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A new class of lattice paths and partitions with *n* copies of *n*

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Abstract. Agarwal and Bressoud (*Pacific J. Math.* **136(2)** (1989) 209–228) defined a class of weighted lattice paths and interpreted several q-series combinatorially. Using the same class of lattice paths, Agarwal (*Utilitas Math.* **53** (1998) 71–80; *ARS Combinatoria* **76** (2005) 151–160) provided combinatorial interpretations for several more q-series. In this paper, a new class of weighted lattice paths, which we call associated lattice paths is introduced. It is shown that these new lattice paths can also be used for giving combinatorial meaning to certain q-series. However, the main advantage of our associated lattice paths is that they provide a graphical representation for partitions with $n + t$ copies of n introduced and studied by Agarwal (Partitions with n copies of n, Lecture Notes in Math., No. 1234 (Berlin/New York: Springer-Verlag) (1985) 1–4) and Agarwal and Andrews (*J. Combin. Theory* **A45(1)** (1987) 40–49).

Keywords. Lattice paths; colored partitions; combinatorial interpretations; q-series.

1. Introduction

Using weighted lattice paths, Agarwal and Bressoud [10] interpreted combinatorially several q-series found by Agarwal *et al* in [9]. A bijection between an appropriate class of lattice paths of weight ν and a set of partitions of ν with $n + t$ copies of n was established in [10] which provided a proof of a partition identity of Agarwal and Andrews [8]. Agarwal [4] used the same class of weighted lattice paths and interpreted many more q-series from Slater's paper [12] combinatorially. Agarwal [1,3] interpreted three generalized q-series as generating functions for certain restricted partition functions with $n + t$ copies of n. Very recently, he extended his results in [7] by using the same lattice paths of Agarwal and Bressoud.

In this paper, we introduce a new class of lattice paths which we call associated lattice paths and interpret the aforementioned three generalized q-series as generating functions for certain restricted associated lattice path functions. However, the main advantage of this switch to new lattice paths is that we obtain a graphical representation of all partitions of ν with $n+t$ copies of n. Associated lattice paths also provide a graphical representation for n-color compositions defined and studied in [5,6,11].

DEFINITION 1.1 [8]

A partition with $n + t$ copies of n, $t \geq 0$ is a partition in which a part of size n, $n \geq 0$ can come in $n + t$ different colors denoted by subscripts: $n_1, n_2, \ldots, n_{n+t}$.

For example, partitions of 2 with $n+1$ copies of n are:

$$2_1, 2_1+0_1, 1_1+1_1, 1_1+1_1+0_1,$$

$$2_2, 2_2+0_1, 1_2+1_1, 1_2+1_1+0_1,$$

$$2_3, 2_3+0_1, 1_2+1_2, 1_2+1_2+0_1.$$

Note that zeros are permitted if and only if t is greater than or equal to one. Also, zeros are not permitted to repeat in any partition. In fact, only one copy of 0 namely 0_t is allowed to appear.

Partitions with n copies of n are also called n-color partitions (see for instance [2]).

DEFINITION 1.2 [8]

The weighted difference of two elements $m_i, n_j, m \geq n$ is defined by $m-n-i-j$ and is denoted by $((m_i - n_j))$.

DEFINITION 1.3 [5]

An ordered n-color partition is called an n-color composition. For example, n-color compositions of $\nu = 3$ are $3_1, 3_2, 3_3, 2_1+1_1, 1_1+2_1, 2_2+1_1, 1_1+2_2, 1_1+1_1+1_1$.

DEFINITION 1.4

For $|q| < 1$ and any constant a, the rising q-factorial is defined by $(a;q)_n = \prod_{i=0}^{\infty} \frac{1-aq^i}{1-aq^{n+i}}$.

Next, we recall the following description of lattice paths of Agarwal and Bressoud [10].

All paths will be of finite length lying in the first quadrant. They begin on the y-axis and terminate on the x-axis. Only three moves are allowed at each step:

Northeast: From (i, j) to $(i+1, j+1)$,
Southeast: From (i, j) to $(i+1, j-1)$, only allowed if $j > 0$,
Horizontal: From $(i, 0)$ to $(i+1, 0)$, only allowed along the x-axis.

All paths are either empty or terminate with a southeast step: from $(i, 1)$ to $(i+1, 0)$.

The following terminologies are used:

Peak: Either a vertex on the y-axis which is followed by a southeast step or a vertex preceded by a northeast step and followed by a southeast step.

Valley: A vertex preceded by a southeast step and followed by a northeast step. Note that a southeast step followed by a horizontal step followed by a northeast step does not constitute a valley.

Mountain: A section of the path which starts on either the x-axis or the y-axis, which ends on the x-axis and does not touch the x-axis anywhere in between the endpoints. Every mountain has at least one peak and may have more than one.

Plain: A section of path consisting of only horizontal steps which starts either on the y-axis or at a vertex preceded by a southeast step and ends at a vertex followed by a northeast step.

The *height* of a vertex is its y-coordinate. The *weight* of a vertex is its x-coordinate. *Weight* of a path is the sum of the weights of its peaks.

Agarwal's results [1,3,7] are contained in the following theorems.

Theorem 1.1. *For $k \geq -1$, let $A_1^k(\nu)$ denote the number of lattice paths of weight ν which starts at a point on the x-axis, have no valley above height 0 if $k = -1$ and no valleys at all if $k \geq 0$ and there is a plain of minimal length $(k+1)$, $k \geq 0$ between any two mountains and $B_1^k(\nu)$ denotes the number of partitions of ν with n copies of n such that the weighted difference of each pair of parts is $> k$. Then $A_1^k(\nu) = B_1^k(\nu)$ for all ν, and*

$$\sum_{\nu=0}^{\infty} A_1^k(\nu) q^{\nu} = \sum_{\nu=0}^{\infty} B_1^k(\nu) q^{\nu} = \sum_{m=0}^{\infty} \frac{q^{m[1+\frac{(k+3)(m-1)}{2}]}}{(q;q)_m (q;q^2)_m}. \tag{1.1}$$

Theorem 1.2. *For $k \geq -1$, let $A_2^k(\nu)$ denote the number of lattice paths of weight ν which starts from $(0, 1)$, have no valley above height 0 if $k = -1$ and no valleys at all if $k \geq 0$ and there is a plain of minimal length $(k+1)$, $k \geq 0$ between any two mountains and $B_2^k(\nu)$ denotes the number of partitions of ν with $n+1$ copies of n such that for some i, i_{i+1} is a part and the weighted difference of each pair of parts is greater than k. Then $A_2^k(\nu) = B_2^k(\nu)$ for all ν, and*

$$\sum_{\nu=0}^{\infty} A_2^k(\nu) q^{\nu} = \sum_{\nu=0}^{\infty} B_2^k(\nu) q^{\nu} = \sum_{m=0}^{\infty} \frac{q^{\frac{m(m+1)(k+3)}{2}}}{(q;q)_m (q;q^2)_{m+1}}. \tag{1.2}$$

Theorem 1.3. *For $k \geq -1$, let $A_3^k(\nu)$ denote the number of lattice paths of weight ν which start from $(0, 2)$, have no valley above height 0 if $k = -1$ and no valleys at all if $k \geq 0$ and there is a plain of minimal length $(k+1)$, $k \geq 0$ between any two mountains and $B_3^k(\nu)$ denotes the number of partitions of ν with $n+2$ copies of n such that for some i, i_{i+2} is a part and the weighted difference of each pair of parts is $> k$. Then $A_3^k(\nu) = B_3^k(\nu)$ for all ν, and*

$$\sum_{\nu=0}^{\infty} A_3^k(\nu) q^{\nu} = \sum_{\nu=0}^{\infty} B_3^k(\nu) q^{\nu} = \sum_{m=0}^{\infty} \frac{q^{m[1+\frac{(m+1)(k+3)}{2}]}}{(q;q)_m (q;q^2)_{m+1}}. \tag{1.3}$$

In our next section, we introduce a new class of weighted lattice paths which we call the associated lattice paths and establish a bijection between an appropriate class of associated lattice paths of weight ν and partitions of ν with $n+t$ copies of n. This leads to a graphical representation of partitions with $n+t$ copies of n. In §3, we interpret the right-hand sides of (1.1)–(1.3) as generating functions for certain restricted associated lattice path functions. This extends Theorems 1.1–1.3 to 3-way combinatorial identities. We conclude in §4 by discussing a particular case.

2. A new class of lattice paths and n-color partitions

A new class of lattice paths which we call associated lattice paths with the following description is introduced in this paper.

All paths are of finite length lying in the first quadrant. They will begin on the y-axis and terminate on the x-axis. Only three moves are allowed at each step.

Northeast: From (i, j) to $(i+1, j+1)$,
Southeast: From (i, j) to $(i+1, j-1)$, only allowed if $j > 0$,
Horizontal: From (i, j) to $(i+1, j)$, only allowed when the first step is preceded by a northeast step and the last is followed by a southeast step.

The following terminology will be used in describing associated lattice paths:

Truncated Isosceles Trapezoidal Section (*TITS*): A section of the path which starts on the x-axis with northeast steps followed by horizontal steps and then followed by southeast steps ending on the x-axis forms what we call a truncated isosceles trapezoidal section. Since the lower base lies on the x-axis and is not a part of the path, hence the term truncated.

Slant Section (*SS*): A section of the path consisting of only southeast steps which starts on the y-axis (origin not included) and ends on the x-axis.

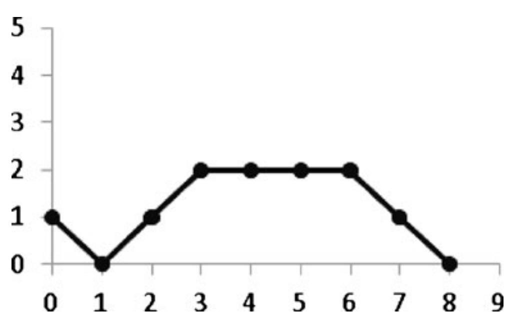
Height of a slant section is 't' if it starts from $(0, t)$. Clearly, a path can have an SS only in the beginning. A lattice path can have at most one SS.

Weight of a TITS: To define this, we shall represent every TITS by an ordered pair $\{a, b\}$ where a denotes its altitude and b the length of the upper base.

Weight of the TITS with ordered pair $\{a, b\}$ is a units.

Weight of a lattice path is the sum of weights of its TITSs. Slant section is assigned weight zero.

For example, in the below figure,



the associated lattice path has one SS of height 1 and one TITS with ordered pair $\{2, 3\}$ and its weight is 2 units.

The following results provide a lattice path representation for partitions with $n + t$ copies of n and for n-color compositions:

Theorem 2.1. *For $t \geq 0$, let $C_t(\nu)$ denote the number of associated lattice paths of weight ν such that for any TITS with ordered pair $\{a, b\}$, b does not exceed $a + t$ and if $t > 0$, then there may be an SS of height t and the TITSs are arranged in the order of* *non decreasing altitudes and the TITSs with the same altitude are ordered by the length of their upper base. Let $B_t(\nu)$ denote the number of partitions of ν with $n+t$ copies of n. Then*

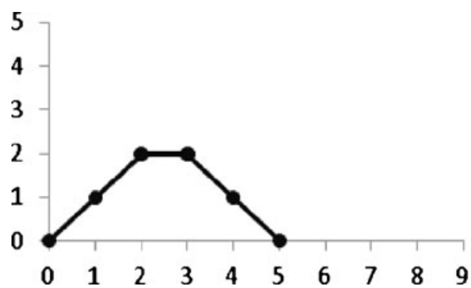
$$B_t(\nu) = C_t(\nu), \quad \text{for all } \nu\,.$$

Proof of Theorem 2.1. We map each part a_b, $a \geq 1$ of a partition π enumerated by $B_t(\nu)$ to a TITS of the associated lattice path with ordered pair $\{a, b\}$. If π contains 0_t; it will be mapped to an SS of height t. As $B_t(\nu)$ enumerates partitions with $n+t$ copies of n, b cannot exceed $a+t$. In this way, corresponding to each partition enumerated by $B_t(\nu)$, we get an associated lattice path enumerated by $C_t(\nu)$. Similarly, going backwards, corresponding to each lattice path enumerated by $C_t(\nu)$, we get a partition enumerated by $B_t(\nu)$. This proves Theorem 2.1.

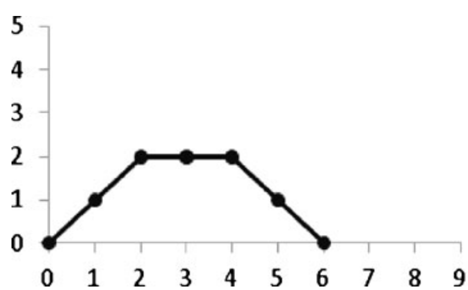
For example, when $\nu = 2$, $t = 1$, we have

$$C_1(2) = 12 = B_1(2).$$

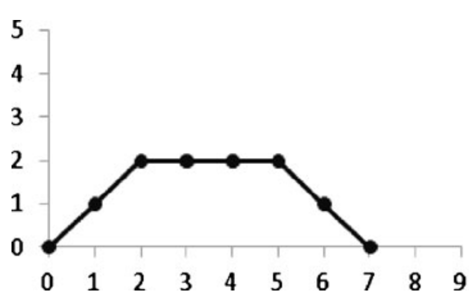
The lattice paths enumerated by $C_1(2)$ and the corresponding partitions enumerated by $B_1(2)$ are shown below:



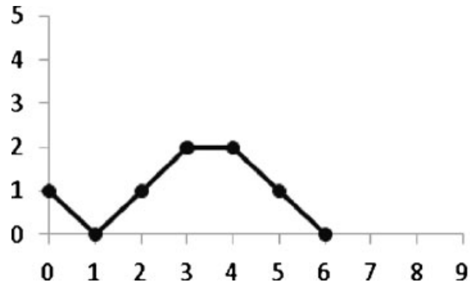
$\longleftrightarrow$ 2_1



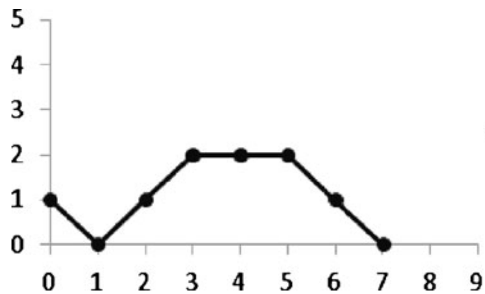
$\longleftrightarrow$ 2_2



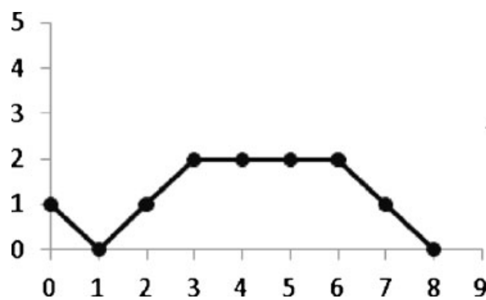
$\longleftrightarrow$ 2_3



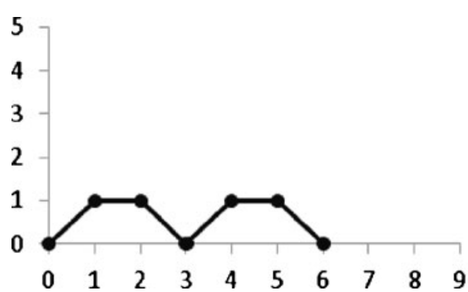
$\longleftrightarrow 0_1 2_1$



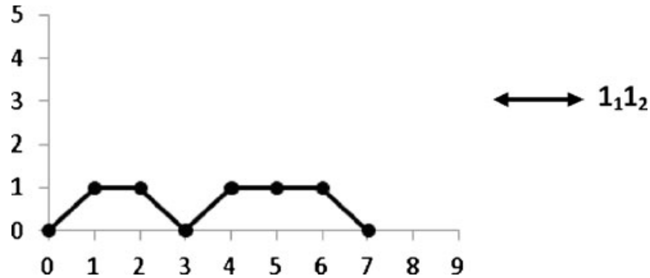
$\longleftrightarrow 0_1 2_2$



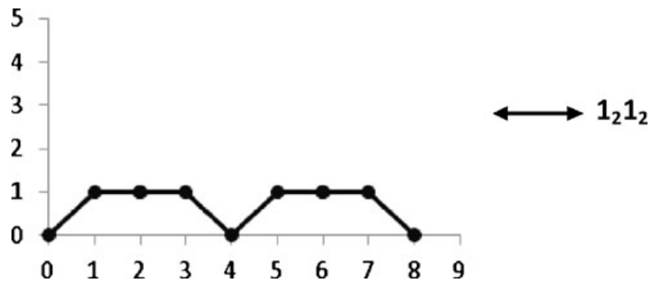
$\longleftrightarrow 0_1 2_3$



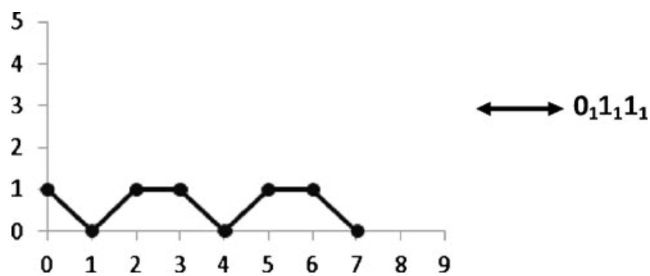
$\longleftrightarrow 1_1 1_1$



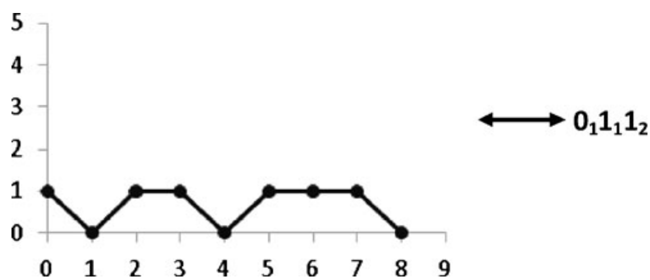
5 4 3 2 1 0
0 1 2 3 4 5 6 7 8 9
$1_1 1_2$



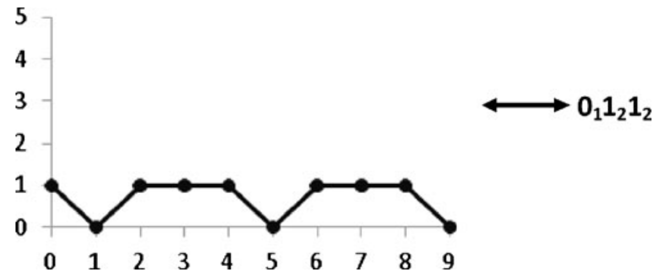
5 4 3 2 1 0
0 1 2 3 4 5 6 7 8 9
$1_2 1_2$



5 4 3 2 1 0
0 1 2 3 4 5 6 7 8 9
$0_1 1_1 1_1$



5 4 3 2 1 0
0 1 2 3 4 5 6 7 8 9
$0_1 1_1 1_2$



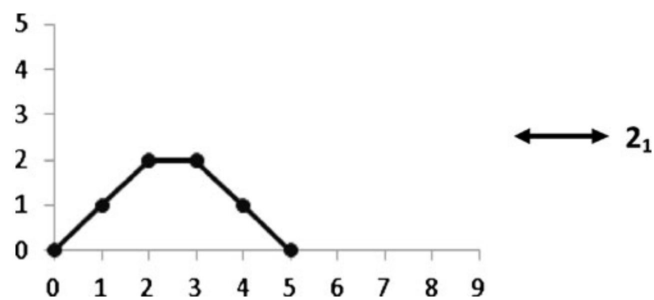
Note: The case $t = 0$ of Theorem 2.1 gives the correspondence between a class of associated lattice paths and n-color partitions and the result can be stated as follows.

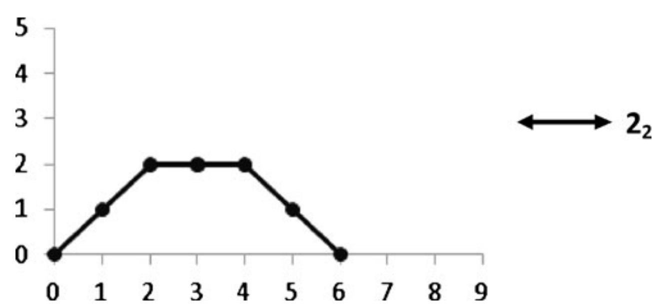
COROLLARY 2.2

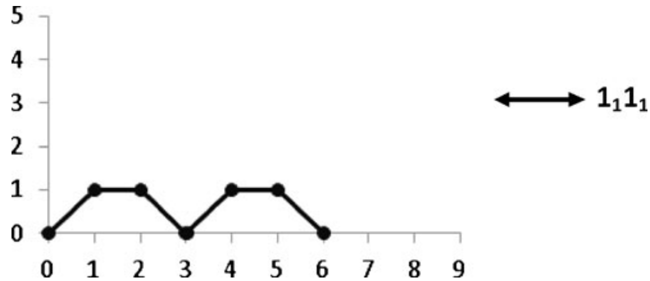
Let $C_0(\nu)$ denote the number of associated lattice paths of weight ν such that for any TITS with ordered pair $\{a, b\}$, b does not exceed a and where TITSs are arranged in the same order as in Theorem 2.1 *and there is no SS. Let $B_0(\nu)$ denote the number of partitions of ν with n copies of n. Then*

$$B_0(\nu) = C_0(\nu), \quad \textit{for all } \nu.$$

Example. For $\nu = 2$, n-color partitions and the corresponding associated lattice paths are:

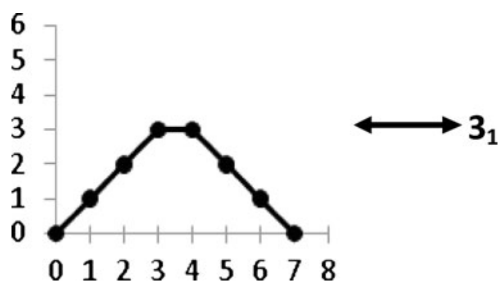


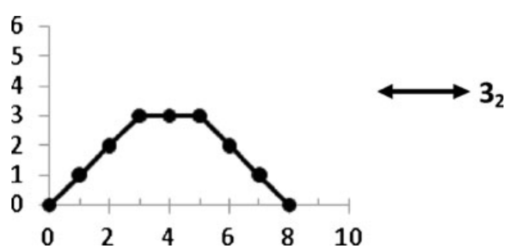


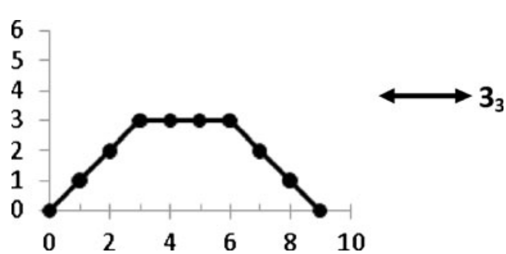


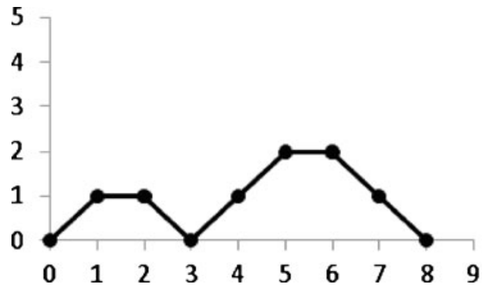
Note: If we lift the condition of arranging the TITSs in a non decreasing order of their altitudes in Theorem 2.1, we get a lattice path representation for n-color compositions from the above corollary.

Example. For $\nu = 3$, n-color compositions and the corresponding associated lattice paths are:

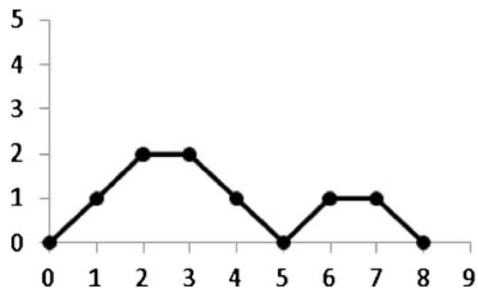




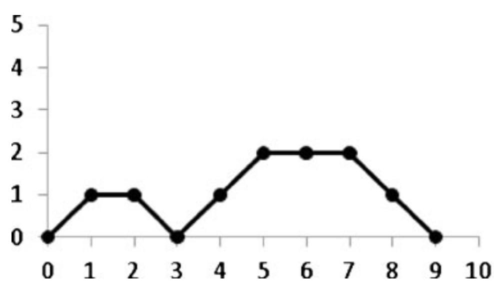




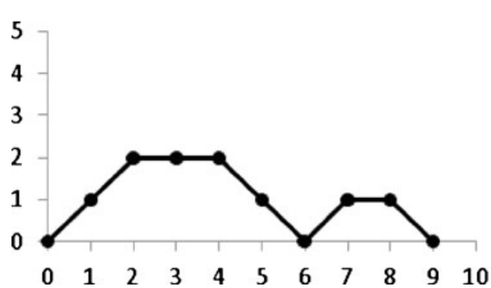
$\longleftrightarrow 1_1 2_1$



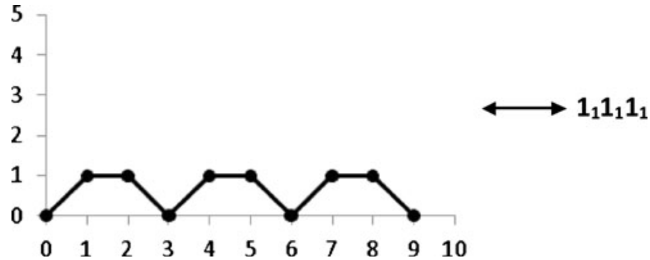
$\longleftrightarrow 2_1 1_1$



$\longleftrightarrow 1_1 2_2$



$\longleftrightarrow 2_2 1_1$



3. Extensions of Theorems 1.1–1.3

Using the associated lattice paths we shall prove the following extensions of Theorems 1.1–1.3.

Theorem 3.1. *For $k \geq -1$, let $C_1^k(\nu)$ denote the number of associated lattice paths of weight ν such that*

(a) *for any TITS with ordered pair $\{a, b\}$, b does not exceed a;*
(b) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with the same altitude are ordered by the length of their upper base, and*
(c) *for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$, $(a_1 \leq a_2)$,*

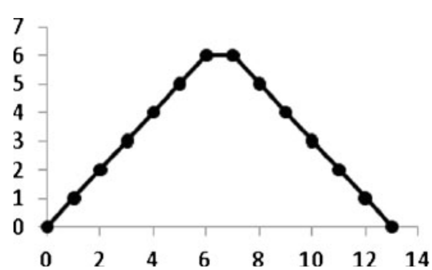
$$a_2 - b_2 > a_1 + b_1 + k.$$

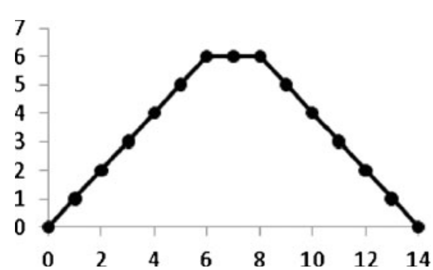
Then

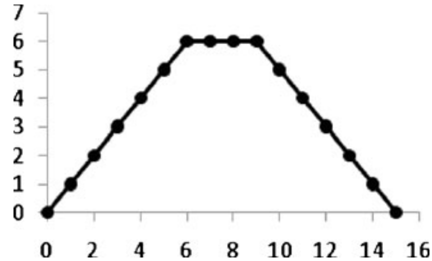
$$A_1^k(\nu) = B_1^k(\nu) = C_1^k(\nu), \tag{3.1}$$

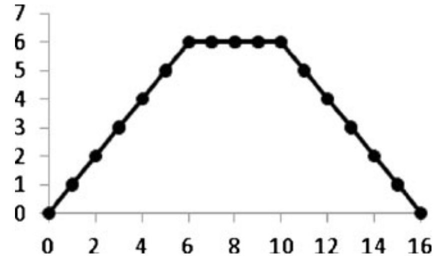
for all ν.

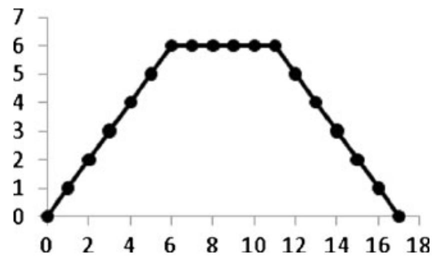
Example. Consider the case when $\nu = 6$, $k = 0$. $B_1^0(6) = 7$ since the relevant partitions of 6 with n copies of n are 6_1, 6_2, 6_3, 6_4, 6_5, 6_6, $5_1 + 1_1$. Also, $C_1^0(6) = 7$, since the relevant associated lattice paths of weight 6 are:

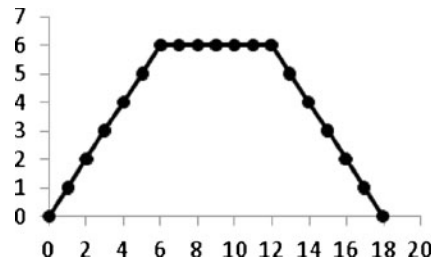


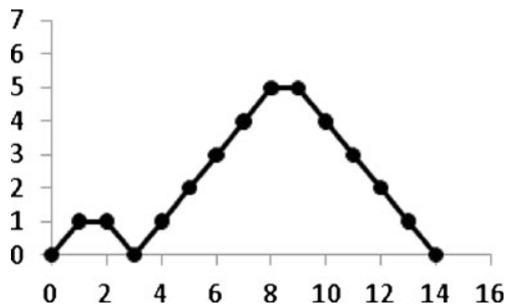












Theorem 3.2. *For $k \geq -1$, let $C_2^k(\nu)$ denote the number of associated lattice paths of weight ν such that*

(a) *for any TITS with ordered pair $\{a, b\}$, b does not exceed $a + 1$;*
(b) *the TITSs are arranged in order of nondecreasing altitudes and the TITSs with the same altitude are ordered by the length of their upper base;*
(c) *for any two TITSs with respective ordered pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$, $(a_1 \leq a_2)$,*

$$a_2 - b_2 > a_1 + b_1 + k, \text{ and}$$

(d) *there is one TITS with ordered pair $\{a, a + 1\}$ or an SS of height 1. Then*

$$A_2^k(\nu) = B_2^k(\nu) = C_2^k(\nu), \text{ for all } \nu. \tag{3.2}$$

Theorem 3.3. *For $k \geq -1$, let $C_3^k(\nu)$ denote the number of associated lattice paths of weight ν such that*

(a) *for any TITS with ordered pair $\{a, b\}$, b does not exceed $a + 2$;*

(b) *the TITSs are arranged in the order of nondecreasing altitudes and the TITSs with the same altitude are ordered by the length of their upper base*;
(c) *for any two TITS with respective ordered pairs* $\{a_1, b_1\}$ *and* $\{a_2, b_2\}$, $(a_1 \le a_2)$,

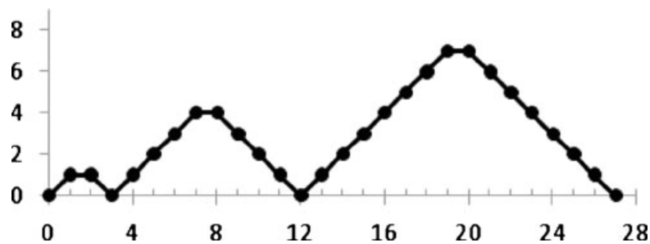
$$a_2 - b_2 > a_1 + b_1 + k, \ \textit{and}$$

(d) *there is one TITS with ordered pair* $\{a, a+2\}$ *or an SS of height* 2. *Then*

$$A_3^k(\nu) = B_3^k(\nu) = C_3^k(\nu), \qquad \textit{for all } \nu. \tag{3.3}$$

We shall prove Theorem 3.1 completely by providing two different proofs. In the first proof we shall show that the extreme right-hand side of equation (1.1) also generates $C_1^k(\nu)$ while the second proof is bijective. We shall sketch the proofs of Theorems 3.2–3.3. The reader can supply the details or obtain them from the authors.

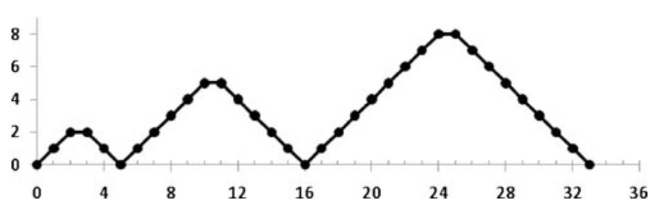
Proof of Theorem 3.1. In $\frac{q^{m[1+\frac{(k+3)(m-1)}{2}]}}{(q;q)_m(q;q^2)_m}$, the factor $q^{m[1+\frac{(k+3)(m-1)}{2}]}$ generates a lattice path having m TITSs, i-th TITS having ordered pair $\{(2i-1)+(i-1)(k+1), 1\}$. Thus the path begins as:



Graph A

In the above graph, if we consider two TITSs, say i-th and $(i+1)$-th. The ordered pairs associated with them are $\{(2i-1)+(i-1)(k+1), 1\}$ and $\{(2i+1)+i(k+1), 1\}$.

The factor $\frac{1}{(q;q)_m}$ generates m non negative integers, say $a_1 \ge a_2 \cdots a_m \ge 0$ which are encoded by increasing the altitude of the i-th TITS by a_{m-i+1}, $1 \le i \le m$. Thus, the ordered pair associated with the i-th TITS becomes $\{(2i-1)+(i-1)(k+1)+a_{m-i+1}, 1\}$. Graph A now becomes

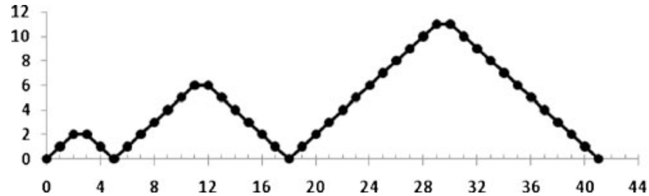


Graph B

The factor $\frac{1}{(q;q^2)_m}$ generates b_i non negative multiples of $(2i-1), 1 \le i \le m$, say $b_1 \times 1, b_2 \times 3, \dots, b_m \times (2m-1)$. This is encoded by increasing the altitude of i-th TITS by $2(b_m + b_{m-1} + \cdots + b_{m-i+2}) + b_{m-i+1}$ and upper base by b_{m-i+1} thus making the associated ordered pair as

$$\{(2i-1)+(i-1)(k+1)+a_{m-i+1}+2(b_m+\cdots+b_{m-i+2})+b_{m-i+1}, b_{m-i+1}+1\}.$$

Graph B now becomes



Graph C

Thus we see that every lattice path enumerated by $C_1^k(\nu)$ is uniquely generated in this manner. This proves that the right-hand side of eq. (1.1) also generates $C_1^k(\nu)$.

Now, we establish a one-to-one correspondence between the associated lattice paths enumerated by $C_1^k(\nu)$ and the partitions with n copies of n enumerated by $B_1^k(\nu)$. To each TITS of the associated lattice path enumerated by $C_1^k(\nu)$, we associate a part of the partition enumerated by $B_1^k(\nu)$. Now i-th TITS of the lattice path has an ordered pair

$$\{(2i-1)+(i-1)(k+1)+a_{m-i+1}+2(b_m+\cdots+b_{m-i+2})+b_{m-i+1}, b_{m-i+1}+1\}.$$

Associate it to the part A_x of the partition with n copies of n where

$$\begin{aligned} A &= (2i-1)+(i-1)(k+1)+a_{m-i+1} \\ &\quad +2(b_m+\cdots+b_{m-i+2})+b_{m-i+1}, \\ x &= b_{m-i+1}+1 \end{aligned}$$

and $(i+1)$-th TITS is associated to part B_y as

$$\begin{aligned} B &= (2i+1)+i(k+1)+a_{m-i}+2(b_m+\cdots+b_{m-i+1})+b_{m-i}, \\ y &= b_{m-i}+1. \end{aligned}$$

The weighted difference of these two parts is

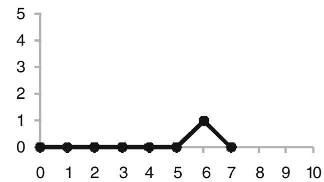
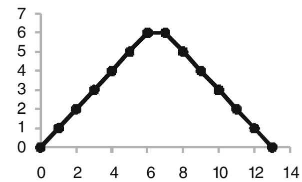
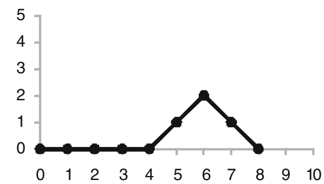
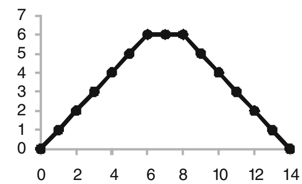
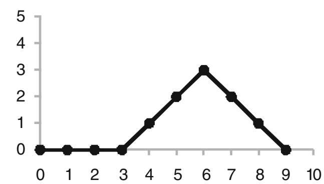
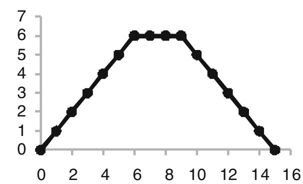
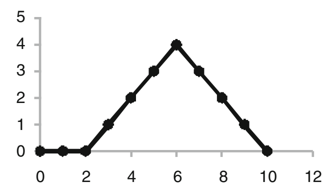
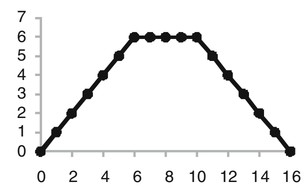
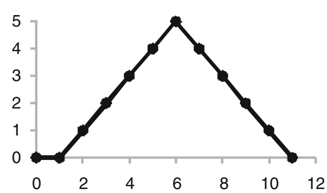
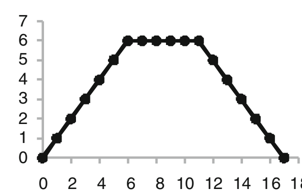
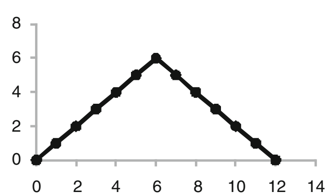
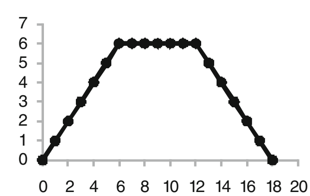
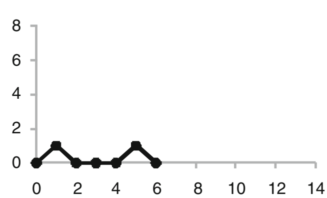
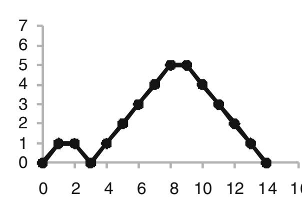
$$((B_y-A_x)) = B-A-x-y = (a_{m-i}-a_{m-i+1})+k+1 > k.$$

For the reverse implication, consider two parts of the partition enumerated by $B_1^k(\nu)$, say C_u and D_v. Associate the part C_u to a TITS with ordered pair $\{C, u\}$ and D_v to a TITS with ordered pair $\{D, v\}$. Without loss of generality, let $D \geq C$. Clearly, $u \leq C$ and $v \leq D$. Also, since C_u and D_v are the parts of a partition enumerated by $B_1^k(\nu)$, therefore, $D-C-v-u > k$ which implies $D-v > C+u+k$. This completes the bijection between the associated lattice paths enumerated by $C_1^k(\nu)$ and the partitions with n copies of n enumerated by $B_1^k(\nu)$.

Finally, we establish a bijection between the lattice paths enumerated by $A_1^0(\nu)$ and the associated lattice paths enumerated by $C_1^0(\nu)$. We do this by mapping each peak of weight a and height b of a lattice path enumerated by $A_1^0(\nu)$ to a TITS with ordered pair $\{a, b\}$ of an associated lattice path enumerated by $C_1^0(\nu)$ and conversely. It is easy to see that under this mapping the conditions on the lattice paths enumerated by $A_1^0(\nu)$ are translated into the conditions on the associated lattice paths enumerated by $C_1^0(\nu)$ and vice-versa.

Note that for $k > 0$, slant section of height t in associated lattice paths will correspond to peak starting from y-axis of height t in the paths given in [10].

The following table illustrates this bijection for $\nu = 6$:

Lattice paths enumerated by $A_1^0(6)$	Corresponding associated lattice paths enumerated by $C_1^0(6)$
	
	
	
	
	
	
	

It is worthwhile to remark here that a bijection between the lattice paths enumerated by $A_1^0(\nu)$ and the partitions with n copies of n enumerated by $B_1^0(\nu)$ is given in [7].

Sketch of the proof of Theorem 3.2. The main point of departure is that in the beginning of the path there is an SS of height 1 or a TITS with ordered pair $\{a, a+1\}$.

Sketch of the proof of Theorem 3.3. In this case there is an SS of height 2 or a TITS with ordered pair $\{a, a+2\}$ in the beginning of the path.

4. Conclusion

Theorems 3.1–3.3 are infinite classes of 3-way combinatorial identities. In some particular cases we get even 4-way identities. Each 4-way identity gives us six combinatorial identities in the usual sense. For example, when $k = 0$, Theorem 3.1 in view of the identity (eq. (46), p. 156 of [12])

$$\sum_{n=0}^{\infty} \frac{q^{n(3n-1)/2}}{(q;q)_n (q;q^2)_n} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{10n})(1-q^{10n-4})(1-q^{10n-6}),$$

yields the following 4-way identity:

$$A_1^0(\nu) = B_1^0(\nu) = C_1^0(\nu) = D_1(\nu), \quad \text{for all } \nu, \tag{4.1}$$

where $D_1(\nu)$ is the number of ordinary partitions of ν into parts $\not\equiv 0, \pm 4 \pmod{10}$. Out of the six identities induced by (4.1), three, viz., $A_1^0(\nu) = B_1^0(\nu)$, $A_1^0(\nu) = D_1(\nu)$, $B_1^0(\nu) = D_1(\nu)$ are found in the literature (cf. [1,10]) while the remaining three involving $C_1^0(\nu)$ are new.

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