

Overlapping domain decomposition methods for elliptic quasi-variational inequalities related to impulse control problem with mixed boundary conditions

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Abstract. In this paper we provide a maximum norm analysis of an overlapping Schwarz method on non-matching grids for quasi-variational inequalities related to impulse control problem with mixed boundary conditions. We provide that the discretization on every sub-domain converges in uniform norm. Furthermore, a result of approximation in uniform norm is given.

Keywords. Domain decomposition; geometrical convergence; quasi-variational inequalities; impulse control; error analysis.

1. Introduction

Schwarz method has been invented by Herman Amandus Schwarz in 1890. This method has been used to solve the stationary or evolutionary boundary value problems on domains which consists of two or more overlapping subdomains (see [1–3, 6, 8–12]). The solution is approximated by an infinite sequence of functions which results from solving a sequence of stationary or evolutionary boundary value problems in each of the subdomain. In this work we provide a maximum norm analysis of an overlapping Schwarz method on non-matching grids for quasi-variational inequalities related to impulse control problem with respect to the mixed boundary conditions.

We can state our problem as follows: Find $u \in H^1(\Omega)$ solution of

$$\left\{ \begin{array}{l} -\Delta u - f \leq 0, \\ u - Mu \leq 0 \quad Mu \geq 0, \\ (\Delta u - f)(u - Mu) = 0 \text{ in } \Omega, \\ \frac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0 \text{ and } u = 0 \text{ in } \Gamma / \Gamma_0, \end{array} \right.$$

where Ω is a smooth bounded domain of \mathbb{R}^2 with boundary Γ and f is a regular function and M is an operator given by $Mu = k + \inf_{\xi \geq 0, x+\xi \in \bar{\Omega}} u(x + \xi)$ where $k > 0$ and $\xi \geq 0$ means that $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ with $\xi_i \geq 0$, and Γ_0 is the part of the boundary defined by $\Gamma_0 = \{x \in \partial\Omega = \Gamma \text{ such that } \forall \xi > 0, x + \xi \notin \bar{\Omega}\}$. Finally, $\frac{\partial u}{\partial \eta} = \nabla u \cdot \vec{\eta}$, such that $\vec{\eta}$ is the normal vector. The symbol $(\cdot, \cdot)_{\Omega}$ stands for the inner product in $L^2(\Omega)$, $(\cdot, \cdot)_{\Gamma_0}$ stands for the inner product in $L^2(\Gamma_0)$.

On the analytical side, quasi-variational inequalities have been extensively studied in the last three decades (see [1, 2, 7–11]) and for the numerical approximations and computational aspects we have a few results (see [3–6, 12]).

In the present paper, we provide a new approach for the finite element approximation of an overlapping Schwarz method on non-matching grids for the quasi-variational inequalities related to impulse control problem. We consider a domain which is the union of two overlapping subdomains, where each sub-domain has its own generated triangulation. The grid points on the subdomain boundaries need not match the grid points from the other sub-domains. Under a discrete maximum principle [10], we show that the discretization on each sub-domain converges quasi-optimally in the L^∞ -norm.

The paper is organized as follows: In §2, we state the continuous alternating Schwarz sequence for quasi-variational inequalities and define their respective finite element counterparts in the context of overlapping grids. In §3, we provide the error analysis of the overlapping domain decomposition methods, where simple proofs to main fundamental theorems are proved. Then the geometrical convergence of the problem is established. Furthermore, an error estimate for each sub-domain is derived in a uniform norm.

2. The Schwarz method for the elliptic quasi-variational inequalities.

In this section, we introduce some definitions and some classical results related to quasi-variational inequalities.

2.1 Elliptic quasi-variational inequalities

Let Ω be a convex domain in \mathbb{R}^2 with sufficiently smooth boundary $\partial\Omega$. We consider the following obstacle problem

$$\left\{ \begin{array}{l} -\Delta u - f \leq 0, \quad \forall u \in H^1(\Omega), \text{ in } \Omega, \\ u - Mu \leq 0, \quad Mu \geq 0, \\ Mu = k + \inf_{\xi \geq 0, x+\xi \in \bar{\Omega}} u(x + \xi), \\ \frac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0 \text{ and } u = 0 \text{ in } \Gamma / \Gamma_0. \end{array} \right. \quad (2.1)$$

We are also given the right-hand side f such that

$$g \in L^\infty(\Omega).$$

M is an operator given by

$$Mu = k + \inf_{\xi \geq 0, x+\xi \in \bar{\Omega}} u(x + \xi). \quad (2.2)$$

Applying Green's formula, (2.1) can be transformed to the following elliptic variational inequalities

$$\left\{ \begin{array}{l} a(u, v - u) - (f, (v - u))_{\Omega} - (\varphi, (v - u))_{\Gamma_0} \geq 0, \quad \text{on } \Omega, \quad v \in V \\ u - Mu \leq 0 \quad Mu \geq 0, \\ Mu = k + \inf_{\xi \geq 0, x+\xi \in \bar{\Omega}} u(x + \xi), \\ \frac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0 \text{ and } u = 0 \text{ in } \Gamma / \Gamma_0, \end{array} \right. \quad (2.3)$$

where V is the non empty convex set defined by

$$V = \left\{ v \in H^1(\Omega) : \frac{\partial v}{\partial \eta} = \varphi \text{ in } \Gamma_0, \quad v \leq Mu \text{ on } \Omega \text{ and } v = 0 \text{ in } \Gamma / \Gamma_0 \right\}, \quad (2.4)$$

where φ is a regular function defined on Γ_0 . Thus, it can be easily deduce that

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx = (\nabla u, \nabla v)_{\Omega},$$

$$(f, v)_{\Omega} = \int_{\Omega} f \cdot v dx$$

and

$$(\varphi, v)_{\Gamma_0} = \int_{\Gamma_0} \varphi \cdot v d\sigma.$$

Theorem 1 [7]. *Under the previous assumptions, the problem (2.1) has an unique solution. Moreover it satisfies:*

$$u \in W^{2,p}, \quad 2 \leq p \leq \infty.$$

Theorem 2 [7]. *If $\varphi \in W^{2,\infty}(\partial\Omega)$ and $M\varphi \in W_{\text{loc}}^{2,\infty}(\Omega)$ (resp. φ is locally semi-concave). Then $u \in W_{\text{loc}}^{1,\infty}(\Omega)$ (resp. $u \in W_{\text{loc}}^{2,\infty}(\Omega)$).*

Let V^h be the space of finite elements consisting of continuous piecewise linear functions and $K_{\varphi h}$ the non-empty discrete convex set associated to K_{φ} defined as

$$K_{\varphi h} = \{u_h \in V^h : u_h = \pi_h \varphi \text{ on } \Gamma_0, \quad u_h \leq r_h M u_h \text{ on } \Omega \text{ and } u = 0 \text{ in } \Gamma / \Gamma_0\}, \quad (2.5)$$

where π_h is an interpolation operator on Γ_0 , and r_h is the usual finite element restriction operator on Ω .

The discrete counterpart of (2.3) consists of finding $u_h \in K_{\varphi h}$ such that

$$\left\{ \begin{array}{l} a(u_h, v_h - u_h) - (f, (v_h - u_h))_{\Omega} - (\varphi, (v_h - u_h))_{\Gamma_0} \geq 0, \\ Mu_h = k + \inf_{\xi \geq 0, x + \xi \in \bar{\Omega}} u_h(x + \xi), \\ u_h - r_h Mu_h \leq 0, \\ \frac{\partial u_h}{\partial \eta} = \varphi \text{ in } \Gamma_0 \text{ and } u_h = 0 \text{ in } \Gamma / \Gamma_0. \end{array} \right. \quad (2.6)$$

The lemma below establishes a monotonicity property of the solution of (2.3) with respect to the obstacle defined as an impulse control problem.

Lemma 1 [7]. *If we have $u_h \leq \tilde{u}_h$ in the $K_{\varphi h}$, then $Mu_h \leq M\tilde{u}_h$ and we have*

$$\forall u \in K_{\varphi}, \lambda \in \mathbb{R}, M(u + \lambda) = Mu + \lambda. \quad (2.7)$$

Remark 1. Under the previous hypotheses we have the following inequality

$$\|Mu - M\tilde{u}\|_{L^{\infty}(\Omega)} \leq \|u - \tilde{u}\|_{L^{\infty}(\Omega)}, \quad \forall u, \tilde{u} \in K_{\varphi}. \quad (2.8)$$

Proof. We have

$$u \leq \tilde{u} + \|u - \tilde{u}\|_{L^{\infty}(\Omega)}.$$

Using the Lemma 1, we get

$$Mu \leq M(\tilde{u} + \|u - \tilde{u}\|_{L^{\infty}(\Omega)}) = M\tilde{u} + \|u - \tilde{u}\|_{L^{\infty}(\Omega)},$$

thus

$$Mu - M\tilde{u} \leq \|u - \tilde{u}\|_{L^{\infty}(\Omega)}.$$

Similarly, interchanging the roles of Mu and $M\tilde{u}$, we get

$$M\tilde{u} - Mu \leq \|u - \tilde{u}\|_{L^{\infty}(\Omega)}.$$

Hence

$$\|Mu - M\tilde{u}\|_{L^{\infty}(\Omega)} \leq \|u - \tilde{u}\|_{L^{\infty}(\Omega)}.$$

□

Notation 1. Let $(M\xi, \varphi)$, $(M\tilde{\xi}, \tilde{\varphi})$ be a pair of data, and $\xi = \sigma(M\xi, \varphi)$, $\tilde{\xi} = \sigma(M\tilde{\xi}, \tilde{\varphi})$ be the corresponding solutions to the following quasi-variational inequalities (QVI):

$$a(\xi, v - \xi) \geq (f, v - \xi)_{\Omega} + (\varphi, (v - \xi))_{\Gamma_0}, \quad \forall v \in K_{\varphi}$$

and

$$a(\tilde{\xi}, v - \tilde{\xi}) \geq (f, v - \tilde{\xi})_{\Omega} + (\tilde{\varphi}, (v - \xi))_{\Gamma_0}, \quad \forall v \in K_{\tilde{\varphi}}.$$

Then, the following comparison result holds.

Lemma 2. If $\varphi \geq \tilde{\varphi}$, then $\sigma(M\xi, \varphi) \geq \sigma(M\tilde{\xi}, \tilde{\varphi})$.

Proof. Let $v = \min(0, \xi - \tilde{\xi})$. In the region where v is negative ($v < 0$), we have

$$\xi \leq \tilde{\xi} \leq M\tilde{\xi} \leq M\xi,$$

which means that the obstacle is not active for u . So, for v , we have

$$a(\xi, v) = (f, v)_{\Omega} + (\varphi, v)_{\Gamma_0}, \quad (2.9)$$

$$\tilde{\xi} + v \leq M\tilde{\xi}. \quad (2.10)$$

So

$$a(\tilde{\xi}, v) \geq (f, v)_{\Omega} + (\varphi, v)_{\Gamma_0}. \quad (2.11)$$

Subtracting (2.9) and (2.11) from each other, we obtain

$$a(\xi - \tilde{\xi}, v) \geq 0. \quad (2.12)$$

But

$$a(v, v) = a(\xi - \tilde{\xi}, v) = -a(\tilde{\xi} - \xi, v) \leq 0, \quad (2.13)$$

so

$$v = 0$$

and consequently,

$$\xi \geq \tilde{\xi}$$

which completes the proof. \square

The proof for the discrete case is similar.

PROPOSITION 1

Under the previous hypotheses, we have the following inequality

$$\|u - \tilde{u}\|_{L^{\infty}(\Omega_i)} \leq \|Mu - M\tilde{u}\|_{L^{\infty}(\Omega_i)} + \|\varphi - \tilde{\varphi}\|_{L^{\infty}(\partial\Omega_i \cap \Omega_j)},$$

such that $i \neq j$, $i, j = 1, 2$. (2.14)

Proof. Setting

$$\beta = \|Mu - M\tilde{u}\|_{L^\infty(\Omega_i)} + \|\varphi - \tilde{\varphi}\|_{L^\infty(\partial\Omega_i \cap \Omega_j)},$$

such that $i \neq j$, $i, j = 1, 2$, (2.15)

we have

$$\begin{aligned} Mu &\leq M\tilde{u} + Mu - M\tilde{u} \leq M\tilde{u} + |Mu - M\tilde{u}| \\ &\leq M\tilde{u} + \|Mu - M\tilde{u}\|_{L^\infty(\Omega_i)} \\ &\leq M\tilde{u} + \|Mu - M\tilde{u}\|_{L^\infty(\Omega_i)} + \|\varphi - \tilde{\varphi}\|_{L^\infty(\partial\Omega_i \cap \Omega_j)}. \end{aligned}$$

Hence

$$Mu \leq M\tilde{u} + \beta.$$

On the other hand, we have

$$\begin{aligned} \varphi &\leq \tilde{\varphi} + \varphi - \tilde{\varphi} \leq \tilde{\varphi} + |\varphi - \tilde{\varphi}| \\ &\leq \tilde{\varphi} + \|\varphi - \tilde{\varphi}\|_{L^\infty(\partial\Omega_i \cap \Omega_j)} \\ &\leq \tilde{\varphi} + \|\varphi - \tilde{\varphi}\|_{L^\infty(\partial\Omega_i \cap \Omega_j)} + \|Mu - M\tilde{u}\|_{L^\infty(\Omega_i)}, \end{aligned}$$

that is to say,

$$\varphi \leq \tilde{\varphi} + \beta. \tag{2.16}$$

Since σ is increasing in $L^\infty(\Omega)$, we have

$$\sigma(Mu, \varphi) \leq \sigma(M\tilde{u} + \beta, \tilde{\varphi} + \beta) \leq \sigma(M\tilde{u}, \tilde{\varphi}) + \beta.$$

Hence

$$\sigma(Mu, \varphi) - \sigma(M\tilde{u}, \tilde{\varphi}) \leq \beta.$$

Similarly, interchanging the roles of the couples (Mu, φ) and $(M\tilde{u}, \tilde{\varphi})$, we get

$$\sigma(M\tilde{u}, \tilde{\varphi}) - \sigma(Mu, \varphi) \leq \beta.$$

The proof for the discrete case is similar. □

Theorem 3 [3, 4]. *Under the previous hypotheses and the maximum principle [5], there exists a constant C independent of h such that*

$$\|u - u_h\|_{L^\infty(\Omega)} \leq Ch^2 |\log h|^2.$$

2.2 The continuous Schwarz sequences

Let Ω be a bounded open domain in \mathbb{R}^2 and we assume that Ω is smooth and connected.

Then we decompose Ω into two sub-domains Ω_1, Ω_2 such that

$$\Omega = \Omega_1 \cup \Omega_2 \tag{2.17}$$

and u satisfies the local regularity condition

$$u|_{\Omega_i} \in W^{2,p}(\Omega_i) \quad (2.18)$$

and we denote by $\Gamma = \partial\Omega$, $\Gamma_1 = \partial\Omega_1$, $\Gamma_2 = \partial\Omega_2$, $\gamma_1 = \partial\Omega_1 \cap \Omega_2$, $\gamma_2 = \partial\Omega_2 \cap \Omega_1$, $\Omega_{1,2} = \Omega_1 \cap \Omega_2$.

We consider the model obstacle problem. Find $u \in K_\varphi$ such that

$$\begin{cases} a(u, v - u) - (f, (v - u))_\Omega - (\varphi, (v - u))_{\Gamma_0} \geq 0, \quad \forall v \in K_\varphi, \\ u - Mu \leq 0 \quad Mu \geq 0, \\ Mu = k + \inf_{\xi \geq 0, x + \xi \in \bar{\Omega}} u(x + \xi), \\ \frac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0 \text{ and } u = 0 \text{ in } \Gamma / \Gamma_0, \end{cases} \quad (2.19)$$

where f is a given function in $L^\infty(\Omega)$, and we will assume in this section that $f \geq 0$, $\varphi \geq 0$. Indeed this enables us to make such an assumption by adding constants to u and φ and a positive function to f .

We define the following process:

Let $u_0 \in K_\varphi$ be given, and we define the alternating Schwarz sequences (u^{2n+1}) on Ω_1 such that $u^{2n+1} \in K_\varphi$ solves

$$\begin{cases} a(u^{2n+1}, v - u^{2n+1}) - (f, (v - u^{2n+1}))_{\Omega_1} - (\varphi, (v - u^{2n+1}))_{\Gamma_0} \geq 0, \quad \forall v \in V, \\ u^{2n+1} - Mu^{2n-1} \leq 0 \quad Mu \geq 0, \\ u^{2n+1} = u^{2n} \text{ on } \partial\Omega_1, \quad v = u^{2n} \text{ on } \partial\Omega_1 \end{cases} \quad (2.20)$$

and (u^{2n}) on Ω_2 such that $u^{2n} \in K_\varphi$ solves

$$\begin{cases} a(u^{2n}, v - u^{2n}) - (f, (v - u^{2n}))_{\Omega_2} - (\varphi, (v - u^{2n}))_{\Gamma_0} \geq 0, \quad \forall v \in V, \\ u^{2n} - Mu^{2n-2} \leq 0, \quad Mu^{2n-2} \geq 0, \\ u^{2n} = u^{2n-1} \text{ on } \partial\Omega_2, \quad v = u^{2n-1} \text{ on } \partial\Omega_2. \end{cases} \quad (2.21)$$

2.3 Discretization

For $i = 1, 2$, let τ^{h_i} be a standard regular and quasi-uniform finite element triangulation in Ω_i ; h_i ($h_1 = h_2 = h$) being the mesh size. We assume that the two triangulations are mutually independent on $\Omega_{1,2}$ in the sense that a triangle belonging to one triangulation does not necessarily belong to the other.

Let V^{h_i} be the space of continuous piecewise linear functions on τ^{h_i} which vanish on $\Omega_i \cap \partial\Omega_j$, $i \neq j$, $i, j = 1, 2$. For $w \in C(\partial\bar{\Omega}_i)$, we define

$$V_w^{h_i} = \{v_h \in V^{h_i} : v_h = \pi_{h_i}(w) \text{ on } \Omega_i \cap \partial\Omega_j; \\ v_h = 0 \text{ in } \Gamma/\Gamma_0; i \neq j, i, j = 1, 2\}, \quad (2.22)$$

where π_{h_i} denotes the interpolation operator on $\partial\Omega_i$.

We assume that the respective matrices resulting from the discretization of problems (2.20) and (2.21) are M -matrix [5].

We define the discrete counterparts of the continuous Schwarz sequences defined in (2.20) and (2.21), respectively by $u_h^{2n+1} \in V_{(u_h^{2n})}^h$ such that

$$\begin{cases} a(u_h^{2n+1}, v_h - u_h^{2n+1}) - (f, (v_h - u_h^{2n+1}))_{\Omega_1} \\ \quad - (\varphi, (v_h - u_h^{2n+1}))_{\Gamma_0} \geq 0, \quad \forall v_h \in V_{(u_h^{2n})}^h, \\ u_h^{2n+1} = u_h^{2n} \text{ on } \partial\Omega_1, \quad v_h = u_h^{2n} \text{ on } \partial\Omega_1, \\ u_h^{2n+1} \leq r_h M u_h^{2n-1} \end{cases} \quad (2.23)$$

and $u_h^{2n} \in V_{(u_h^{2n-1})}^h$ such that

$$\begin{cases} a(u_h^{2n}, v_h - u_h^{2n}) - (f, (v_h - u_h^{2n}))_{\Omega_2} \\ \quad - (\varphi, (v_h - u_h^{2n}))_{\Gamma_0} \geq 0, \quad \forall v_h \in V_{(u_h^{2n-1})}^h, \\ u_h^{2n} = u_h^{2n-1} \text{ on } \partial\Omega_2, \quad v_h = u_h^{2n-1} \text{ on } \partial\Omega_2, \\ u_h^{2n} \leq r_h M u_h^{2n-2}. \end{cases} \quad (2.24)$$

3. Error analysis

This section is devoted to the proof of the main result of the present paper. We need to introduce an auxiliary sequence of discrete quasi-variational inequalities first and then prove the two fundamental theorems.

For $\zeta_h^0 = u_h^0 \in K_{\varphi h}$, we define the sequences (ζ_h^{2n+1}) such that $\zeta_h^{2n+1} \in V_{(u_h^{2n})}^h$ solves

$$\begin{cases} a(\zeta_h^{2n+1}, v_h - \zeta_h^{2n+1}) - (f, (v_h - \zeta_h^{2n+1}))_{\Omega_1} \\ \quad - (\varphi, (v_h - \zeta_h^{2n+1}))_{\Gamma_0} \geq 0, \quad \forall v_h \in V_{(u_h^{2n})}^h, \\ \zeta_h^{2n+1} = u_h^{2n} \text{ on } \partial\Omega_1, \quad v_h = u_h^{2n} \text{ on } \partial\Omega_1, \\ \zeta_h^{2n+1} \leq r_h M u_h^{2n-1} \end{cases} \quad (3.1)$$

and (ζ_h^{2n}) such that $\zeta_h^{2n} \in V_{(u_h^{2n-1})}^h$ solves

$$\left\{ \begin{array}{l} a(\zeta_h^{2n}, v_h - \zeta_h^{2n}) - (f, (v_h - \zeta_h^{2n}))_{\Omega_2} \\ \quad - (\varphi, (v - \zeta_h^{2n}))_{\Gamma_0} \geq 0, \quad \forall v_h \in V_{(u_h^{2n-1})}^h, \\ \zeta_h^{2n} = u_h^{2n-1} \text{ on } \partial\Omega_2, \quad v_h = u_h^{2n-1} \text{ on } \partial\Omega_2, \\ \zeta_h^{2n} \leq r_h M u_h^{2n-2}. \end{array} \right. \quad (3.2)$$

3.1 Convergence proof via the maximum principle

We introduce the sets

$$T^{2n} = \left\{ \begin{array}{l} u_h^{2n} \in V_{(u_h^{2n-1})}^h : -\Delta u_h^{2n} \leq f, \quad u_h^{2n} = u_h^{2n-1} \text{ on } \partial\Omega_2, \\ u_h^{2n} \leq r_h M u_h^{2n-2} \text{ on } \Omega_2 \text{ and } u_h^{2n} = 0 \text{ in } \Gamma / \Gamma_0 \end{array} \right\}$$

and

$$T^{2n+1} = \left\{ \begin{array}{l} u_h^{2n+1} \in V_{(u_h^{2n})}^h : -\Delta u_h^{2n+1} \leq f, \quad u_h^{2n+1} = u_h^{2n} \text{ on } \partial\Omega_1, \\ u_h^{2n+1} \leq r_h M u_h^{2n-1} \text{ on } \Omega_1 \text{ and } u_h^{2n+1} = 0 \text{ in } \Gamma / \Gamma_0. \end{array} \right\}.$$

Lemma 3 [12]. If A is the M -matrix and u_h^{2n} (resp. u_h^{2n+1}) is the solution (2.23), (resp. (2.24)), then u_h^{2n} (resp. ∇u_h^{2n+1}) is the minimal of T^{2n} (resp. T^{2n+1}).

Theorem 4. Let u_h be a solution of (2.6). Then the iterative sequence $\{u_h^{2n}\}$ (resp. $\{u_h^{2n+1}\}$) is monotone; that is, $u_h^{2n} \in T^{2n}$ (resp. $u_h^{2n+1} \in T^{2n+1}$) and $u_h \leq u_h^{2n+2} \leq u_h^{2n} \leq \dots \leq u_h^0$.

Proof. We take $u_h^0 = u_h \mid \Omega_2$ such that $-\Delta u_h^0 = f$. We know that if $u_h^0 \leq r_h M u_h$ then $(-\Delta u_h^0 - f) \mid \Omega_2 \leq 0$, i.e.

$$(\nabla u_h^0, \nabla(v_h - u_h^0))_{\Omega_2} - (f, (v_h - u_h^0))_{\Omega_2} - (\varphi, (v - u_h^0))_{\Gamma_0} \geq 0.$$

Therefore $u_h^0 \in T^0$. From Lemma 2 we know that u_h^2 is the minimal element of T^0 . So $u_h^2 \leq r_h M u_h^0$ and yields $u_h^2 \leq u_h^0$.

By induction, for index n we obtain

$$u_h^{2n} \leq u_h^{2n-2} \leq \dots \leq u_h^2 \leq u_h^0 = u_h.$$

We know that if $u_h^3 \leq r_h M u_h^1$, then $(-\Delta u_h^3 - f) \mid \Omega_1 \leq 0$, i.e.

$$(\nabla u_h^3, \nabla(v_h - u_h^3))_{\Omega_1} - (f, (v_h - u_h^3))_{\Omega_1} - (\varphi, (v - u_h^3))_{\Gamma_0} \geq 0.$$

Therefore $u_h^3 \in T^3$. From Lemma 4 we know that u_h^3 is the minimal element of T^3 which yields $u_h^3 \leq u_h^1$.

By induction, for index n we obtain

$$u_h^{2n+1} \leq u_h^{2n-1} \leq \dots \leq u_h^1.$$

□

Lemma 4. If $A = (a_{ij})_{i,j=\{1\dots N\}}$ is the M -matrix, then there exists two constants k_1, k_2 , i.e., $k_1 = \sup \{w_h(x), x \in \gamma_2\} \in (0, 1)$, and $k_2 = \sup \{w_h(x), x \in \gamma_1\} \in (0, 1)$ such that

$$\sup_{\gamma_1} |u_h - u_h^{2n+1}| \leq k_1 \sup_{\gamma_1} |u_h - u_h^{2n}| \quad (3.3)$$

and

$$\sup_{\gamma_2} |u_h - u_h^{2n+1}| \leq k_2 \sup_{\gamma_2} |u_h - u_h^{2n}|. \quad (3.4)$$

Proof. Setting $M_1 = \sup_{\gamma_1} |u_h^{2n+1} - u_h|$ and $M = \sup_{\gamma_1} |u_h - u_h^{2n}|$, we may suppose that $M_1 \neq 0$. We prove

$$M_1 < M. \quad (3.5)$$

If (3.3) is not true then there exists $x_{i_0} \in \gamma_1$ such that

$$|u_h^{2n+1}(x_{i_0}) - u_h(x_{i_0})| = M_1 \geq M.$$

Hence, we have (noting $a_{ii} > 0$, $a_{ij} \leq 0$ for $i \neq j$ because A is the M -matrix)

$$0 \geq \sum_{i=1}^N a_{ii_0} (u_h^{2n+1} - u_h) \geq \sum_{i=1}^N a_{ii_0} \geq 0.$$

We know by Theorem 3 that $u_h^{2n+1} \geq u_h$ which implies that

$$\sum_{i \neq i_0} a_{ii_0} |u_h^{2n+1} - u_h| - M_1 = 0.$$

Therefore

$$|u_h^{2n+1} - u_h| = M_1, \quad \text{if } a_{ii_0} \neq 0. \quad (3.6)$$

Since $(a_{ij})_{i,j=\{1\dots N\}} = A$ is irreducible, there exists $x_{i_1}, x_{i_2}, \dots, x_{i_s} \in \gamma_2$ such that

$$a_{i_0 i_1}, a_{i_1 i_2}, \dots, a_{i_s i_k} \neq 0.$$

We know by (3.6) that $|u_h^{2n+1}(x_{i_1}) - u_h(x_{i_1})| = M_1$. Similarly, we get

$$\begin{aligned} |u_h^{2n+1}(x_{i_2}) - u_h(x_{i_2})| &= \dots = |u_h^{2n+1}(x_{i_s}) - u_h(x_{i_s})| \\ &= |u_h^{2n+1}(x_{i_k}) - u_h(x_{i_k})| = M_1. \end{aligned}$$

Hence we have

$$0 \geq \sum_{i=1}^N a_{li} |u_h^{2n+1} - u_h| \geq \sum_{i=1}^N a_{li} M_1 > 0,$$

which is a contradiction with (3.6).

The proof for (3.4) case is similar. \square

Remark 2. The demonstration of Lemma 4 is an adaptation of the one in [12] given for the problem of variational inequality. This lemma remains true for the problem introduced in this paper.

The main convergence result is given by the following theorem.

Theorem 5. *The sequences (u_h^{2n+1}) , (u_h^{2n}) , $n \geq 0$ produced by the Schwarz alternating method converge geometrically to the solution u of the obstacle problem (2.3). More precisely, there exist $k_1, k_2 \in (0, 1)$ which depend on (Ω_1, γ_2) and (Ω_2, γ_1) such that for all $n \geq 0$,*

$$\sup_{\bar{\Omega}_1} |u_h - u_h^{2n+1}| \leq k_1^n k_2^n \sup_{\gamma_1} |u_h - u_h^0| \quad (3.7)$$

and

$$\sup_{\bar{\Omega}_2} |u_h - u_h^{2n}| \leq k_1^n k_2^{n-1} \sup_{\gamma_2} |u_h - u_h^0|. \quad (3.8)$$

Proof. Under Lemma 4, we have

$$|u_h - u_h^{2n+1}| \leq w_h(x) \sup_{\gamma_1} |u_h - u_h^{2n}|.$$

Hence

$$\sup_{\gamma_1} |u_h - u_h^{2n+1}| \leq k_1 \sup_{\gamma_1} |u_h - u_h^{2n}|.$$

$$\text{Thus} \quad \sup_{\gamma_1} |u_h - u_h^{2n+1}| \leq k_1^{n+1} k_2^n \sup_{\gamma_1} |u_h - u_h^0|$$

and we also have

$$|u_h - u_h^{2n}| \leq w(x) \sup_{\gamma_2} |u_h - u_h^{2n-1}|.$$

Hence

$$\sup_{\gamma_2} |u_h - u_h^{2n}| \leq k_2 \sup_{\gamma_2} |u_h - u_h^{2n-1}|,$$

that is to say,

$$\sup_{\gamma_2} |u_h - u_h^{2n}| \leq k_1^n k_2^n \sup_{\gamma_2} |u_h - u_h^0|.$$

Equations (3.7) and (3.8) follow from the maximum principle which yields

$$\sup_{\bar{\Omega}_1} |u_h - u_h^{2n+1}| = \sup_{\gamma_1} |u_h - u_h^{2n+1}| = \sup_{\gamma_1} |u_h - u_h^{2n}| \text{ for } n \geq 0$$

and

$$\sup_{\bar{\Omega}_2} |u_h - u_h^{2n+1}| = \sup_{\gamma_2} |u_h - u_h^{2n+1}| = \sup_{\gamma_2} |u_h - u_h^{2n}| \text{ for } n \geq 0.$$

Hence

$$\sup_{\bar{\Omega}_1} |u_h - u_h^{2n+1}| \leq k_1^n k_2^n \sup_{\gamma_1} |u_h - u_h^0|$$

and

$$\sup_{\bar{\Omega}_2} |u_h - u_h^{2n}| \leq k_1^n k_2^{n-1} \sup_{\gamma_2} |u_h - u_h^0|.$$

□

3.2 Error estimate for the quasi-variational inequalities

Theorem 6. *Let u be a solution of problem (2.3). Then there exists a constant C independent of both h and n such that*

$$\|u - u_h^{2n+1}\|_{L^\infty(\bar{\Omega}_1)} \leq Ch^2 |\log h|^3 \quad (3.9)$$

and

$$\|u - u_h^{2n}\|_{L^\infty(\bar{\Omega}_2)} \leq Ch^2 |\log h|^3. \quad (3.10)$$

Proof. Setting $k = k_1 = k_2$, and using Theorems 2 and 4 we have

$$\begin{aligned} \|u - u_h^{2n+1}\|_{L^\infty(\bar{\Omega}_1)} &\leq \|u - u_h\|_{L^\infty(\bar{\Omega}_1)} + \|u_h - u_h^{2n+1}\|_{L^\infty(\bar{\Omega}_1)} \\ &\leq \|u - u_h\|_{L^\infty(\bar{\Omega}_1)} + k^{2n} \|u_h - u_h^0\|_{L^\infty(\gamma_1)} \\ &\leq Ch^2 |\log h|^2 + k^{2n} \|u_h - u_h^0\|_{L^\infty(\gamma_1)} \\ &\leq Ch^2 |\log h|^2 \\ &\quad + k^{2n} (\|u - u_h\|_{L^\infty(\gamma_1)} + \|u - u_h^0\|_{L^\infty(\gamma_1)}) \\ &\leq Ch^2 |\log h|^2 + Ch^2 k^{2n} |\log h|^2 \end{aligned}$$

and also setting

$$k^{2n} \leq |\log h|,$$

we get

$$\|u - u_h^{2n+1}\|_{L^\infty(\bar{\Omega}_1)} \leq Ch^2 |\log h|^3.$$

The proof for the (3.10) case is similar.

□

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References

- [1] Badea L, On the Schwarz alternating method with more than two sub-domains for monotone problems, *SIAM J. Numer. Anal.* **28**(1) (1991) 179–204
- [2] Bensoussan A and Lions J L, *Contrôle impulsif et Inéquations Quasi-variationnelles* (Dunod) (1982)
- [3] Boulbrachene M and Saadi S, Maximum norm analysis of an overlapping nonmatching grids method for the obstacle problem (Hindawi Publishing Corporation) (2006) pp. 1–10
- [4] Boulbrachene M and Haiour M, The finite element approximation of Hamilton–Jacobi–Belman equations, *Comput. Math. Appl.* **41** (2001) 993–1007
- [5] Ciarlet P and Raviart P, Maximum principle and uniform convergence for the finite element method. *Commun. Math. Appl. Mech. Eng.* **2** (1973) 1–20
- [6] Haiour M and Hadidi E, Uniform convergence of Schwarz method for noncoercive variational inequalities. *Int. J. Contemp. Math. Sci.* **4**(29) (2009) 1423–1434
- [7] Kuznetsov Yu A, Neitaanmaki P and Tarvainen P, Schwarz methods for obstacle problems with convection-diffusion operator, *Domain decomposition Methods in Scientific and Engineering Computing* (University Park. Pa. 1993) (eds) D E Keyes and J Xu. *Contemp. Math.* (Rhode Island: American Mathematical Society) (1994) vol. 180, pp. 251–256
- [8] Lions P L and Perthame B, Une remarque sur les opérateurs non linéaires intervenant dans les inéquations quasi-variationnelles. *Annales de la faculté des sciences de Toulouse 5^e serie, tome 5, n^o 3–4* (1983) pp. 259–263
- [9] Lions P L, On the Schwarz alternating method I, *First International Symposium on Domain Decomposition Methods for Partial Differential Equations* (Paris, 1987) (SIAM, Philadelphia) (1988) pp. 1–42
- [10] Lions P L, On the Schwarz alternating method II, *Stochastic interpretation and order properties, Domain Decomposition Methods* (Los Angeles, California, 1988) (SIAM, Philadelphia) (1989) pp. 47–70
- [11] Perthame B, Some remarks on quasi-variational inequalities and the associated impulsive control problem, *Annales de l’I. H. P. Section C. Tome 2, n^o 3* (1985) pp. 237–260
- [12] Zeng Jinping and Zhou Shuzi, Schwarz algorithm for the solution of variational inequalities with nonlinear source terms, *Appl. Math. Comput.* **97** (1998) 23–35