Proc. Indian Acad. Sci. (Math. Sci.) Vol. 121, No. 4, November 2011, pp. 435–446.

On hypersurfaces with two distinct principal curvatures in space forms

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MS received 28 December 2009; revised 18 August 2011

Abstract. We investigate the immersed hypersurfaces in space forms $\mathbb{N}^{n+1}(c)$, $n \geq 4$ with two distinct non-simple principal curvatures without the assumption that the (high order) mean curvature is constant. We prove that any immersed hypersurface in space forms with two distinct non-simple principal curvatures is locally conformal to the Riemannian product of two constant curved manifolds. We also obtain some characterizations for the Clifford hypersurfaces in terms of the trace free part of the second fundamental form.

Keywords. Hypersurface; space forms; principal curvature; mean curvature.

1. Introduction

Let $\mathbb{N}^{n+1}(c)$ be an $(n+1)$-dimensional simply connected space form of constant curvature c, namely,

$$\mathbb{N}^{n+1}(c) = \begin{cases} \mathbb{S}^{n+1}(c) = \left\{ x \in \mathbb{R}^{n+2} : \langle x, x \rangle = \dfrac{1}{c} \right\}, & c > 0; \\ \mathbb{R}^{n+1}, \quad c = 0; & \\ \mathbb{H}^{n+1}(c) = \left\{ x \in \mathbb{R}_1^{n+2} : \langle x, x \rangle_1 = \dfrac{1}{c}, x^{n+2} > 0 \right\}, & c < 0. \end{cases}$$

Let us first recall the definition of an important class of hypersurfaces in space forms, namely, the Clifford hypersurfaces.

Example 1.1 (*The Clifford hypersurfaces in* $\mathbb{N}^{n+1}(c)$). Let us first consider the case when $c > 0$. In this case, $\mathbb{N}^{n+1}(c) = \mathbb{S}^{n+1}(c) = \left\{ x \in \mathbb{R}^{n+2} : \langle x, x \rangle = \frac{1}{c} \right\}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^{n+2}$. For $1 \leq m \leq n-1, t \in \left(0, \frac{\pi}{2}\right)$, let $M_{m,n-m}(c, t) = \mathbb{S}^m\left(\frac{c}{\sin^2 t}\right) \times \mathbb{S}^{n-m}\left(\frac{c}{\cos^2 t}\right)$. We view $x = (x_1, x_2) \in M_{m,n-m}(c, t)$ as a vector in $\mathbb{R}^{n+2} = \mathbb{R}^{m+1} \times \mathbb{R}^{n-m+1}$, then $x \in \mathbb{S}^{n+1}(c)$. This is the standard isometric embedding of $M_{m,n-m}(c, t)$ into $\mathbb{S}^{n+1}(c)$. In this situation, for suitably chosen unit normal vector field, $M_{m,n-m}(c, t)$ has two distinct principal curvatures $\lambda = \sqrt{c} \cot t$ of multiplicity m and $\mu = -\sqrt{c} \tan t$ of multiplicity $n - m$.

Similarly, when $c = 0$, $N^{n+1}(0) = \mathbb{R}^{n+1}$. For $1 \le m \le n-1$, $t \in (0, +\infty)$, let $M_{m,n-m}(0,t) = \mathbb{R}^m \times \mathbb{S}^{n-m}(t^2)$. Then $M_{m,n-m}(0,t)$ is an embedded hypersurface in $\mathbb{R}^{n+1}$, and it has two distinct principal curvatures $\lambda = 0$ with multiplicity m and $\mu = t$ with multiplicity $n-m$.

Finally, when $c < 0$, $\mathbb{N}^{n+1}(c) = \mathbb{H}^{n+1}(c) = \left\{x \in \mathbb{R}_1^{n+2} : \langle x, x\rangle_1 = \frac{1}{c}, x^{n+2} > 0\right\}$. Here $\langle x, y\rangle_1 = x^1y^1 + \cdots + x^{n+1}y^{n+1} - x^{n+2}y^{n+2}$ is the standard Lorentzian inner product on $\mathbb{R}_1^{n+2}$. For $1 \le m \le n-1$, $t \in (0, +\infty)$, let $M_{m,n-m}(c,t) = \mathbb{S}^m\left(\frac{-c}{\sinh^2 t}\right) \times \mathbb{H}^{n-m}\left(\frac{c}{\cosh^2 t}\right)$. Then $M_{m,n-m}(c,t)$ is an embedded hypersurface in $\mathbb{H}^{n+1}(c)$, and for suitably chosen unit normal vector field, it has two distinct principal curvatures $\lambda = \sqrt{-c}\coth t$ of multiplicity m and $\mu = \sqrt{-c}\tanh t$ of multiplicity $n-m$.

There has been a long history for the study of hypersurfaces in space forms with two distinct principal curvatures. In 1970, Otsuki [4] studied the minimal hypersurfaces in $\mathbb{S}^{n+1}(1)(n \ge 3)$ with two distinct principal curvatures and proved that if the multiplicities of the two principal curvatures are both greater than 1 (namely, the two principal curvatures are both non-simple), then they are the Clifford minimal hypersurfaces. This result can be generalized to the case of constant (high order) mean curvature and other space forms (see e.g., [3,5,6]).

In this paper, we shall study the hypersurfaces in $\mathbb{N}^{n+1}(c)(n \ge 4)$ with two distinct non-simple principal curvatures without the assumption that the (high order) mean curvature is constant. For convenience, we shall denote by $M^k(c)$ or $M_1^k(c)$, etc, the k-dimensional complete Riemannian manifolds with constant curvature c. Our first result is the local structure theorem for such hypersufaces.

Theorem 1.2. *Any (connected) hypersurface in $\mathbb{N}^{n+1}(c)$, $n \ge 4$ with two distinct non-simple principal curvatures is locally conformal to $M_1^m(c_1) \times M_2^{n-m}(c_2)$ with $1 < m < n-1$ and $c_1 + c_2 = 1$.*

Now let M be a hypersurface in $\mathbb{N}^{n+1}(c)$ with two distinct principal curvatures λ, μ of multiplicities $m, n-m$. Denote by (h_{ij}) the second fundamental form of M, by $H = \frac{1}{n}\sum h_{ii}$ the mean curvature of M, and by ϕ_{ij} the tensor $h_{ij} - H\delta_{ij}$ of the trace free part of the second fundamental form (h_{ij}). Let Φ be the square of the length of (ϕ_{ij}), and $\epsilon = \mathrm{sgn}(\lambda - \mu)$ be the signature of $\lambda - \mu$. For each H, m and $\epsilon = \pm 1$, set

$$P_{m,\epsilon}(H, x) = x^2 - \epsilon\frac{n(n-2m)}{\sqrt{nm(n-m)}}Hx - n(c + H^2). \tag{1.1}$$

Suppose that $c + H^2 > 0$, and let $B_{m,\epsilon}(H)$ be the square of the positive root of $P_{m,\epsilon}(H, x) = 0$, i.e.,

$$\sqrt{B_{m,\epsilon}(H)} = \frac{n(n-2m)\epsilon H + n\sqrt{n^2H^2 + 4m(n-m)c}}{2\sqrt{nm(n-m)}}. \tag{1.2}$$

Our second result provides a characterization for Clifford hypersurfaces in terms of Φ.

Theorem 1.3. *Let M be a complete hypersurface immersed in $\mathbb{N}^{n+1}(c)$, $n \ge 4$ with two distinct non-simple principal curvatures λ, μ of multiplicities $m, n-m$. Suppose in addition that* $\inf|\lambda - \mu| > 0$, $c + H^2 > 0$ *and* $\Phi \ge B_{m,\epsilon}(H)$, *here* $\epsilon = \mathrm{sgn}(\lambda - \mu)$, *then H is constant, $\Phi = B_{m,\epsilon}(H)$, and M is isometric to the Clifford hypersurface as described in Example* 1.1.

Especially, for hypersurfaces in spheres and Euclidean space, we have the following two results.

Theorem 1.4. *Let M be a compact hypersurface immersed in $\mathbb{S}^{n+1}(c)$, $n \geq 4$ with two distinct non-simple principal curvatures λ, μ of multiplicities $m, n-m$. If one of the following three conditions holds, then M is isometric to the Clifford hypersurface as described in Example* 1.1.

(1) *M has nonnegative sectional curvature*;
(2) $\Phi \geq B_{m,\epsilon}(H)$;
(3) $\Phi \leq B_{m,\epsilon}(H)$.

Theorem 1.5. *The only complete noncompact hypersurfaces in $\mathbb{R}^{n+1}$, $n \geq 4$ with two distinct bounded non-simple principal curvatures λ, μ satisfying* $\inf|\lambda - \mu| > 0$ *are Clifford hypersurfaces in $\mathbb{R}^{n+1}$ as described in Example* 1.1.

Remark.

(1) Under the additional assumption that the mean curvature H is constant, (2) and (3) of Theorem 1.4 has been verified by Chang [2].
(2) The basic idea of the present paper can be used to study space-like hypersurfaces in Lorentzian space forms and we can obtain the Lorentzian versions of the main results of this paper (see [7]).

2. Preliminaries

Let M be an n-dimensional hypersurface in a space form $\mathbb{N}^{n+1}(c)$ of constant curvature c. For any $p \in M$, we choose a local orthonormal frame $e_1, \ldots, e_n, e_{n+1}$ in $\mathbb{N}^{n+1}(c)$ around p such that $e_1, \ldots, e_n$ are tangent to M. Take the corresponding dual coframe $\omega_1, \ldots, \omega_n, \omega_{n+1}$ with the connection 1-forms ω_{AB}, $1 \leq A, B \leq n+1$. We make the convention on the range of indices that $1 \leq A, B, \ldots \leq n+1$, $1 \leq i, j, \ldots \leq n$. The structure equations of $\mathbb{N}^{n+1}(c)$ are

$$\mathrm{d}\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$\mathrm{d}\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D, \tag{2.2}$$

$$K_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}), \tag{2.3}$$

where K_{ABCD} is the curvature tensor of $\mathbb{N}^{n+1}(c)$. When restricted to M, we have $\omega_{n+1} = 0$, and thus $0 = \mathrm{d}\omega_{n+1} = -\sum_i \omega_{n+1\,i} \wedge \omega_i$. By Cartan's lemma, there exist local functions h_{ij} such that

$$\omega_{n+1\,i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.4}$$

The *second fundamental form* is $h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j$. We also write $h = (h_{ij})_{n\times n}$ and call the eigenvalues of matrix (h_{ij}) the *principal curvatures* of M. The *mean curvature* of M is given by $H = \frac{1}{n}\mathrm{tr}(h) = \frac{1}{n}\sum_i h_{ii}$. From (2.1)–(2.4) we obtain the structure equations of M,

$$\mathrm{d}\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.5}$$

$$\mathrm{d}\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l \tag{2.6}$$

and the Gauss equations

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{2.7}$$

where R is the Riemannian curvature tensor of M. The covariant derivative of h_{ij} is defined by

$$\sum_k h_{ijk}\omega_k = \mathrm{d}h_{ij} - \sum_k \left(h_{kj}\omega_{ki} + h_{ik}\omega_{kj}\right). \tag{2.8}$$

Thus, by exterior differentiation of (2.4), we obtain the Codazzi equation

$$h_{ijk} = h_{ikj}. \tag{2.9}$$

3. Conformal structure for hypersurfaces

Now, let M be a (connected) hypersurface in $\mathbb{N}^{n+1}(c)$ with two distinct non-simple principal curvatures with multiplicities $m, n-m$, here $1 < m < n-1$. In this situation, we can choose local frame field $e_1, \ldots, e_n$ such that

$$h_{ij} = \lambda_i \delta_{ij}, \tag{3.1}$$

where

$$\lambda_1 = \cdots = \lambda_m = \lambda, \quad \lambda_{m+1} = \cdots = \lambda_n = \mu. \tag{3.2}$$

By means of (2.8) and (3.1), we obtain

$$\sum_k h_{ijk}\omega_k = \delta_{ij} d\lambda_i + (\lambda_j - \lambda_i)\omega_{ij}. \tag{3.3}$$

In the following we shall use the convention on the ranges of indices: $1 \le a, b, c, \ldots \le m$, $m+1 \le r, s, t, \ldots \le n$. From (2.9), (3.2) and (3.3) we easily get

$$h_{abi} = 0, \quad \forall a \neq b, i, \tag{3.4}$$

$$h_{rsi} = 0, \quad \forall r \neq s, i, \tag{3.5}$$

$$\lambda_{,a} = \mu_{,r} = 0, \quad \forall a, r, \tag{3.6}$$

$$h_{aar} = \lambda_{,r}, \quad h_{rra} = \mu_{,a}, \quad \forall a, r. \tag{3.7}$$

Here $\lambda_{,i} = e_i(\lambda)$, $\mu_{,i} = e_i(\mu)$. Combining (2.9), (3.3)–(3.7), we have

$$\sum_i h_{ari}\omega_i = (\lambda - \mu)\omega_{ra} = \lambda_{,r}\omega_a + \mu_{,a}\omega_r,$$

and consequently,

$$\omega_{ra} = \frac{\lambda_{,r}}{\lambda - \mu}\omega_a + \frac{\mu_{,a}}{\lambda - \mu}\omega_r = \frac{(\lambda - \mu)_{,r}}{\lambda - \mu}\omega_a - \frac{(\lambda - \mu)_{,a}}{\lambda - \mu}\omega_r. \tag{3.8}$$

Now we consider a new Riemannian metric $d\bar{s}^2$ on M by

$$d\bar{s}^2 = \sum_i \bar{\omega}_i^2, \quad \bar{\omega}_i = (\lambda - \mu)\omega_i. \tag{3.9}$$

Clearly, (M, ds^2) is conformal to $(M, d\bar{s}^2)$. In the following we are going to prove that locally $(M, d\bar{s}^2)$ is isometric to the Riemannian product of two constant curved manifolds of dimensions m and $n-m$. For smooth function f on M, let $f_{,i}$ and $f_{,\bar{i}}$ be the components of the first covariant derivative of f with respect to the metric ds^2 and $d\bar{s}^2$, respectively. By definition, we have

$$df = \sum_i f_{,i}\omega_i = \sum_i f_{,\bar{i}}\bar{\omega}_i,$$

and thus

$$f_{,i} = (\lambda - \mu)f_{,\bar{i}}. \tag{3.10}$$

Let $\bar{\omega}_{ij}$ be the connection 1-forms of $d\bar{s}^2$. Then by the structure equations of ds^2 and $d\bar{s}^2$, it is easy to see that

$$\begin{aligned} \omega_{ij} &= \bar{\omega}_{ij} + \frac{(\lambda - \mu)_{,\bar{i}}}{\lambda - \mu}\bar{\omega}_j - \frac{(\lambda - \mu)_{,\bar{j}}}{\lambda - \mu}\bar{\omega}_i \\ &= \bar{\omega}_{ij} + \frac{(\lambda - \mu)_{,i}}{\lambda - \mu}\omega_j - \frac{(\lambda - \mu)_{,j}}{\lambda - \mu}\omega_i. \end{aligned} \tag{3.11}$$

Combining (3.6), (3.8), (3.10) and (3.11) we have

$$\omega_{ab} = \bar{\omega}_{ab} - \frac{\mu_{,\bar{a}}}{\lambda - \mu}\bar{\omega}_b + \frac{\mu_{,\bar{b}}}{\lambda - \mu}\bar{\omega}_a, \tag{3.12}$$

$$\omega_{rs} = \bar{\omega}_{rs} + \frac{\lambda_{,\bar{r}}}{\lambda - \mu}\bar{\omega}_s - \frac{\lambda_{,\bar{s}}}{\lambda - \mu}\bar{\omega}_r, \tag{3.13}$$

$$\omega_{ar} = -\frac{\mu_{,\bar{a}}}{\lambda - \mu}\bar{\omega}_r - \frac{\lambda_{,\bar{r}}}{\lambda - \mu}\bar{\omega}_a, \quad \bar{\omega}_{ar} = 0. \tag{3.14}$$

Since $\bar{\omega}_{ar} = 0$, $(M, d\bar{s}^2)$ is locally isometric to the Riemannian product $\left(M_1^m, d\bar{s}_1^2\right) \times \left(M_2^{n-m}, d\bar{s}_2^2\right)$ of two manifolds of dimensions m and $n-m$, and λ and μ can be viewed as a function on $\left(M_2^{n-m}, d\bar{s}_2^2\right)$ and $\left(M_1^m, d\bar{s}_1^2\right)$, respectively. Here $d\bar{s}_1^2 = \sum_a \bar{\omega}_a^2$, $d\bar{s}_2^2 =$

$\sum_r \bar{\omega}_r^2$. Taking exterior differentiation on the first equality of (3.14) and using (2.6), (2.7) and (3.12)–(3.14) yields

$$\begin{aligned} \mathrm{d}\omega_{ar} &= -\sum_b \omega_{ab}\wedge\omega_{br} - \sum_s \omega_{as}\wedge\omega_{sr} + \frac{1}{2}R_{arij}\omega_i\wedge\omega_j \\ &= \sum_b \left(\bar{\omega}_{ab} - \frac{\mu_{,\bar{a}}}{\lambda-\mu}\bar{\omega}_b + \frac{\mu_{,\bar{b}}}{\lambda-\mu}\bar{\omega}_a\right)\wedge\left(\frac{\mu_{,\bar{b}}}{\lambda-\mu}\bar{\omega}_r + \frac{\lambda_{,\bar{r}}}{\lambda-\mu}\bar{\omega}_b\right) \\ &\quad + \sum_s \left(\frac{\mu_{,\bar{a}}}{\lambda-\mu}\bar{\omega}_s + \frac{\lambda_{,\bar{s}}}{\lambda-\mu}\bar{\omega}_a\right)\wedge\left(\bar{\omega}_{sr} + \frac{\lambda_{,\bar{s}}}{\lambda-\mu}\bar{\omega}_r - \frac{\lambda_{,\bar{r}}}{\lambda-\mu}\bar{\omega}_s\right) \\ &\quad + \frac{c+\lambda\mu}{(\lambda-\mu)^2}\bar{\omega}_a\wedge\bar{\omega}_r = -\mathrm{d}\left(\frac{\mu_{,\bar{a}}}{\lambda-\mu}\right)\wedge\bar{\omega}_r + \frac{\mu_{,\bar{a}}}{\lambda-\mu}\sum_s \bar{\omega}_{rs}\wedge\bar{\omega}_s \\ &\quad - \mathrm{d}\left(\frac{\lambda_{,\bar{r}}}{\lambda-\mu}\right)\wedge\bar{\omega}_a + \frac{\lambda_{,\bar{r}}}{\lambda-\mu}\sum_b \bar{\omega}_{ab}\wedge\bar{\omega}_b. \end{aligned} \tag{3.15}$$

Note that as a function on $(M_1^m, \mathrm{d}\bar{s}_1^2)$, the second covariant derivative of μ is defined by

$$\sum_b \mu_{,\bar{a}\bar{b}}\bar{\omega}_b = \mathrm{d}\mu_{,\bar{a}} - \sum_b \mu_{,\bar{b}}\bar{\omega}_{ba}, \tag{3.16}$$

Similarly, as a function on $\left(M_2^{n-m}, \mathrm{d}\bar{s}_2^2\right)$, the second covariant derivative of λ is given by

$$\sum_s \lambda_{,\bar{r}\bar{s}}\bar{\omega}_s = \mathrm{d}\lambda_{,\bar{r}} - \sum_s \lambda_{,\bar{s}}\bar{\omega}_{sr}. \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.15), after simplifying we reach at

$$\begin{aligned} &\frac{1}{(\lambda-\mu)^2}\left(c+\lambda\mu+|\bar{\nabla}\lambda|^2+|\bar{\nabla}\mu|^2\right)\bar{\omega}_a\wedge\bar{\omega}_r \\ &\quad = -\frac{1}{\lambda-\mu}\left(\sum_b \mu_{,\bar{a}\bar{b}}\bar{\omega}_b\wedge\bar{\omega}_r + \sum_s \lambda_{,\bar{r}\bar{s}}\bar{\omega}_s\wedge\bar{\omega}_a\right), \end{aligned}$$

where $|\bar{\nabla}\lambda|^2 = \sum_r \lambda_{,\bar{r}}^2$, $|\bar{\nabla}\mu|^2 = \sum_a \mu_{,\bar{a}}^2$. By comparison we get

$$c+\lambda\mu+|\bar{\nabla}\lambda|^2+|\bar{\nabla}\mu|^2+(\lambda-\mu)(\mu_{,\bar{a}\bar{a}}-\lambda_{,\bar{r}\bar{r}}) = 0, \quad \forall a, r, \tag{3.18}$$

$$\lambda_{,\bar{r}\bar{s}} = 0, \quad \forall r\neq s; \quad \mu_{,\bar{a}\bar{b}} = 0, \quad \forall a\neq b. \tag{3.19}$$

From (3.18) it is clear that

$$\lambda_{,\bar{r}\bar{r}} = \lambda_{,\bar{s}\bar{s}}, \quad \forall r, s; \quad \mu_{,\bar{a}\bar{a}} = \mu_{,\bar{b}\bar{b}}, \quad \forall a, b. \tag{3.20}$$

Let $\bar{R}_{abcd}$ be the curvature tensor of $(M_1^m, \mathrm{d}\bar{s}_1^2)$, it is determined by the structure equation

$$\mathrm{d}\bar{\omega}_{ab} = -\sum_c \bar{\omega}_{ac}\wedge\bar{\omega}_{cb} + \frac{1}{2}\sum_{c,d}\bar{R}_{abcd}\bar{\omega}_c\wedge\bar{\omega}_d. \tag{3.21}$$

Similarly, the curvature tensor $\bar{R}_{rstw}$ of $\left(M_2^{n-m}, \mathrm{d}\bar{s}_2^2\right)$ is determined by

$$\mathrm{d}\bar{\omega}_{rs} = -\sum_t \bar{\omega}_{rt} \wedge \bar{\omega}_{ts} + \frac{1}{2}\sum_{t,w} \bar{R}_{rstw}\bar{\omega}_t \wedge \bar{\omega}_w. \tag{3.22}$$

Differentiating (3.12) and (3.13) and using (2.6), (2.7) and (3.20)–(3.22), after simplifying we get

$$\bar{R}_{abcd} = c_1(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}), \quad \bar{R}_{rstw} = c_2(\delta_{rt}\delta_{sw} - \delta_{rw}\delta_{st}). \tag{3.23}$$

Here

$$c_1 = \frac{1}{(\lambda-\mu)^2}\left(c + \lambda^2 + |\bar{\nabla}\lambda|^2 + |\bar{\nabla}\mu|^2 + 2(\lambda-\mu)\mu_{,\bar{a}\bar{a}}\right), \tag{3.24}$$

$$c_2 = \frac{1}{(\lambda-\mu)^2}\left(c + \mu^2 + |\bar{\nabla}\lambda|^2 + |\bar{\nabla}\mu|^2 - 2(\lambda-\mu)\lambda_{,\bar{r}\bar{r}}\right). \tag{3.25}$$

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By (3.23)–(3.25), we need only to prove that c_1, c_2 are constant with $c_1 + c_2 = 1$. By (3.18) it is easy to see that $c_1 + c_2 = 1$. On the other hand, by (3.18) and (3.19) we see that

$$\begin{aligned}(c_1)_{,\bar{r}} \;=\; & \frac{1}{(\lambda-\mu)^2}(2\lambda\lambda_{,\bar{r}} + 2\lambda_{,\bar{r}}\lambda_{,\bar{r}\bar{r}} + 2\lambda_{,\bar{r}}\mu_{,\bar{a}\bar{a}}) \\ & -\frac{2\lambda_{,\bar{r}}}{(\lambda-\mu)^3}\left(c + \lambda^2 + |\bar{\nabla}\lambda|^2 + |\bar{\nabla}\mu|^2 + 2(\lambda-\mu)\mu_{,\bar{a}\bar{a}}\right) = 0.\end{aligned}$$

Similarly, $(c_2)_{,\bar{a}} = 0$. Since $c_1 + c_2 = 1$, we have $(c_1)_{,\bar{r}} = (c_1)_{,\bar{a}} = (c_2)_{,\bar{a}} = (c_2)_{,\bar{r}} = 0$, namely, c_1, c_2 are constant. Hence we have proved the Theorem. □

4. Auxiliary lemmas

In order to prove Theorems 1.3–1.5, we need some auxiliary lemmas. At first we have the following:

Lemma 4.1 (*Euler's lemma*) [1]. *Let* $f : \mathbb{R}^k \to \mathbb{R}$ *be a smooth function defined on a Euclidean k-space. If* f *is positively homogeneous of degree* s, *namely*, $f(t \cdot x) = t^s \cdot f(x), \forall t > 0$, *then*

$$\sum_{A=1}^{k} x^A \frac{\partial f}{\partial x^A} = s \cdot f.$$

Lemma 4.2. *Let* $f : \mathbb{N}^m(c) \to \mathbb{R}, m \geq 2$ *be a smooth function, and* $f_{,\bar{a}\bar{b}}, 1 \leq a, b \leq m$ *be the components of the second covariant derivative of* f *with respect to the local orthonormal frame of* $\mathbb{N}^m(c)$. *If* f *satisfies*

$$f_{,\bar{a}\bar{b}} = g \cdot \delta_{ab}. \tag{4.1}$$

Here $g : \mathbb{N}^m(c) \to \mathbb{R}$ *is a smooth function, then*

(1) *when* $c > 0, \mathbb{N}^m(c) = \mathbb{S}^m(c) \hookrightarrow \mathbb{R}^{m+1}$, *there is a constant vector* $p \in \mathbb{R}^{m+1}$ *and a constant* b *such that* $f(x) = \langle x, p\rangle + b, g(x) = -c\langle x, p\rangle, \forall x \in \mathbb{S}^m(c)$;
(2) *when* $c = 0, \mathbb{N}^m(0) = \mathbb{R}^m$, *there are two constants* a, b *and a vector* $p \in \mathbb{R}^m$ *such that* $f(x) = a\langle x - p, x - p\rangle + b, g(x) = a, \forall x \in \mathbb{R}^m$; *consequently,* f *is constant if it is bounded*;
(3) *when* $c < 0, \mathbb{N}^m(c) = \mathbb{H}^m(c) \hookrightarrow \mathbb{R}_1^{m+1}$, *there is a constant vector* $p \in \mathbb{R}_1^{m+1}$ *and a constant* b *such that* $f(x) = \langle x, p\rangle_1 + b, g(x) = -c\langle x, p\rangle_1, \forall x \in \mathbb{H}^m(c)$. *Consequently, if* f *has upper bound or lower bound, then* p *is time-like or* $p = 0$; *and* f *is constant if it is bounded.*

Proof. It should be noted that this lemma has been verified in [7], here we include the proof for readers' convenience. We shall only prove (3), and (1) and (2) can be proved similarly. Without loss of generality, we assume that $c = -1$. Let $x^1, \ldots, x^{m+1}$ be the global co-ordinates of $\mathbb{R}_1^{m+1}$ so that the Lorentzian inner product on $\mathbb{R}_1^{m+1}$ is given by

$$\langle x, y\rangle_1 = \sum_{A=1}^{m+1} \epsilon_A x^A y^A, \quad \forall x = (x^1, \ldots, x^{m+1}), y = (y^1, \ldots, y^{m+1}).$$

Here $\epsilon_1 = \cdots = \epsilon_m = 1 = -\epsilon_{m+1}$. Let $\mathcal{C}^+ = \{x \in \mathbb{R}_1^{m+1} : \langle x, x\rangle_1 < 0, x^{m+1} > 0\}$, then $r = \sqrt{-\langle x, x\rangle_1} : \mathcal{C}^+ \to \mathbb{R}$ is a smooth function, and the hyperbolic m-space of constant sectional curvature -1 is defined by $\mathbb{H}^m(-1) = r^{-1}(1)$. We choose the local orthonormal frame $e_1, \ldots, e_m, e_{m+1} = \frac{\partial}{\partial r} = \frac{x}{r}$ of $\mathcal{C}^+$, then when restricted on $\mathbb{H}^m(-1)$, $e_1, \ldots, e_m$ are tangent to $\mathbb{H}^m(-1)$, and the standard orthonormal basis $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{m+1}}$ of $\mathbb{R}_1^{m+1}$ can be expressed by

$$\frac{\partial}{\partial x^A} = \sum_a c_A^a e_a - \frac{\epsilon_A x^A}{r} e_{m+1}. \tag{4.2}$$

From (4.2) it is clear that

$$\epsilon_A \delta_{AB} = \left\langle \frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B} \right\rangle_1 = \sum_a c_A^a c_B^a - \frac{\epsilon_A \epsilon_B x^A x^B}{r^2}. \tag{4.3}$$

By the definition of r we have

$$\frac{\partial^2 r}{\partial x^A \partial x^B} = Hr\left(\frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B}\right) = -\frac{1}{r}\left(\epsilon_A \delta_{AB} + \frac{\epsilon_A \epsilon_B x^A x^B}{r^2}\right). \tag{4.4}$$

Here H denotes the Hessian operator on $\mathbb{R}_1^{m+1}$. Let D be the Levi-Civita connection of $\mathbb{R}_1^{m+1}$. Then

$$D_{e_a} e_b = \bar{\nabla}_{e_a} e_b + \frac{1}{r}\delta_{ab} e_{m+1}, \quad D_{e_a} e_{m+1} = \frac{e_a}{r}, \quad D_{e_{m+1}} e_{m+1} = 0. \tag{4.5}$$

Here $\bar{\nabla}_{e_a} e_b$ is the component of $D_{e_a} e_b$ which is orthogonal to e_{m+1}. It is clear that when restricted on $\mathbb{H}^m(c)$, $\bar{\nabla}$ is the Levi-Civita connection of $\mathbb{H}^m(c)$, here $c < 0$. Let f be the function given by the lemma. We can extend it to a function $\tilde{f}$ which is positively homogeneous of degree zero on $\mathcal{C}^+$ by $\tilde{f}(x) = f(\frac{x}{r}), \forall x \in \mathcal{C}^+$. Let $\bar{H}\tilde{f}(e_a, e_b) = e_a e_b \tilde{f} - \bar{\nabla}_{e_a} e_b \tilde{f}$. Then $\bar{H}\tilde{f}(e_a, e_b)$ is positively homogeneous of degree -2, and it is the component of the Hessian of $\tilde{f}$ when restricted on $\mathbb{H}^m(c)$ for any $c < 0$. Therefore, by (4.1) we have

$$\bar{H}\tilde{f}(e_a, e_b) = \tilde{g} \cdot \frac{\delta_{ab}}{r^2}. \tag{4.6}$$

Here $\tilde{g}$ is defined in the same way as $\tilde{f}$. Now we define a function F on $\mathcal{C}^+$ by $F = r \cdot \tilde{f}$. It is clear that $e_a(r) = e_{m+1}(\tilde{f}) = 0$, $e_{m+1}(r) = 1$. By (4.5) we have

$$HF(e_a, e_b) = e_a e_b(F) - D_{e_a} e_b(F) = r \cdot \bar{H}\tilde{f}(e_a, e_b) - \delta_{ab} \cdot \frac{\tilde{f}}{r},$$

which together with (4.6) yields

$$HF(e_a, e_b) = G \cdot \delta_{ab}, \quad G = \frac{\tilde{g} - \tilde{f}}{r}. \tag{4.7}$$

By (4.6) we also have

$$HF(e_a, e_{m+1}) = HF(e_{m+1}, e_{m+1}) = 0. \tag{4.8}$$

Combining (4.2)–(4.4), (4.7) and (4.8) we get

$$\frac{\partial^2 F}{\partial x^A \partial x^B} = HF\left(\frac{\partial}{\partial x^A}, \frac{\partial}{\partial x^B}\right) = \sum_a c_A^a c_B^a G = -\frac{\partial^2 r}{\partial x^A \partial x^B} \cdot rG. \tag{4.9}$$

It is clear from (4.9) that

$$\frac{\partial^2 r}{\partial x^A \partial x^B} \cdot \frac{\partial (rG)}{\partial x^C} = \frac{\partial^2 r}{\partial x^A \partial x^C} \cdot \frac{\partial (rG)}{\partial x^B},$$

which together with (4.4) implies that

$$\left(\delta_{AB} + \frac{\epsilon_B x^A x^B}{r^2}\right) \frac{\partial (rG)}{\partial x^C} = \left(\delta_{AC} + \frac{\epsilon_C x^A x^C}{r^2}\right) \frac{\partial (rG)}{\partial x^B}. \tag{4.10}$$

It is clear that $rG = \tilde{g} - \tilde{f}$ is positively homogeneous of degree zero, thus by Lemma 4.1 we have

$$\sum_A x^A \frac{\partial (rG)}{\partial x^A} = 0. \tag{4.11}$$

Letting $A = C$ in (4.10) and then taking the sum, by using (4.11) it is easy to get

$$(m-1)\frac{\partial (rG)}{\partial x^B} = 0.$$

Since $m \geq 2$, we see that $\frac{\partial(rG)}{\partial x^B} = 0, \forall B$. As the result, $rG = a_1$ (a constant). Now (4.9) shows that the function $F + a_1 r$ is a linear function on $\mathcal{C}^+$, namely, there is a constant a_2 and a constant vector $p \in \mathbb{R}_1^{m+1}$ such that $F(x) + a_1 r(x) = \langle p, x\rangle_1 + a_2$, and consequently, $f(x) = \langle p, x\rangle_1 + b, \forall x \in \mathbb{H}^m(-1)$. Here $b = a_2 - a_1$. Now it is clear that $f_{,\bar{a}\bar{b}} = \langle p, x\rangle_1 \cdot \delta_{ab}$, and this implies that $g(x) = \langle p, x\rangle_1, \forall x \in \mathbb{H}^m(-1)$, and thus we are done. □

By Lemma 4.2 we can prove

Lemma 4.3. *Let M be a complete hypersurface immersed in $\mathbb{N}^{n+1}(c)(n \geq 4)$ with two distinct non-simple principal curvatures λ, μ with* $\inf|\lambda - \mu| > 0$. *If $c + \lambda\mu \leq 0$, then M is isometric to the Clifford hypersurface as described in Example* 1.1.

Proof. Let $\mathrm{d}s^2$ be the original Riemannian metric on M. By assumption, $\mathrm{d}s^2$ is complete, and $\inf|\lambda - \mu| > 0$, it is clear that the new metric $\mathrm{d}\bar{s}^2 = (\lambda - \mu)^2 \mathrm{d}s^2$ is also complete, and thus by Theorem 1.2, $(M, \mathrm{d}\bar{s}^2)$ is isometric to $M_1^m(c_1) \times M_2^{n-m}(c_2)$, where m is the multiplicity of λ. We shall prove the lemma when $\lambda - \mu > 0$, the case when $\lambda - \mu < 0$ can be shown similarly. In this situation, since λ and μ can be viewed as functions on $M_2^{n-m}(c_2)$ and $M_1^m(c_1)$, respectively, we conclude that λ has lower bound while μ has upper bound. Now we claim that if λ is not a constant, then it attains its minimum at some point $v_0 \in M_2^{n-m}(c_2)$, and $\lambda_{,\bar{r}\bar{r}} > 0$ for any r at v_0. To prove this claim, we can assume that $M_2^{n-m}(c_2) \cong \mathbb{N}^{n-m}(c_2)$ without loss of generality (we may consider the lift of function λ to the universal covering space if necessary). By (3.19) and (3.20), λ satisfies $\lambda_{,\bar{r}\bar{s}} = \nu \cdot \delta_{rs}$, here ν is a smooth function on $\mathbb{N}^{n-m}(c_2)$. If $c_2 > 0$, then $\mathbb{N}^{n-m}(c_2) = \mathbb{S}^{n-m}(c_2)$, and by Lemma 4.2, $\lambda(v) = a\langle v, p\rangle + b, \forall v \in \mathbb{S}^{n-m}(c_2)$ for some constants a, b and $p \in \mathbb{S}^{n-m}(c_2)$. Since λ is not a constant, we have $a \neq 0$, and without loss of generality, we assume that $a > 0$. In this situation, λ attains its minimum at $v_0 = -p$, and since $\lambda_{,\bar{r}\bar{r}} = -ac_2\langle v, p\rangle$, one has $\lambda_{,\bar{r}\bar{r}} > 0$ at $v_0 = -p$ for any r. When $c_2 = 0$ or $c_2 < 0$, we can show that the claim still holds, by using Lemma 4.2. Similarly, if μ is not a constant, then it attains its maximum at some point $u_0 \in M_1^m(c_1)$, and $\mu_{,\bar{a}\bar{a}} < 0$ for any a at u_0. Now let $(u_0, v_0) \in M_1^m(c_1) \times M_2^{n-m}(c_2)$ be the point such that λ attains its minimum and μ attains its maximum, then since $c + \lambda\mu \leq 0$, by (3.18) and the maximum principal we have

$$0 \geq c + \lambda\mu = (\lambda - \mu)(\lambda_{,\bar{r}\bar{r}} - \mu_{,\bar{a}\bar{a}}) \geq 0$$

at (u_0, v_0). Noting that $\inf(\lambda - \mu) > 0$, we must have $\lambda_{,\bar{r}\bar{r}} = \mu_{,\bar{a}\bar{a}} = 0, \forall a, r$. Thus by the above discussion, λ, μ are both constants, and consequently M is isometric to the Clifford hypersurface, and the lemma is proved. □

5. The proof of the main results

In this last section we shall complete the proof of Theorems 1.3–1.5.

Proof of Theorem 1.3. By assumption, the second fundamental form (h_{ij}) has two non-simple eigenvalues λ and μ with multiplicities m and $n - m$, thus $(\phi_{ij}) = (h_{ij} - H\delta_{ij})$

has two eigenvalues $\tilde{\lambda} = \lambda - H$ and $\tilde{\mu} = \mu - H$ of multiplicities m and $n - m$, and consequently,

$$0 = \operatorname{trace}(\phi_{ij}) = m\tilde{\lambda} + (n-m)\tilde{\mu}, \tag{5.1}$$

$$\Phi = m\tilde{\lambda}^2 + (n-m)\tilde{\mu}^2. \tag{5.2}$$

By (5.1) and (5.2) it is easy to see that

$$\tilde{\lambda} = \epsilon\sqrt{\frac{n-m}{mn}\Phi}, \quad \tilde{\mu} = -\epsilon\sqrt{\frac{m}{(n-m)n}\Phi}. \tag{5.3}$$

Here $\epsilon = \operatorname{sgn}(\tilde{\lambda} - \tilde{\mu}) = \operatorname{sgn}(\lambda - \mu)$. From (5.3) we have

$$c + \lambda\mu = c + (\tilde{\lambda} + H)(\tilde{\mu} + H) = c + H^2 + \frac{n-2m}{\sqrt{nm(n-m)}}\epsilon H\sqrt{\Phi} - \frac{1}{n}\Phi,$$

and thus

$$-n(c+\lambda\mu) = \Phi - \epsilon\frac{n(n-2m)}{\sqrt{nm(n-m)}}H\sqrt{\Phi} - n(c+H^2) = P_{m,\epsilon}(H, \sqrt{\Phi}). \tag{5.4}$$

Notice that $\Phi \geq B_{m,\epsilon}(H)$. We see that $n(c + \lambda\mu) = -P_{m,\epsilon}(H, \sqrt{\Phi}) \leq 0$, thus by Lemma 4.3 we conclude that M is isometric to the Clifford hypersurface as described in Example 1.1. $\square$

Proof of Theorem 1.4. Note that Case (2) is the special case of Theorem 1.3. We need only to prove Cases (1) and (3). By (3.20), (3.18) can be re-written as

$$\frac{c + \lambda\mu + |\bar{\nabla}\lambda|^2 + |\bar{\nabla}\mu|^2}{\lambda - \mu} + \frac{1}{m}\bar{\Delta}\mu - \frac{1}{n-m}\bar{\Delta}\lambda = 0.$$

Here $\bar{\Delta}$ denotes the Laplace operator on $(M, \mathrm{d}\bar{s}^2)$. Since M is compact, by Stokes theorem we get

$$\int_{(M,\mathrm{d}\bar{s}^2)} \frac{c + \lambda\mu + |\bar{\nabla}\lambda|^2 + |\bar{\nabla}\mu|^2}{\lambda - \mu} = 0. \tag{5.5}$$

If M has nonnegative sectional curvature, by Gauss equation it is equivalent to $c+\lambda\mu \geq 0$, which together with (5.5) yields $|\bar{\nabla}\lambda| = |\bar{\nabla}\mu| = 0$, namely, λ and μ are constants, and thus M is isometric to the Clifford hypersurface. If $\Phi \leq B_{m,\epsilon}(H)$, then by (5.4) one has $n(c + \lambda\mu) = -P_{m,\epsilon}(H, \sqrt{\Phi}) \geq 0$, again by (5.5) M is isometric to the Clifford hypersurface. $\square$

Proof of Theorem 1.5. Since $\inf|\lambda - \mu| > 0$, by Theorem 1.2, $(M, \mathrm{d}\bar{s}^2 = (\lambda - \mu)^2\mathrm{d}s^2)$ is isometric to $M_1^m(c_1) \times M_2^{n-m}(c_2)$ with $c_1 + c_2 = 1$. Here by (3.24) and (3.25), c_1 and c_2 are given by

$$c_1 = \frac{1}{(\lambda - \mu)^2}\left(\lambda^2 + |\bar{\nabla}\lambda|^2 + |\bar{\nabla}\mu|^2 + 2(\lambda - \mu)\mu_{,\bar{a}\bar{a}}\right), \tag{5.6}$$

$$c_2 = \frac{1}{(\lambda - \mu)^2}\left(\mu^2 + |\bar{\nabla}\lambda|^2 + |\bar{\nabla}\mu|^2 - 2(\lambda - \mu)\lambda_{,\bar{r}\bar{r}}\right). \tag{5.7}$$

Note that M is noncompact, and we must have $c_1 \leq 0$ or $c_2 \leq 0$. Without loss of generality, we assume that $c_1 \leq 0$. Since μ is bounded, by (3.19), (3.20) and Lemma 4.2, μ is constant, which together with (5.6) yields $\lambda = c_1 = 0$, namely, both λ and μ are constants, and consequently M is isometric to the Clifford hypersurface. □

Acknowledgments

This project was supported by the Natural Science Foundation of China (No. 11171139) and the Natural Science Foundation of Fujian Province of China (No. 2010J01009).

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