by

$$\begin{aligned}\Omega &= \{\alpha \in \Lambda : [M_\alpha, M_{-\alpha}] \neq 0\} \\ &\cup \{\alpha \in \Lambda : [[M_\beta, M_{-\beta}], M_\alpha] \neq 0 \text{ for some } \beta \in \Lambda\}.\end{aligned}$$

We associate to any $\alpha \in \Omega$ the symbol θ_α and denote $\Theta_\Omega = \{\theta_\alpha : \alpha \in \Omega\}$. Let us define the mapping $+ : (\Lambda \cup \Theta_\Omega) \times \Lambda \to H^* \cup \Theta_\Omega$, where H^* is the dual space of H, as follows:

- For $\alpha \in \Lambda$, $\alpha + (-\alpha) = \begin{cases} \theta_\alpha, & \text{if } \alpha \in \Omega, \\ 0, & \text{if } \alpha \notin \Omega \end{cases}$.
- For $\alpha, \beta \in \Lambda$ with $\beta \neq -\alpha$, we define $\alpha + \beta \in H^*$ as the usual sum of linear functionals, that is $(\alpha + \beta)(h) = (\alpha + \beta)(h) = \alpha(h) + \beta(h)$ for any $h \in H$.
- For $\theta_\alpha \in \Theta_\Omega$ and $\beta \in \Lambda$,

$$\theta_\alpha + \beta = \begin{cases} \beta, & \text{if either } [[M_\alpha, M_{-\alpha}], M_\beta] \neq 0 \text{ or } [[M_\beta, M_{-\beta}], M_\alpha] \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

  where 0 denotes the zero root.

Since $\alpha \in \Omega$ implies $-\alpha \in \Omega$, we get that if $\alpha + (-\alpha) = \theta_\alpha$ then $-\alpha + \alpha = \theta_{-\alpha}$. The below lemma is a direct consequence of the above definition.

Lemma 2.1. *The following assertions hold.*

(1) *For any $\alpha \in \Omega$ and $\beta \in \Lambda$ such that $\theta_\alpha + \beta = \beta$ we have*

  (i) $\beta \in \Omega$ *and* $\beta + (-\beta) = \theta_\beta$.

  (ii) $\theta_\beta + \alpha = \alpha$ *and* $\theta_{-\alpha} + (-\beta) = -\beta$.

(2) *For any $\alpha, \beta, \gamma, \delta \in \Lambda$ we have*

  (i) if $\alpha + \beta = \delta$, *then* $\delta + (-\beta) = \alpha$ *and* $-\alpha + (-\beta) = -\delta$;

  (ii) if $(\alpha + \beta) + \gamma = \delta$ *with* $\alpha + \beta \in \Theta_\Omega$, *then* $\beta = -\alpha$, $\delta = \gamma$, $\delta + (-\gamma) = \theta_\gamma$, $(\delta + (-\gamma)) + (-\beta) = \alpha$, $-\alpha + (-\beta) = \theta_{-\alpha}$ *and* $(-\alpha + (-\beta)) + (-\gamma) = -\delta$.

DEFINITION 2.1

Let α and β be two nonzero roots. We say that α is connected to β if there exist $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that

(1) $\alpha_1 = \alpha$;
(2) $\{\alpha_1 + \alpha_2, (\alpha_1 + \alpha_2) + \alpha_3, \ldots, (\cdots((\alpha_1 + \alpha_2) + \alpha_3) + \cdots) + \alpha_{n-1}\} \subset \Lambda \cup \Theta_\Omega$;
(3) $((\cdots((\alpha_1 + \alpha_2) + \alpha_3) + \cdots) + \alpha_{n-1}) + \alpha_n \in \pm\beta$.

We also say that $\{\alpha_1, \ldots, \alpha_n\}$ is a connection from α to β.

PROPOSITION 2.1

The relation $\sim$ in Λ, defined by $\alpha \sim \beta$ if and only if α is connected to β, is of equivalence.

Proof. $\{\alpha\}$ is a connection from α to itself and therefore $\alpha \sim \alpha$. Let us see the symmetric character of $\sim$. If $\alpha \sim \beta$, there is a connection $\{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}, \alpha_n\}$ from α to β. If $n = 1$, then $\alpha_1 = \alpha = \pm\beta$ and so $\{\beta\}$ is a connection from β to α. Suppose $n \geq 2$. We can distinguish two possibilities. In the first one $((\cdots((\alpha_1+\alpha_2)+\alpha_3)+\cdots)+\alpha_{n-1})+\alpha_n = \beta$ and in the second one

$$((\cdots((\alpha_1 + \alpha_2) + \alpha_3) + \cdots) + \alpha_{n-1}) + \alpha_n = -\beta. \tag{1}$$

Suppose we have the first one. By the symmetry of Λ, we can consider the set of nonzero roots $\{\beta, -\alpha_n, -\alpha_{n-1}, \ldots, -\alpha_3, -\alpha_2\} \subset \Lambda$. Let us show that this set is a connection from β to α. Definition 2.1(2) gives us two options for the expression $(\cdots((\alpha_1 + \alpha_2) + \alpha_3) + \cdots) + \alpha_{n-1}$. If $(\cdots((\alpha_1 + \alpha_2) + \alpha_3) + \cdots) + \alpha_{n-1} \in \Lambda$, Lemma 2.1(2(i)) implies

$$\beta + (-\alpha_n) = (\cdots((\alpha_1 + \alpha_2) + \alpha_3) + \cdots) + \alpha_{n-1} \in \Lambda. \tag{2}$$

If $(\cdots((\alpha_1 + \alpha_2) + \alpha_3) + \cdots) + \alpha_{n-1} \in \Theta_\Omega$, then necessarily $n \geq 3$ and

$$(\cdots((\alpha_1 + \alpha_2) + \alpha_3) + \cdots) + \alpha_{n-2} \in \Lambda.$$

Lemma 2.1(2(ii)) shows $\beta + (-\alpha_n) = \theta_{\alpha_n} \in \Theta_\Omega$ and

$$(\beta + (-\alpha_n)) + (-\alpha_{n-1}) = (\cdots((\alpha_1 + \alpha_2) + \alpha_3) + \cdots) + \alpha_{n-2} \in \Lambda. \tag{3}$$

Now, we can argue in a similar way from equations (2) and (3), taking into account Lemma 2.1(2), to conclude $(\cdots((\beta + (-\alpha_n)) + (-\alpha_{n-1})) + \cdots) + (-\alpha_2) = \alpha_1$ and so $\{\beta, -\alpha_n, -\alpha_{n-1}, \ldots, -\alpha_3, -\alpha_2\}$ is a connection from β to α.

Suppose we are in the second possibility, that is, as given by equation (1). Let us show that $\{\beta, \alpha_n, \alpha_{n-1}, \ldots, \alpha_3, \alpha_2\}$ is a connection from β to α. We begin by observing that, taking into account condition (2) in Definition 2.1, a recursive argument with Lemma 2.1(2) and the fact that if $\alpha + (-\alpha) = \theta_\alpha$, then $-\alpha + \alpha = \theta_{-\alpha}$. Let us assert that if $(\cdots((\alpha_1 + \alpha_2) + \alpha_3) + \cdots) + \alpha_i \in \Lambda$ for $i = 2, \ldots, n$, then

$$\begin{aligned}&(\cdots(((-\alpha_1) + (-\alpha_2)) + (-\alpha_3)) + \cdots) + (-\alpha_i)\\ &\quad = -((\cdots((\alpha_1 + \alpha_2) + \alpha_3) + \cdots) + \alpha_i),\end{aligned}$$

and that if $(\cdots((\alpha_1 + \alpha_2) + \alpha_3) + \cdots) + \alpha_i = \theta_{-\alpha_i} \in \Theta_\Omega$ for $i = 2, \ldots, n-1$, with $((\cdots((\alpha_1 + \alpha_2) + \alpha_3) + \cdots) + \alpha_i) + \alpha_{i+1} \in \Lambda$, then

$$(\cdots(((-\alpha_1) + (-\alpha_2)) + (-\alpha_3)) + \cdots) + (-\alpha_i) = \theta_{\alpha_i}$$

and

$$\begin{aligned}&((\cdots(((-\alpha_1) + (-\alpha_2)) + (-\alpha_3)) + \cdots) + (-\alpha_i)) + (-\alpha_{i+1})\\ &\quad = -(((\cdots((\alpha_1 + \alpha_2) + \alpha_3) + \cdots) + \alpha_i) + \alpha_{i+1}).\end{aligned}$$

In particular, by considering equation (1), we have

$$\beta = ((\cdots(((-\alpha_1) + (-\alpha_2)) + (-\alpha_3)) + \cdots) + (-\alpha_{n-1})) + (-\alpha_n). \tag{4}$$

Taking into account the above observation, we can argue as in the first possibility that $\{\beta, \alpha_n, \alpha_{n-1}, \ldots, \alpha_3, \alpha_2\}$ is a connection from β to α and to conclude that $\sim$ is symmetric.

Finally, suppose $\alpha \sim \beta$ and $\beta \sim \gamma$, and write $\{\alpha_1, \ldots, \alpha_n\}$ for a connection from α to β and $\{\beta_1, \ldots, \beta_m\}$ for a connection from β to γ. If $m = 1$, then $\gamma \in \pm\beta$ and so $\{\alpha_1, \ldots, \alpha_n\}$ is a connection from α to γ. If $m \geq 2$, we have that $\{\alpha_1, \ldots, \alpha_n, \beta_2, \ldots, \beta_m\}$ is a connection from α to γ in case $(\cdots(\alpha_1 + \alpha_2) + \cdots) + \alpha_n = \beta$, and taking into account the observation given by equations (1) and (4), that $\{\alpha_1, \ldots, \alpha_n, -\beta_2, \ldots, -\beta_m\}$ is a connection from α to γ in case $(\cdots(\alpha_1 + \alpha_2) + \cdots) + \alpha_n = -\beta$. Therefore $\alpha \sim \gamma$ and $\sim$ is of equivalence. □

Given $\alpha \in \Lambda$, we denote by $\Lambda_\alpha = \{\beta \in \Lambda : \alpha \text{ and } \beta \text{ are connected}\}$, and define $H_{\Lambda_\alpha} := \text{span}_{\mathbb{K}}\{[M_\beta, M_{-\beta}] : \beta \in \Lambda_\alpha\}$ and $V_{\Lambda_\alpha} := \bigoplus_{\beta \in \Lambda_\alpha} M_\beta$. It is easy to verify that $M_{\Lambda_\alpha} := H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}$ is a Malcev subalgebra of M that we call the Malcev subalgebra *associated* to Λ_α.

PROPOSITION 2.2

If $\gamma \notin \Lambda_\alpha$, then $[M_\beta, M_\gamma] = 0$ and $[[M_\beta, M_{-\beta}], M_\gamma] = 0$ for any $\beta \in \Lambda_\alpha$.

Proof. Let us suppose that there exists $\beta \in \Lambda_\alpha$ such that $[M_\beta, M_\gamma] \neq 0$ with $\gamma \notin \Lambda_\alpha$. Then $\gamma \neq \pm\beta$ and $\beta + \gamma \in \Lambda$. From here, we easily get that α is connected to $\beta + \gamma$, that is, $\beta + \gamma \in \Lambda_\alpha$. Taking into account $-\beta, \beta + \gamma \in \Lambda_\alpha$, we deduce $\gamma \in \Lambda_\alpha$, a contradiction. Therefore $[M_\beta, M_\gamma] = 0$ for any $\beta \in \Lambda_\alpha$ and $\gamma \notin \Lambda_\alpha$. Finally, suppose $[[M_\beta, M_{-\beta}], M_\gamma] \neq 0$. Then $\{\beta, -\beta, \gamma\}$ is a connection from β to γ and so $\gamma \in \Lambda_\alpha$, a contradiction. Hence, $[[M_\beta, M_{-\beta}], M_\gamma] = 0$. □

Theorem 2.1. *The following assertions hold*:

1. *For any $\alpha \in \Lambda$, the Malcev subalgebra $M_{\Lambda_\alpha} = H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}$ of M associated to Λ_α is an ideal of M.*
2. *If M is simple, then there exists a connection from α to β for any $\alpha, \beta \in \Lambda$ and $H = \sum_{\alpha \in \Lambda} [M_\alpha, M_{-\alpha}]$.*
3. *For a vector space complement $\mathcal{U}$ of* $\text{span}_{\mathbb{K}}\{[M_\alpha, M_{-\alpha}] : \alpha \in \Lambda\}$ *in H, we have $M = \mathcal{U} + \sum_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}$, where any $I_{[\alpha]}$ is one of the ideals M_{Λ_α} of M described in item* (1), *satisfying $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$.*

Proof.

(1) By Proposition 2.2, we get

$$[M_{\Lambda_\alpha}, M] = \left[\bigoplus_{\beta \in \Lambda_\alpha} [M_\beta, M_{-\beta}] \oplus \bigoplus_{\beta \in \Lambda_\alpha} M_\beta, H \oplus \left(\bigoplus_{\beta \in \Lambda_\alpha} M_\beta\right)\right.$$

$$\left. \oplus \left(\bigoplus_{\gamma \notin \Lambda_\alpha} M_\gamma\right)\right] \subset M_{\Lambda_\alpha}.$$

(2) The simplicity of M implies $M_{\Lambda_\alpha} = M$. Therefore $\Lambda_\alpha = \Lambda$ and $H = \sum_{\alpha\in\Lambda}[M_\alpha, M_{-\alpha}]$.
(3) Consequence of Proposition 2.1(1) and Proposition 2.2.

COROLLARY 2.1

If the center of M is zero, ($\mathcal{Z}(M) = 0$), and $[M, M] = M$, then M is the direct sum of the ideals given in Theorem 2.1, $M = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$.

Proof. From $[M, M] = M$, we have $M = \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$. The direct character of the sum now follows from the facts $[I_{[\alpha]}, I_{[\beta]}] = 0$, if $[\alpha] \neq [\beta]$ and $\mathcal{Z}(M) = 0$. □

3. The simple components

Recall that any simple Malcev algebra over $\mathbb{K}$ is either a Lie algebra or a seven-dimensional algebra over its centroid, denoted by $\mathfrak{C}_0$. This simple non-Lie Malcev algebra is a split one under the decomposition $\mathfrak{C}_0 = \mathfrak{H} \oplus (\mathfrak{C}_0)_\rho \oplus (\mathfrak{C}_0)_{-\rho}$, where $\mathfrak{H}$ is a one-dimensional MASA of $\mathfrak{C}_0$.

The following lemma is consequence of the fact that the set of multiplications by elements in H is a commuting set of diagonalizable endomorphisms, and I is invariant under this set.

Lemma 3.1. *Let $M = H \oplus (\bigoplus_{\alpha\in\Lambda}(I \cap M_\alpha)$ be a split Malcev algebra. If I is an ideal of M, then $I = (I \cap H) \oplus (\bigoplus_{\alpha\in\Lambda}(I \cap M_\alpha))$.*

We take the following definitions from the theory of split Lie algebras and split Lie triple systems [2, 6].

DEFINITION 3.1

We say that a split Malcev algebra M is root-multiplicative if $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$, such that $\alpha + \beta \in \Lambda$ or $\theta_\alpha + \beta \in \Lambda$. Then $[M_\alpha, M_\beta] = M_{\alpha+\beta}$ or $[[M_\alpha, M_{-\alpha}], M_\beta] = M_\beta$ respectively.

We also say that a nonzero root α of a split Malcev algebra M is *abelian* if there exists $0 \neq e_\alpha \in M_\alpha$ such that $[e_\alpha, M_{-\alpha}] = 0$.

We are interested in split Malcev algebras with no abelian nonzero roots. As examples of root-multiplicative split Malcev algebras satisfying this fact we have the non-Lie simple Malcev algebra $\mathfrak{C}_0$ (see the multiplication table in (§6 of [8]), and so all are of finite dimensional semisimple Malcev algebras (over an algebraically closed field). We also have semisimple separable L^*-algebras and semisimple locally finite split Lie algebras over a field of characteristic zero [6].

Theorem 3.1. *Let M be a root-multiplicative split Malcev algebra with no abelian nonzero roots and with $\mathcal{Z}(M) = 0$. Then M is simple if and only if it has all its nonzero roots connected and $H = \sum_{\alpha\in\Lambda}[M_\alpha, M_{-\alpha}]$.*

Proof. The first implication is Theorem 2.1(2). To prove the converse, consider I a nonzero ideal of L. By Lemma 3.1, we can write $I = (I \cap H) \oplus (\bigoplus_{\alpha\in\Lambda_I}(I \cap M_\alpha))$ with

$\Lambda_I = \{\alpha \in \Lambda : I \cap M_\alpha \neq 0\}$ and, taking into account $\mathcal{Z}(M) = 0$, with $\Lambda_I \neq \emptyset$. We can assert that there exists $\alpha \in \Lambda^I$ such that $\alpha(I \cap H) \neq 0$. Indeed, first observe that $I \cap H \neq 0$, because in the opposite case we have that for any $0 \neq e_\alpha \in I \cap M_\alpha$ necessarily $[e_\alpha, M_{-\alpha}] = 0$, α is an abelian root of M which is a contradiction. Second, if $\alpha(I \cap H) = 0$ for any $\alpha \in \Lambda^I$, then $[I \cap H, M_\alpha] = 0$. As we also have $[I \cap H, H] = 0$ and $[I \cap H, M_\beta] = 0$ for any $\beta \in \Lambda \setminus \Lambda^I$, we conclude $I \cap H \subset \mathcal{Z}(M) = 0$, which contradicts the fact that $I \cap H \neq 0$. Let us show that $H \subset I$. By the above, we can take $\alpha_0 \in \Lambda_I$ satisfying $\alpha_0(I \cap H) \neq 0$. From here, $[I \cap H, M_{\alpha_0}] = M_{\alpha_0}$ and so $M_{\alpha_0} \subset I$. Now, for any $\beta \in \Lambda \setminus \{\pm 2\alpha_0\}$, the fact that α_0 and β are connected and the root-multiplicativity of M give us a connection $\{\gamma_1, \ldots, \gamma_r\}$ from α_0 to β such that

$$\gamma_1 = \alpha_0, \gamma_1+\gamma_2, (\gamma_1+\gamma_2)+\gamma_3, \ldots, (\cdots(\gamma_1+\gamma_2)+\gamma_3)+\cdots)+\gamma_{r-1} \in \Lambda \cup \Theta_\Omega,$$

$(\cdots(\gamma_1 + \gamma_2) + \gamma_3) + \cdots) + \gamma_r \in \pm\beta$ and $[[\cdots[[M_{\alpha_0}, M_{\gamma_2}], M_{\gamma_3}], \cdots], M_{\gamma_r}] = M_{\epsilon\beta}$, with $\epsilon \in \pm 1$. From here, we deduce that either $M_\beta \subset I$ or $M_{-\beta} \subset I$. In both cases $[M_\beta, M_{-\beta}] \subset I$. From here, the fact that $H = \sum_{\beta \in \Lambda} [M_\beta, M_{-\beta}]$ finally gives us $H \setminus \{[M_{2\alpha_0}, M_{-2\alpha_0}]\} \subset I$. Consider now the bracket $[M_{\alpha_0}, M_{-\alpha_0}]$. Since α_0 is a non abelian root, this product is nonzero and taking into account $\mathcal{Z}(M) = 0$, there exists $\delta \in \Lambda$ such that $[[M_{\alpha_0}, M_{-\alpha_0}], M_\delta] \neq 0$, being so $M_\delta \subset I$. If $\delta \in \pm\alpha_0$ then $[[M_{\alpha_0}, M_{-\alpha_0}], M_{2\alpha_0}] = M_{2\alpha_0} \subset I$, and in case $\delta \notin \pm\alpha_0$, then $2\delta \notin \pm 2\alpha_0$ and we can argue from δ as we did above with α_0 to get $[M_{2\alpha_0}, M_{-2\alpha_0}] \subset I$. Consequently we can assert $H \subset I$. Given now any $\alpha \in \Lambda$, the facts $\alpha \neq 0$ and $H \subset I$ show $[H, M_\alpha] = M_\alpha \subset I$. We conclude $I = M$ and therefore M is simple. $\square$

Theorem 3.2. *Let M be a root-multiplicative split Malcev algebra with no abelian nonzero roots and satisfying $\mathcal{Z}(M) = 0, [M, M] = M$. Then M is the direct sum of the family of its minimal ideals, each one being a simple split Malcev algebra having all its nonzero roots connected.*

Proof. By Corollary 2.1, $M = \bigoplus_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}$ is the direct sum of the ideals $I_{[\alpha]} = H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}$ having any $I_{[\alpha]}$ its root system, Λ_α, with all of its roots connected. Even more, Λ_α has all of its roots Λ_α-connected (connected through roots in Λ_α). We also have that any of the $I_{[\alpha]}$ is root-multiplicative as a consequence of the root-multiplicativity of M. Clearly $I_{[\alpha]}$ have no abelian nonzero roots, and finally $\mathcal{Z}_{I_{[\alpha]}}(I_{[\alpha]}) = 0$ where $\mathcal{Z}_{I_{[\alpha]}}(I_{[\alpha]})$ denotes the center $I_{[\alpha]}$ in $I_{[\alpha]}$, as a consequence of $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$ (Theorem 2.1(3)), and $\mathcal{Z}(M) = 0$. We can apply Theorem 3.1 to any $I_{[\alpha]}$ so as to conclude $I_{[\alpha]}$ is simple. It is clear that the decomposition $M = \bigoplus_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}$ satisfies the assertions of the theorem. $\square$

In the following corollary the term *semisimple algebra* means direct sum of simple algebras.

COROLLARY 3.1

Let M be a root-multiplicative split Malcev algebra with no abelian nonzero roots and satisfying $\mathcal{Z}(M) = 0, [M, M] = M$. Then M is the direct sum of a semisimple split Lie algebra and a direct sum of simple non-Lie Malcev algebras (seven dimensional over their centroid).

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