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The cohomology of orbit spaces of certain free circle group actions

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Abstract. Suppose that $G = \mathbb{S}^1$ acts freely on a finitistic space X whose (mod p) cohomology ring is isomorphic to that of a lens space $L^{2m-1}(p; q_1, \ldots, q_m)$ or $\mathbb{S}^1 \times \mathbb{C}P^{m-1}$. The mod p index of the action is defined to be the largest integer n such that $\alpha^n \neq 0$, where $\alpha \in H^2(X/G; \mathbb{Z}_p)$ is the nonzero characteristic class of the $\mathbb{S}^1$-bundle $\mathbb{S}^1 \hookrightarrow X \to X/G$. We show that the mod p index of a free action of G on $\mathbb{S}^1 \times \mathbb{C}P^{m-1}$ is $p-1$, when it is defined. Using this, we obtain a Borsuk–Ulam type theorem for a free G-action on $\mathbb{S}^1 \times \mathbb{C}P^{m-1}$. It is note worthy that the mod p index for free G-actions on the cohomology lens space is not defined.

Keywords. Characteristic class; finitistic space; free action; index; spectral sequence.

1. Introduction

Let X be a topological space and G a topological group acting continuously on X. The set $\hat{x} = \{gx | g \in G\}$ is called the orbit of x. The set of all orbits $\hat{x}$, $x \in X$ is denoted by X/G and assigned the quotient topology induced by the natural projection $\pi : X \to X/G$, $x \to \hat{x}$. An action of G on X is said to be free if $g(x) = x$, for any $x \in X \Rightarrow g = e$, the identity element of G. The orbit space of a free transformation group (G, S^n), where G is a finite group, has been studied extensively [2, 7, 8, 10, 15]. However, a little is known if the total space X is a compact manifold other than a sphere [3, 6, 9, 14]. The orbit space of a free involution on a real or complex projective space has been studied in [13]. We have also determined the cohomology algebra of the orbit space of free actions of Z_p on a generalized lens space $L^{2m-1}(p; q_1, q_2, \ldots, q_m)$ in [12]. In this note, we determine the mod p cohomology algebra of obrit spaces of free actions of circle group $\mathbb{S}^1$ on real projective space, lens space and $\mathbb{S}^1 \times \mathbb{C}P^{m-1}$. Note that $\mathbb{S}^1$ can not act freely on a ‘finitistic’ space having integral cohomology of a finite-dimensional complex projective space or a quaternionic projective space (Theorem 7.10 of Chapter III, [1]). We recall that a paracompact Hausdorff space is finitistic if every open covering has a finite-dimensional refinement.

Throughout this paper, $H^*(X)$ will denote the Čech cohomology of the space X with coefficients in a field $F = \mathbb{Z}_p$ or $\mathbb{Q}$ (the field of rational numbers). It is known that $H^*(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2[a]/\langle a^{n+1}\rangle$, where deg $a = 1$, and $H^*(L^{2m-1}(p; q_1, \ldots, q_m); \mathbb{Z}_p) =$

$\wedge(a) \otimes \mathbb{Z}_p[b]/\langle b^m \rangle$, $\deg a = 1$, $\beta(a) = b$, where β is the Bockstein homomorphism associated with the coefficient sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$. By $X \sim_F Y$, we shall mean that $H^*(X; F)$ and $H^*(Y; F)$ are isomorphic. We prove the following results.

Theorem 1.1. *Let $G = \mathbb{S}^1$ act freely on a finitistic space $X \sim_F \mathbb{S}^1 \times \mathbb{C}P^{m-1}$, $F = \mathbb{Z}_p$ or $\mathbb{Q}$. Then $H^*(X/G; F)$ is*

(i) $F[z]/\langle z^m \rangle$, $\deg z = 2$.

(ii) $\dfrac{\mathbb{Z}_p[x, y_1, y_3, \ldots, y_{2p-3}, z]}{\langle x^p, z^n, xy_q, y_q y_{q'} - A_{qq'} x^{\frac{q+q'}{2}} - B_{qq'} z x^{\frac{q+q'-2p}{2}} \rangle}$, *where* $m = np$, $\deg x = 2$, $\deg y_q = q$, $\deg z = 2p$, $A_{qq'} = 0$ *when* $q + q' > 2p$, $B_{qq'} = 0$ *when* $q + q' < 2p$ *and both* $A_{qq'}$ *and* $B_{qq'}$ *are zero when* $q = q'$ *or* $q + q' = 2p$. *If* $F = \mathbb{Q}$, *then we have only the case* (i).

For free actions of circle group on a cohomology lens space, we have the following theorem.

Theorem 1.2. *Let $G = \mathbb{S}^1$ act freely on a finitistic space X with* mod p *cohomology of the lens space $L^{2m-1}(p; q_1, q_2, \ldots, q_m)$, p a prime. Then*

$$H^*(X/G; \mathbb{Z}_p) = \mathbb{Z}_p[z]/\langle z^m \rangle, \quad \deg z = 2.$$

Let $G = \mathbb{S}^1$ act freely on a space X. Then there is an orientable 1-sphere bundle $\mathbb{S}^1 \hookrightarrow X \xrightarrow{\nu} X/G$, where ν denotes the orbit map. Let $\alpha \in H^2(X/G; \mathbb{Z})$ be its characteristic class. Jaworowski [4] has defined the (integral) index of a free $\mathbb{S}^1$-action on the space X to be the largest integer n (if it exists) such that $\alpha^n \neq 0$. Similarly, one can define mod p index of a free $\mathbb{S}^1$-action on a space X. Jaworowski has shown that the (integral or rational) $\mathbb{S}^1$-index of $L^{2m-1}(p; q_1, q_2, \ldots, q_m)$ is $m - 1$. It follows from the Thom-Gysin sequence for bundle $\mathbb{S}^1 \hookrightarrow X \to X/G$ that the characteristic class is zero for $X \sim_p L^{2m-1}(p; q_1, q_2, \ldots, q_m)$. So, the mod p index is not defined for a cohomology lens space. We show that the mod p index of a free action of $\mathbb{S}^1$ on a $\mathbb{S}^1 \times \mathbb{C}P^{m-1}$ is $p - 1$, provided that characteristic class is nonzero. It should be noted that $G = \mathbb{S}^1$ can not act freely on $X \sim_2 \mathbb{R}P^{2m}$.

2. Preliminaries

Let $G = \mathbb{S}^1$ act on a paracompact Hausdorff space X. Then there is an associated fibration $X \xrightarrow{i} X_G \xrightarrow{\pi} B_G$, where $X_G = (E_G \times X)/G$ and $E_G = \mathbb{S}^\infty \to B_G = \mathbb{C}P^\infty$ are universal G-bundles. It is known that B_G is a CW-complex with $2N$-skeleton $\mathbb{C}P^N$ for all N and E_G is a CW-complex with $2N + 1$-skeleton $\mathbb{S}^{2N+1}$. Write $E_G^N = \mathbb{S}^{2N+1}$ and $B_G^N = \mathbb{C}P^N$. Then, $H^i(E_G^N) = 0$ for $0 < i < 2N + 1$. Let $X_G^N = X \times_G E_G^N$ be the associated bundle over B_G^N with fibre X. Then the equivariant projection $X \times E_G^N \to X$ induces the map $\phi : X_G^N \to X/G$. Let G act freely on X. Then

$$\phi^* : H^i(X/G) \to H^i(X_G)$$

is an isomorphism for all $i < 2N + 1$ with coefficient group $\mathbb{Z}_p$, p a prime, by Vietoris–Begle mapping theorem. By $H^i(X_G)$ we mean $H^i(X_G^N)$, N large.

To compute $H^*(X_G)$ we exploit the Leray–Serre spectral sequence of the map $\pi : X_G \to B_G$ with coefficients in $\mathbb{Z}_p$, p being a prime. The edge homomorphisms

$$H^p(B_G) = E_2^{p,0} \to E_3^{p,0} \to \cdots \to E_{p+1}^{p,0} = E_\infty^{p,0} \subseteq H^p(X_G),$$

and

$$H^q(X_G) \to E_\infty^{0,q} = E_{q+1}^{0,q} \subset \ldots \subset E_2^{0,q} = H^q(X)$$

are the homomorphisms

$$\pi^* : H^p(B_G) \to H^p(X_G) \quad \text{and} \quad i^* : H^q(X_G) \to H^q(X),$$

respectively. We also recall the fact that the cup product in E_{r+1} is induced from that in E_r via the isomorphism $E_{r+1} \cong H^*(E_r)$. For the above facts, we refer to McCleary [5].

3. Proofs

To prove our theorems, we need the following:

PROPOSITION 3.1

Let $G = \mathbb{S}^1$ act freely on a finitistic space X with $H^i(X) = 0$ for all $i > n$. Then $H^i(X/G) = 0$ for all $i \geq n$ with coefficient group $\mathbb{Z}_p$, p being a prime.

Proof. We recall that the bundle $\mathbb{S}^1 \hookrightarrow X \xrightarrow{\nu} X/G$ is orientable, where $\nu : X \to X/G$ is the orbit map. Consider, the Thom–Gysin sequence

$$\cdots \to H^i(X/G) \xrightarrow{\nu^*} H^i(X) \xrightarrow{\lambda^*} H^{i-1}(X/G) \xrightarrow{\mu^*} H^{i+1}(X/G) \to \cdots$$

of the bundle, where μ^* is the multiplication by a characteristic class $\alpha \in H^2(X/G)$. This implies that $H^i(X/G) \xrightarrow{\mu^*} H^{i+2}(X/G)$ is an isomorphism for all $i \geq n$. Since X is finitistic, X/G is also finitistic [11]. Therefore, $H^*(X/G)$ can be defined as the direct limit of $H^*(K(\mathcal{U}))$, where $K(\mathcal{U})$ denotes the nerve of $\mathcal{U}$ and $\mathcal{U}$ runs over all finite dimensional open coverings of X/G. Let $\beta \in H^i(X/G)$ be arbitrary. Then, we find a finite dimensional covering $\mathcal{V}$ of X/G and elements $\alpha' \in H^2(K(\mathcal{V}))$, $\beta' \in H^i(K(\mathcal{V}))$ such that $\rho(\alpha') = \alpha$ and $\rho(\beta') = \beta$ where $\rho : \sum_{\mathcal{U}} H^i(K(\mathcal{U})) \longrightarrow H^i(X/G)$ is the canonical map. Consequently, we have $(\alpha')^k \beta' = 0$ for $2k + i > \dim \mathcal{V}$, which implies that $(\mu^*)^k(\beta) = \alpha^k \beta = 0$. Thus $\beta = 0$, and the proposition follows.

Now, we prove our main theorems.

Proof of Theorem 1.1. The case $m = 1$ is trivial. So we assume $m > 1$. Since $G = \mathbb{S}^1$ acts freely on X, the Leray–Serre spectral sequence of the map $\pi : X_G \to B_G$ does not collapse at the E_2-term. As $\pi_1(B_G)$ is trivial, the fibration $X \xrightarrow{i} X_G \xrightarrow{\pi} B_G$ has a simple system of local coefficients on B_G. So the spectral sequence has

$$E_2^{k,l} \cong H^k(B_G) \otimes H^l(X).$$

Let $a \in H^1(X)$ and $b \in H^2(X)$ be generators of the cohomology ring $H^*(X)$. Then $a^2 = 0$ and $b^m = 0$. Consequently, we have either ($d_2(1 \otimes a) = t \otimes 1$ and $d_2(1 \otimes b) = 0$) or ($d_2(1 \otimes a) = 0$ and $d_2(1 \otimes b) = t \otimes a$).

Case I. If $d_2(1 \otimes a) = t \otimes 1$ and $d_2(1 \otimes b) = 0$, then

$$d_2 : E_2^{k,l} \to E_2^{k+2,l-1}$$

is an isomorphism for k even and l odd and trivial homomorphism for the remaining values of k and l. Obviously, $E_3^{k,l} \cong F$ for $k = 0$ and $l = 0, 2, 4, \ldots, 2m-2$. So $E_\infty = E_3$. Therefore, we have

$$E_\infty^{k,l} = \begin{cases} F, & k = 0 \text{ and } l = 0, 2, 4, \ldots, 2m-2 \\ 0, & \text{otherwise.} \end{cases}$$

The element $1 \otimes b \in E_2^{0,2}$ is a permanent cocycle and determines an element $z \in E_\infty^{0,2}$. We have $i^*(z) = b$ and $z^m = 0$. Therefore, the total complex Tot $E_\infty^{*,*}$ is the graded commutative algebra

$$\text{Tot } E_\infty^{*,*} = F[z]/\langle z^m \rangle, \quad \deg z = 2.$$

It follows that

$$H^*(X_G) = F[z]/\langle z^m \rangle, \quad \deg z = 2.$$

Case II. If $d_2(1 \otimes a) = 0$ and $d_2(1 \otimes b) = t \otimes a$, then we have $d_2(1 \otimes b^q) = qt \otimes ab^{q-1}$ and $d_2(1 \otimes ab^q) = 0$ for $1 \le q < m$. So $0 = d_2[(1 \otimes b^{m-1}) \cup (1 \otimes b)] = mt \otimes ab^{m-1}$. This is clearly not true if $F = \mathbb{Q}$. Now suppose that $F = \mathbb{Z}_p$. Then $m = np$ for some integer $n > 0$. The differential

$$d_2 : E_2^{k,l} \to E_2^{k+2,l-1}$$

is an isomorphism if l is even and $2p$ does not divide l; and trivial homomorphism if l is odd or $2p$ divides l. So $E_3^{k,l} \cong E_2^{k,l} \cong \mathbb{Z}_p$ for even k and $l = 2qp$ or $2(q+1)p - 1$, $0 \le q < n$; $k = 0$, l is odd and $2p$ does not divide l; and $E_3^{k,l} = 0$, otherwise. Clearly, all the differentials $d_3, d_4, \ldots, d_{2p-1}$ are trivial. Obviously,

$$d_{2p} : E_{2p}^{k,2qp} \to E_{2p}^{k+2p,2(q-1)p+1}$$

are the trivial homomorphisms for $q = 1, 2, \ldots, n-1$. If

$$d_{2p} : E_{2p}^{0,2p-1} \to E_{2p}^{2p,0}$$

is also trivial, then

$$d_{2p} : E_{2p}^{k,2qp-1} \to E_{2p}^{k+2p,2(q-1)p}$$

is the trivial homomorphism for $q = 2, \ldots, n-1$, because every element of $E_{2p}^{k,2qp-1}$ (even k) can be written as the product of an element of $E_{2p}^{k,2(q-1)p}$ by $[1 \otimes ab^{p-1}] \in$

$E_{2p}^{0,2p-1}$. It follows that $d_r = 0, \forall\, r > 2p$ so that $E_\infty = E_3$. This contradicts the fact that $H^i(X_G) = 0$ for all $i \geq 2m-1$. Therefore,

$$d_{2p} : E_{2p}^{0,2p-1} \to E_{2p}^{2p,0}$$

must be non-trivial. Assume that $d_{2p}([1 \otimes ab^{p-1}]) = [t^p \otimes 1]$. Then

$$d_{2p} : E_{2p}^{k,2qp-1} \to E_{2p}^{k+2p,2(q-1)p}$$

is an isomorphism for all k and $1 \leq q \leq n$. Now, it is clear that $E_\infty = E_{2p+1}$. Also, $E_{2p+1}^{k,l} \cong \mathbb{Z}_p$ for ((even) $k < 2p, l = 2qp, (0 \leq q < n)$) and ($k = 0$, l is odd and $2p$ does not divide l). Thus

$$H^j(X_G) = \begin{cases} 0, & j = 2qp - 1 (1 \leq q \leq n) \text{ or } j > 2np - 2, \\ \mathbb{Z}_p, & \text{otherwise.} \end{cases}$$

The elements $1 \otimes b^p \in E_2^{0,2p}$ and $1 \otimes ab^{(h-1)/2} \in E_2^{0,h}$, for $h = 1, 3, \ldots, 2p-3$ are permanent cocycles. So they determine $z \in E_\infty^{0,2p}$ and $y_q \in E_\infty^{0,q}$, $q = 1, 3, \ldots, 2p-3$, respectively. Obviously, $i^*(z) = b^p$, $z^n = 0$ and $y_q y_{q'} = 0$. Let $x = \pi^*(t) \in E_\infty^{2,0}$. Then $x^p = 0$. It follows that the total complex Tot $E_\infty^{*,*}$ is the graded commutative algebra

$$\text{Tot } E_\infty^{*,*} = \frac{\mathbb{Z}_p[x, y_1, y_3, \ldots, y_{2p-3}, z]}{\langle x^p, y_q y_{q'}, x y_q, z^n \rangle},$$

where $q, q' = 1, 3, \ldots, 2p-3$.

Then $i^*(y_q) = ab^{\frac{(q-1)}{2}}$, $y_q^2 = 0$ and $y_q y_{2p-q} = 0$. It follows that

$$H^*(X_G) = \frac{\mathbb{Z}_p[x, y_1, y_3, \ldots, y_{2p-3}, z]}{\langle x^p, z^n, x y_q, y_q y_{q'} - A_{qq'} x^{\frac{q+q'}{2}} - B_{qq'} z x^{\frac{q+q'-2p}{2}} \rangle},$$

where $m = np$, $A_{qq'} = 0$ when $q + q' > 2p$, $B_{qq'} = 0$ when $q + q' < 2p$ and both $A_{qq'}$ and $B_{qq'}$ are zero when $q = q'$ or $q + q' = 2p$, $\deg x = 2$, $\deg z = 2p$, $\deg y_q = q$.

Since the action of G on X is free, the mod p cohomology rings of X_G and X/G are isomorphic. This completes the proof. □

Proof of Theorem 1.2. For prime $p > 2$, we proceed as in Theorem 1.1, and observe that Case II can not occur here. In the Gysin cohomology sequence of the $\mathbb{S}^1$-bundle $\mathbb{S}^1 \hookrightarrow X \xrightarrow{\nu} X/G$, the homomorphism $H^k(X/G) \xrightarrow{\nu^*} H^k(X)$ is an isomorphism for $k = 1$, and trivial homorphism for $k = 2$. By the naturality of Bockstein homomorphism, we see that $\beta : H^1(X) \longrightarrow H^2(X)$ is trivial. In particular, we have $b = \beta(a) = 0$, a contradiction.

For $p = 2$, X is mod 2 cohomology real projective space. Let $a \in H^1(X)$ be the generator of the cohomology ring $H^*(X)$. If $d_2(1 \otimes a) = 0$, then $d_2(1 \otimes a^q) = 0$, by the multiplicative structure of spectral sequence. It follows that the spectral sequence degenerates and hence there are fixed points. Therefore, we must have $d_2(1 \otimes a) = t \otimes 1$. It is easily seen that

$$d_2 : E_2^{k,l} \to E_2^{k+2,l-1}$$

is an isomorphism for k even and l odd; and trivial homomorphism otherwise. So

$$E_\infty^{k,l} \cong \begin{cases} \mathbb{Z}_2, & k = 0 \text{ and } l = 0, 2, 4, \ldots, 2m-2, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $H^*(X_G)$ and Tot $E_\infty^{*,*}$ are same as the graded commutative algebra. The case $m = 1$ is obvious, so assume that $m > 1$.

The element $1 \otimes a^2 \in E_2^{0,2}$ is a permanent cocycle and determines an element $z \in E_\infty^{0,2} = H^2(X_G)$. We have $i^*(z) = a^2$ and $z^m = 0$. Therefore, the total complex Tot $E_\infty^{*,*}$ is the graded commutative algebra,

$$\text{Tot } E_\infty^{*,*} = \mathbb{Z}_2[z]/\langle z^m \rangle, \quad \text{where } \deg z = 2.$$

Thus $H^*(X_G) = \mathbb{Z}_2[z]/\langle z^m \rangle$, where $\deg z = 2$. This completes the proof. □

4. Examples

Consider the $(2m-1)$ sphere $\mathbb{S}^{2m-1} \subset \mathbb{C} \times \ldots \times \mathbb{C}$ (m times). The map $(\xi_1, \ldots, \xi_m) \to (z\xi_1, \ldots, z\xi_m)$, where $z \in \mathbb{S}^1$ defines a free action of $G = \mathbb{S}^1$ on $\mathbb{S}^{2m-1}$ with the orbit space $\mathbb{S}^{2m-1}/\mathbb{S}^1$ the complex projective space. Let $N = \langle z \rangle$, where $z = \mathrm{e}^{2\pi i/p}$. Then the orbit space $\mathbb{S}^{2m-1}/N$ is the lens space $L^{2m-1}(p; 1, \ldots, 1)$ (resp. real projective space $\mathbb{R}P^{2m-1}$ for $p = 2$). It follows that there is a free action of $\mathbb{S}^1 = G/N$ on a lens space with the complex projective space as the orbit space. Thus we realize Theorem 1.2. For Theorem 1.1, we have the diagonal action of G on $\mathbb{S}^1 \times \mathbb{C}P^{m-1}$ with the orbit space $\mathbb{C}P^{m-1}$, where G acts freely on $\mathbb{S}^1$ and trivially on $\mathbb{C}P^{m-1}$. This realizes the first case of the theorem.

5. A Borsuk–Ulam type theorem for free G-actions on $\mathbb{S}^1 \times \mathbb{C}P^{m-1}$

First, we find the mod p index of a free action of G on $\mathbb{S}^1 \times \mathbb{C}P^{m-1}$.

Theorem 5.1. *Let $G = \mathbb{S}^1$ act freely on a finitistic space $X \sim_p \mathbb{S}^1 \times \mathbb{C}P^{m-1}$, p being a prime. Then either the characteristic class of the bundle $\mathbb{S}^1 \hookrightarrow X \to X/G$ is zero, or* mod p *index of X is $p-1$.*

Proof. The Gysin sequence of the bundle $\mathbb{S}^1 \hookrightarrow X \to X/G$ begins with

$$0 \to H^1(X/G; \mathbb{Z}_p) \xrightarrow{\nu^*} H^1(X; \mathbb{Z}_p) \to H^0(X/G; \mathbb{Z}_p)$$
$$\xrightarrow{\psi^*} H^2(X/G; \mathbb{Z}_p) \xrightarrow{\nu^*} \cdots .$$

The characteristic class of the bundle is defined to be the element $\psi^*(1) \in H^2(X/G; \mathbb{Z}_p)$, where 1 is the unity of $H^0(X/G; \mathbb{Z}_p)$. In Theorem 1.1(i), the characteristic class of the bundle $\mathbb{S}^1 \hookrightarrow X \to X/G$ is zero. In Theorem 1.1(ii), $H^i(X/G; \mathbb{Z}_p) \cong \mathbb{Z}_p$ for $i < 2p-1$. Also, we have $H^i(X; \mathbb{Z}_p) \cong \mathbb{Z}_p$ for $i \le 2m-1$. Thus, we have the following exact sequence:

$$0 \to \mathbb{Z}_p \xrightarrow{\nu^*} \mathbb{Z}_p \to \mathbb{Z}_p \xrightarrow{\psi^*} \mathbb{Z}_p \xrightarrow{\nu^*} \mathbb{Z}_p \to \cdots$$

Clearly, the map

$$\psi^* : H^0(X/G; \mathbb{Z}_p) \to H^2(X/G; \mathbb{Z}_p)$$

is an isomorphism. So, the characteristic class $\alpha = \psi^*(1) \in H^2(X/G; \mathbb{Z}_p)$ is nonzero. By Theorem 1.1, $\alpha^p = 0$ but $\alpha^{p-1} \neq 0$. Thus, mod p index of X is $p - 1$. □

If $\eta \to X$ is a real vector bundle, we write $w_i(\eta)$ to denote its i-th Stiefel–Whitney classes. We have the following results.

Theorem 5.2. *Let X be a finitistic space whose* mod p *cohomology isomorphic to $\mathbb{S}^1 \times \mathbb{C}P^{m-1}$, p being a prime. Suppose X is equipped with an arbitrary free $\mathbb{S}^1$-action, and $\mathbb{S}^{2m+1}$ is equipped with the standard (complex multiplication) $\mathbb{S}^1$-action. In case, the characteristic class of the bundle $\mathbb{S}^1 \hookrightarrow X \to X/\mathbb{S}^1$ is nonzero, then there is no equivariant map $\mathbb{S}^{2m+1} \to X$ if $m \geq p$.*

Proof. Suppose, on the contrary, there exists a $\mathbb{S}^1$-equivariant map $f : \mathbb{S}^{2m+1} \to X$, and let $\bar{f} : \mathbb{S}^{2m+1}/\mathbb{S}^1 = \mathbb{C}P^m \to X/\mathbb{S}^1$ be the map induced by f. Write $\lambda \to X/\mathbb{S}^1$ for the complex line bundle associated to the free $\mathbb{S}^1$-action on X. In this case, the total space of λ is the orbit space of $X \times \mathbb{C}$ by the diagonal action of $\mathbb{S}^1$, coming from the free action of $\mathbb{S}^1$ on X and the complex multiplication on $\mathbb{C}$; in the subsequent approach, λ will be considered as a 2-dimensional real vector bundle. Denote by $\xi \to B_{\mathbb{S}^1} = \mathbb{C}P^\infty$ the universal complex line bundle and by $\xi' \to \mathbb{C}P^m$ its restriction to $\mathbb{C}P^m$, both also considered as the 2-dimensional real vector bundles. In Theorem 1.1(i), $x \in H^2(X/\mathbb{S}^1; \mathbb{Z}_p)$ is the image of t under

$$\pi^* : H^2(\mathbb{C}P^\infty; \mathbb{Z}_p) \to H^2(X/\mathbb{S}^1; \mathbb{Z}_p),$$

since also in this case π is a classifying map for $\lambda \to X/\mathbb{S}^1$, this gives that $x = \pi^*(t) = \pi^*(w_2(\xi)) = w_2(\lambda)$. Again, because f is equivariant, $\xi' \to \mathbb{C}P^m$ is the pullback of λ by $\bar{f}$, and so naturality gives $\bar{f}^*(w_2(\lambda)) = w_2(\xi') = t' \in H^2(\mathbb{C}P^m; \mathbb{Z}_p)$, t' the generator of $H^2(\mathbb{C}P^m; \mathbb{Z}_p)$. In Theorem 1.1(ii), the characteristic class of the bundle is nonzero, and $x^p = 0$. Thus, $0 = \bar{f}^*(x^p) = t'^p$. However, because $m \geq p$, and thus $t'^p \in H^{2p}(\mathbb{C}P^m; \mathbb{Z}_p)$ is nonzero. This gives the desired contradiction. □

We remark that the mod p index of a lens space $L^{2m-1}(p; q_1, q_2, \ldots, q_m)$, where p a prime, is not defined. In fact, *the characteristic class of the bundle $\mathbb{S}^1 \hookrightarrow X \xrightarrow{\nu} X/G$, where $X \sim_p L^{2m-1}(p; q_1, q_2, \ldots, q_m)$, is zero*. By Theorem 1.2, $H^1(X/G; \mathbb{Z}_p) = 0$ and $H^0(X/G; \mathbb{Z}_p) = H^2(X/G; \mathbb{Z}_p) \cong \mathbb{Z}_p$. The Gysin sequence of the bundle $\mathbb{S}^1 \hookrightarrow X \to X/G$ reduces to the exact sequence,

$$0 \to 0 \xrightarrow{\nu^*} \mathbb{Z}_p \to \mathbb{Z}_p \xrightarrow{\psi^*} \mathbb{Z}_p \xrightarrow{\nu^*} \mathbb{Z}_p \to \cdots$$

It is now immediate that

$$\psi^* : H^0(X/G; \mathbb{Z}_p) \to H^2(X/G; \mathbb{Z}_p)$$

is trivial homomorphism. Therefore, the characteristic class of the bundle $\mathbb{S}^1 \hookrightarrow X \to X/G$ is zero.

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