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On the normal subgroup with coprime G-conjugacy class sizes

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Abstract. Let N be a normal subgroup of a group G. The positive integers m and n are the two longest sizes of the non-central G-conjugacy classes of N with $m > n$ and $(m, n) = 1$. In this paper, the structure of N is determined when n divides $|N/N \cap Z(G)|$. Some known results are generalized.

Keywords. Finite group; normal subgroup; G-conjugacy class size; Frobenius group.

1. Introduction

All groups considered in this paper are finite. Let G be a group, and x an element of G. We use x^G to denote the G-conjugacy class containing x, and $|x^G|$ the size of x^G. Among all G-conjugacy class sizes, the first and the second largest sizes of conjugacy classes are called the two longest conjugacy class sizes.

In recent years, the relationship between certain arithmetical conditions on conjugacy class sizes and the structures of finite groups has been widely studied, see, for example, [1–9]. Among these results, the paper [3] asserts that $G/Z(G)$ is a Frobenius group under the condition that the two largest sizes of conjugacy classes of the non-central elements in G are coprime.

On the other hand, let N be a normal subgroup of a group G. It is clear that N is a union of some conjugacy classes of a group G. So, it is interesting to explore the structure of the normal subgroup N if G-conjugacy class sizes of N are given, see [6,7,9].

Enlightened by [3] and [6,7,9], we are interested in the following question.

Question. Let G be a group and let N be a normal subgroup of G. Suppose that the positive integers m and n are the two largest sizes of the non-central G-conjugacy classes of N with $m > n$ and $(m, n) = 1$, is $N/Z(N)$ a Frobenius group?

Since there is no necessary relationship between $|x^G|$ and $|x^N|$ except the divisibility that $|x^N|$ divides $|x^G|$ for an element x of N, it is very difficult to obtain a positive answer to the above question. But, by considering the relationship between n and $|N|$, we have the following:

Theorem A. *Let N be a normal subgroup of a group G. For $a, b \in N$, let $m = |b^G|$ and $n = |a^G|$. Suppose that m and n are the two longest sizes of the non-central G-conjugacy classes of N with $m > n$ and $(m, n) = 1$. If n divides $|N/(Z(G) \cap N)|$, then either $N/(Z(G) \cap N)$ is a prime power order group or*

(i) *$C_N(a)$ and $C_N(b)$ are abelian and $C_N(a) \cap C_N(b) = Z(G) \cap N$.*
(ii) *The G-conjugacy class sizes of elements in N are exactly 1, m and n.*
(iii) *$N/Z(N)$ is a Frobenius group with the kernel $C_N(a)/Z(N)$ and complement $C_N(b)/Z(N)$.*

So the main result in [3] is generalized.

Let π be a set of some primes. We use x_π and $x_{\pi'}$ for π-component and π'-component of x, respectively. Moreover, G_π denotes a Hall π-subgroup of G, $G_{\pi'}$ a Hall π'-subgroup of G, n_π the π-part of n whenever n is a positive integer. Apart from these, we call an element x non-central if $x \notin Z(G)$, where $Z(G)$ is the centre of G.

2. Preliminaries

We first list some lemmas that are useful in the proof of our main result.

Lemma 2.1. *Let N be a normal subgroup of a group G and x an element of G. Then*

(a) *$|x^N|$ divides $|x^G|$;*
(b) *$|(Nx)^{G/N}|$ divides $|x^G|$.*

Proof. See Lemma 1.1 in [5]. □

Lemma 2.2. *Let N be a normal of a group G and $B = b^G$, $C = c^G$ with $(|B|, |C|) = 1$, where $b, c \in N$. Then*:

(a) *$G = C_G(b)C_G(c)$.*
(b) *$BC = CB$ is a G-conjugacy class of N and $|BC|$ divides $|B||C|$.*

Proof. Applying Lemma 1 in [1], it is enough to prove that $BC \subseteq N$. In fact $BC = (bc)^G$ by Lemma J(b) in [2], so we have that $BC \subseteq N$ as $b, c \in N$. □

Lemma 2.3. *Let N be a normal subgroup of a group G, and B_0 a non-central G-conjugacy class of N with the largest size m. Then the following properties hold*:

(a) *Let C be a G-conjugacy class of N with $(|B_0|, |C|) = 1$, then $|\langle C^{-1}C\rangle|$ divides $|B_0|$.*
(b) *Let n, m be two largest G-conjugacy class sizes of N with $m > n$ and $(m, n) = 1$, and D a G-conjugacy class of N with $|D| > 1$. If $(|D|, n) = 1$, then $|D| = m$.*

Proof.

(a) Lemma 2.2(b) implies that CB_0 is a G-conjugacy class of N. Clearly $|CB_0| \geq |B_0|$. By the hypotheses, we have that $|CB_0| = |B_0|$, from which it follows that $C^{-1}CB_0 = B_0$, and hence $\langle C^{-1}C\rangle B_0 = B_0$. Consequently $|\langle C^{-1}C\rangle|$ divides $|B_0|$.

(b) Let A be a G-conjugacy class and $|A| = n$. Note that DA is a G-conjugacy class by Lemma 2.2(b). Clearly $|DA| \geq |A|$. The hypotheses of this lemma forces $|DA| = n$ or m. If $|DA| = n$, then $D^{-1}DA$ is a G-conjugacy class and hence $D^{-1}DA = A$, so $\langle D^{-1}D\rangle A = A$. It leads to $|\langle D^{-1}D\rangle|$ divides $|A|$. On the other hand, $\langle D^{-1}D\rangle \subseteq \langle A^{-1}A\rangle$, so $|\langle D^{-1}D\rangle|$ divides $|\langle A^{-1}A\rangle|$. By (a), we have that $|\langle A^{-1}A\rangle|$ divides $|B_0|$, hence $|\langle D^{-1}D\rangle|$ divides $|B_0|$, a contradiction. Consequently $|DA| = m$, that is to say $|B_0|$ divides $|A||D|$, therefore $|D| = |B_0|$. □

Lemma 2.4. *Suppose that N is a normal subgroup of a group G. Let B_0 be a non-central G-conjugacy class of N with the largest size. Let*

$$M = \langle D | D \textit{ is a } G\textit{-conjugacy class of } N \textit{ with } (|D|, |B_0|) = 1\rangle.$$

Then M is abelian. Furthermore, if $(Z(G) \cap N) < M$, then $\pi(M/(Z(G) \cap N)) \subseteq \pi(B_0)$.

Proof. Let

$$K = \langle D^{-1}D | D \text{ is a } G\text{-conjugacy class of } N \text{ with } (|D|, |B_0|) = 1\rangle.$$

By the definition of M and K, it is clear that $K = [M, G]$. Let $d \in D$, where D is a G-conjugacy class of N with $(|D|, |B_0|) = 1$. By Lemma 2.3(a), we have that $\pi(K) \subseteq \pi(B_0)$, so $(|K|, |D|) = 1$, hence $|d^K| = 1$. It shows that $K = C_K(d)$, so $K \leq Z(M)$. Moreover, $M/K \leq Z(G/K)$. It follows that M is nilpotent. Obviously, $(Z(G) \cap N) \leq M$. If $(Z(G) \cap N) < M$, let $r \in \pi(M/(Z(G) \cap N))$, $R \in \mathrm{Syl}_r(M)$. Then $R \trianglelefteq G$. Notice that $1 \neq [R, G] \leq [M, G] = K$, we know $r \in \pi(K) \subseteq \pi(B_0)$, so $\pi(M/(Z(G) \cap N)) \subseteq \pi(B_0)$. Suppose that D is a generating class of M and $d \in D$, then $|d^R| | (|R|, |D|)$, but $(|R|, |D|) = 1$, so $R = C_R(d)$. Therefore $R \leq Z(M)$, and consequently M is abelian. □

3. Proof of Theorem A

In this section we are equipped to prove the main result.

Proof of Theorem A. Assume that $N/Z(G) \cap N$ is not a prime power order group. For convenience, we write

$$M = \langle D | D \text{ is a } G\text{-conjugacy class of } N \text{ with } (|D|, m) = 1\rangle.$$

We will complete the proof by the following steps:

Step 1. We may assume that $N_p \nleq Z(G)$ for every prime factor p of $|N|$.

Otherwise, there exists a prime factor p of $|N|$ such that $N_p \leq Z(G)$. Then $N = N_{p'}N_p$. Without loss of generality, we replace N with $N_{p'}$.

Step 2. If the element $x \in Z(C_G(b)) \cap N$, then either $x \in Z(G)$, or $C_G(x) = C_G(b)$.

Obviously, it follows that $C_G(b) \leq C_G(x)$, so $|x^G|$ divides $|b^G|$. If $x \notin Z(G)$, then $|x^G|$ divides m, so $(|x^G|, n) = 1$ by Lemma 2.3. Therefore $|x^G| = m$, whence $C_G(x) = C_G(b)$.

Step 3. We may assume that b is a prime power order p-element.

Let p be a prime factor of the order of b, b_p the p-component and $C_G(b_p) \neq G$. It is clear that $C_G(b) = C_G(b_p b_{p'}) = C_G(b_p) \cap C_G(b_{p'}) \subseteq C_G(b_p)$. By Step 2, we have that $C_G(b_p) = C_G(b)$. Replacing b with b_p, Step 3 follows.

Step 4. $C_N(b) = P \times L$, where P is a *Sylow* p-subgroup of $C_N(b)$, L is the p'-Hall subgroup of $C_N(b)$ with $L \leq Z(C_G(b))$. If $L \nleq Z(G)$, then $C_N(b) \leq Z(C_G(b))$.

Let $x \in C_N(b)$ be a p'-element. Then $C_G(bx) = C_G(b) \cap C_G(x) \leq C_G(b)$, from which it follows that $|b^G|$ divides $|(bx)^G|$. Now, the maximality of $|b^G|$ implies that $|b^G| = |(bx)^G|$, which forces that $C_G(bx) = C_G(b) \leq C_G(x)$, and hence $x \in Z(C_G(b))$. Consequently, $C_N(b) = P \times L$, where P is a Sylow p-subgroup of $C_N(b)$, and L is the p'-Hall subgroup of $C_N(b)$ with $L \leq Z(C_G(b))$.

Particularly, if $L \nleq Z(G)$, let $y \in L$ be a non-central prime power order q-element. By Step 2, one has that $|y^G| = m$, and hence $C_G(y) = C_G(b)$. By the above argument, we have that $C_N(y) = Q \times L_q$, where Q is a Sylow q-subgroup of $C_N(y)$, and L_q is the q'-Hall subgroup of $C_N(b)$ with $L_q \leq Z(C_G(b))$. Notice that $C_N(b) = LL_q$, we have that $C_N(b) \leq Z(C_G(b))$, as required.

Step 5. $p \nmid m$.

Otherwise, if $p|m$, then $p \nmid n$. In view of Lemma 2.1, we have that $p \nmid |a^N|$. Notice that $|a^N| = |N : C_N(a)| = |C_N(b) : C_N(a) \cap C_N(b)|$. We have that $P \leq C_N(a) \cap C_N(b)$, which implies that $a \in C_N(b)$. To complete the proof, we distinguish two cases according to whether L is contained in $Z(G)$ or not.

(1) If $L \nleq Z(G)$, notice that $a \in C_N(b)$, so application of Step 4 yields $C_G(b) \leq C_G(a)$, which implies that $|a^G|$ divides $|b^G|$, a contradiction to the hypotheses.

(2) If $L \leq Z(G)$, by the argument of the first paragraph, we have that $C_N(b) \leq C_N(a)$, which yields $|a^N|$ divides $|b^N|$. Notice that $(|a^G|, |b^G|) = 1$, we have that $|a^N| = 1$ by Lemma 2.1, equivalently, $N = C_N(a)$. Obviously, we may assume that a is a p-element since $a \in C_N(b) = P \times L$. For every p'-element $x \in N = C_N(a)$, we have that $C_G(ax) = C_G(a) \cap C_G(x) \leq C_G(a)$, so the hypotheses of the theorem imply that $C_G(ax) = C_G(a) \leq C_G(x)$, from which it follows that $x \in Z(C_G(a)) \leq Z(N)$. Of course we have $x \in C_N(b)$ and hence $x \in L$. Consequently $N/(N \cap Z(G))$ is a prime power order group, a contradiction to the hypothesis.

Step 6. We may assume that a is a p'-element.

Let $a = a_p a_{p'}$, where $a_p, a_{p'}$ are the p- and p'-components of a, respectively. Since $C_G(a) = C_G(a_p) \cap C_G(a'_p) \subseteq C_G(a_p)$, it follows that $a_p \in M$ by Lemma 2.1. If $a_p \notin Z(G)$, then $p \in \pi(M/(Z(G) \cap N)) \subseteq \pi(m)$ by Lemma 2.4, a contradiction.

Step 7. $C_N(a)_p \leq Z(G)_p$.

Suppose that there exists a non-central p-element $y \in C_N(a)$, then $C_G(ay) = C_G(a) \cap C_G(y) \subseteq C_G(a)$, so we have $C_G(ay) = C_G(a)$ by the hypothesis of the theorem. It follows that $ay \in M$, therefore $y \in M$ as $a \in M$. Now one has that $p \in \pi(M/(Z(G) \cap N)) \subseteq \pi(m)$, a contradiction. Hence $C_N(a)_p \leq Z(G)_p$.

Step 8. $C_N(a) \cap C_N(b) \leq Z(G)$, and therefore $Z(N) \leq Z(G)$.

If there exists a non-central element $y \in C_N(a) \cap C_N(b)$, we distinguish two cases as in the proof of Step 5:

(1) If $L \leq Z(G)$, then we may assume that y is a p-element. So y is a p-element of $C_N(a)$, a contradiction to Step 7.

(2) If $L \nleq Z(G)$, we have that $y \in C_N(b) \leq Z(C_G(b))$ by Step 4, so $C_G(y) = C_G(b)$. Therefore $a \in C_N(b) = C_N(y)$, and it follows that $C_G(b) \leq C_G(a)$ by Step 4 again, a contradiction.

Finally, notice that $Z(N) \leq C_N(a) \cap C_N(b)$, and we surely have $Z(N) \leq Z(G)$.

Step 9. We have that $\pi(|a^N|) = \pi(|a^G|)$ and $\pi(|b^N|) = \pi(|b^G|)$. Hence, if $L \leq Z(G)$, $|a^G|$ is a power of p.

We begin by showing that $\pi(|a^N|) = \pi(|a^G|)$. If there exists a prime q satisfying that $q||a^G|$ but $q \nmid |a^N|$, then $q \nmid |b^G|$, and consequently $q \nmid |b^N|$. It follows that $C_N(a)$ and $C_N(b)$ contain a Sylow q-subgroup of N. Notice that $N = C_N(a)C_N(b)$, and we have $C_N(a) \cap C_N(b)$ contains a Sylow q-subgroup of N, say N_q, hence $N_q \leq C_N(a) \cap C_N(b) \leq Z(G)$, a contradiction to Step 1. Therefore $q||a^N|$ for every $q||a^G|$, so $\pi(|a^N|) = \pi(|a^G|)$ by Lemma 2.1. Similarly, we have that $\pi(|b^N|) = \pi(|b^G|)$.

Next we show that $|a^G|$ is a power of p. Obviously, $|a^N| = |N : C_N(a)| = |C_N(b) : C_N(a) \cap C_N(b)| = |PL : Z(G) \cap N|$. If $L \leq Z(G)$, then $|a^N|$ is a power of p. Hence $|a^G|$ is a power of p by $\pi(|a^N|) = \pi(|a^G|)$.

Step 10. $|x^G| = m$ for any non-central p-element $x \in N$, and therefore $C_N(b)$ is abelian.

As $C_N(b)$ contains a Sylow p-subgroup of N and $p \nmid m = |b^G|$, without loss of generality, we may assume that $x \in P \setminus Z(G)$. We distinguish two cases as in the proof of Step 5:

(1) If $L \nleq Z(G)$, then $x \in C_N(b) \leq Z(C_G(b))$ by Step 4, so $C_N(b)$ is abelian and $|x^G| = m$.

(2) If $L \leq Z(G)$, we know that $n = |a^G|$ is a power of p by Step 9. On the other hand, M is abelian, and $Z(G)_p \leq M \leq C_N(a)$, so we can write $M = M_{p'} \times Z(G)_p$ by Step 7. Notice that $\langle x \rangle$ acts coprimely on the abelian subgroup $M_{p'}$, and we obtain a direct product:

$$M_{p'} = [M_{p'}, \langle x \rangle] \times C_{M_{p'}}(x).$$

Denote by $U = [M_{p'}, \langle x \rangle]$. As $a \in M_{p'}$ and $M_{p'} = [M_{p'}, \langle x \rangle] \times C_{M_{p'}}(x)$, we can write $a = uw$ with $u \in U$ and $w \in C_{M_{p'}}(x)$. Consider the element $g = wx$. We have that

$$C_G(g) = C_G(w) \cap C_G(x) \leq C_G(x).$$

If $|g^G| = m$, then $|x^G| = m$ by Lemma 2.3 since $|x^G|$ divides $|g^G|$.

If $|g^G| = n$, then $|x^G|$ divides $|g^G|$, namely, $|x^G|$ divides n, so $x \in M$. However, $M_p \leq C_N(a)_p = Z(G)_p$, a contradiction.

If $|g^G| < n$, let P_0 be a Sylow p-subgroup of G such that $P \leq P_0$. Then $M_{p'}P_0$ is a subgroup of G. Now we have following inequalities:

$$\begin{aligned} |M_{p'}P_0 : C_{M_{p'}P_0}(g)| \leq |g^G| < n &= |G : C_G(a)| \\ &= |G : C_G(a)|_p \\ &= |C_G(b) : C_G(a) \cap C_G(b)|_p \\ &= |C_G(b)|_p : |C_G(a) \cap C_G(b)|_p \\ &= |P_0| : |C_G(a)|_p. \end{aligned} \tag{3.1}$$

Moreover, since $M_{p'} \trianglelefteq G$, $M_{p'}$ is abelian, $M_{p'} \cap P_0 = 1$, and $M_{p'} \leq C_G(w)$. We have that

$$\begin{aligned} C_{M_{p'}P_0}(g) &= C_{M_{p'}P_0}(w) \cap C_{M_{p'}P_0}(x) \\ &= M_{p'}C_{P_0}(w) \cap C_{Mp'}(x)C_{P_0}(x) \\ &= C_{M_{p'}}(x)(M_{p'}C_{P_0}(w) \cap C_{P_0}(x)) \\ &= C_{M_{p'}}(x)(C_{P_0}(w) \cap C_{P_0}(x)) \end{aligned} \tag{3.2}$$

Set $D = C_{P_0}(w) \cap C_{P_0}(x)$. Combining equations (3.1) and (3.2), we have that

$$\frac{|P_0|}{|C_G(a)|_p} > \frac{|M_{p'}||P_0|}{|C_{Mp'}(x)||D|}.$$

This implies that $|D| : |C_G(a)|_p > |M_{p'} : C_{M_{p'}}(x)| = |U|$.

On the other hand, since $D \leq C_G(x)$, and $U \times C_{M_{p'}}(x) = M_{p'} \trianglelefteq G$, we have that D normalizes U. Also, it follows that

$$C_D(u) = C_G(u) \cap D = C_{P_0}(u) \cap C_{P_0}(w) \cap C_{P_0}(x) \leq C_{P_0}(a) \cap C_{P_0}(x).$$

Thus

$$|C_D(u)| \leq |C_{P_0}(a) \cap C_{P_0}(x)| \leq |C_G(a)|_p.$$

Therefore

$$|u^D| = |D : C_D(u)| = |D| : |C_D(u)| \geq |D| : |C_G(a)|_p > |U|,$$

a contradiction.

Thus, the above arguments imply that $|x^G| = m$.

Next, we show that $C_N(b) = P \times L$ ($L \leq Z(G)$) is abelian. For any non-central element $x \in P$, we have that $C_{M_{p'}}(x) \leq Z(G)_{p'}$. Otherwise, we may replace b with x for it is proved that $|x^G| = m$, which leads to a contradiction that $C_N(a) \cap C_N(b) \nleq Z(G)$. Therefore $P/P \cap Z(G)$ acts on the group $M_{p'}/M_{p'} \cap Z(G)$ fixed-point freely. Hence, $P/P \cap Z(G)$ is a cyclic group or a generalized quaternion group. So, if P is not abelian, then $p = 2$ and $P/P \cap Z(G)$ is a generalized quaternion group. Now $b \in Z(P)$ but $b \notin Z(G)$. There exists an element $y \in P$ and $y \notin Z(P)$ such that $b = y^2c$, where $c \in Z(G) \cap P$. So $C_G(y) \leq C_G(b)$, which shows that $y \in Z(C_G(b))$, of course, we have that $y \in Z(P)$, a contradiction. Therefore P is abelian, so $C_N(b)$ is abelian, as required.

Step 11. If d and t are two non-central elements of N such that $|t^G| \neq m = |d^G|$, then

(11.1) $C_N(t) \cap C_N(d) \leq Z(G)$.
(11.2) $|t^G| = n$, consequently, the G-conjugacy class sizes of N are 1, m and n.

(11.1) Suppose that there exists a non-central element $y \in C_N(t) \cap C_N(d)$. To accomplish the proof of (11.1), we will distinguish two cases by Step 4 for L.

(1) If $L \nleq Z(G)$, Step 4 implies that $C_N(b) \leq Z(C_G(d))$, from which it follows that $y \in Z(C_G(d))$. Note that y is non-central, so $C_G(y) = C_G(d)$, and hence $t \in C_N(d) \leq Z(C_G(d))$. By Step 2, we have that $C_G(t) = C_G(d)$, a contradiction. Therefore $C_N(t) \cap C_N(d) \leq Z(G)$.

(2) If $L \leq Z(G)$, we know that n is a power of p. In this case, we assert that t can be chosen as a p'-element. In fact, let t_p be the p-component of t, obviously, $C_G(t) \leq C_G(t_p) \leq G$. If $t_p \notin Z(G)$, by Step 10, we have that $|t_p{}^G| = m$, and therefore $|t^G| = m$, against the hypothesis of this step. Therefore, $t_p \in Z(G)$, so $C_G(t) = C_G(t_{p'})$, where $t_{p'}$ is the p'-component of t. Thus, without loss of generality, we may assume that t is a p'-element.

Now, we may assume that y is a p-element. Again by Step 10, we have that $|y^G| = m$. Moreover,

$$C_G(ty) = C_G(t) \cap C_G(y) \subseteq C_G(y),$$

which implies that $|(ty)^G| = m$ by the maximality of m. Hence $C_G(ty) = C_G(y) \leq C_G(t)$, and therefore $(|t^G|, n) = 1$, from which it follows that $|t^G| = m$ by Lemma 2.3(b), a contradiction. Thus, $C_N(d) \cap C_N(t) \leq Z(G)$.

(11.2) Consider the quotient group $C_N(b)/Z(N)$ and the set t^N. For any $\bar{x} \in C_N(b)/Z(N)$ and $y \in t^N$, without loss of generality, we may assume that $\bar{x} = xZ(N)$ where $x \in C_N(b)$. Define

$$y^{\bar{x}} = y^x. \tag{3.3}$$

Clearly, the definition (3.3) indicates that $C_N(b)/Z(N)$ acts as a group on the set t^N. Obviously, $t^N \cap C_N(b) = \emptyset$ and $C_N(t) \cap C_N(b) = Z(N)$, from which it follows that the group $C_N(b)/Z(N)$ acts on the set t^N fixed-point freely. Therefore $|C_N(b)/Z(N)|$ divides $|t^N|$. Also $|C_N(b)/Z(N)| = |a^N|$. Since $N = C_N(a)C_N(b)$, we have that $|N| = |a^N||b^N||Z(N)|$. Because n divides $|N/(Z(G) \cap N)| = |N/Z(N)|$, we have $|a^N| = |a^G| = n$. Therefore $|t^N| = |t^G| = n$. Consequently, the G-conjugacy class sizes of N are $1, m$ and n.

Step 12. $C_N(a)$ is an abelian group and $N/Z(N)$ is a Frobenius group.

For any non-central element $x \in C_N(a)$, we have that $|x^G| \neq m$. In fact, if $|x^G| = m$, then $x \in C_N(a) \cap C_N(x)$, a contradiction to (11.1). Hence $|x^G| = n$ by Step (11.2). Therefore $C_N(a) = M$ is a normal abelian group. On the other hand, $C_N(b)/Z(N)$ acts on $C_N(a)/Z(N)$ fixed-point freely. Since $N = C_N(a)C_N(b)$, we have that $N/Z(N)$ is a Frobenius group with the kernel $C_N(a)/Z(N)$ and the complement $C_N(b)/Z(N)$. □

COROLLARY 1

Let N be a normal subgroup of a group G. For a, $b \in N$, set $m = |b^G|$ and $b = |a^G|$. Suppose that m and n are the two longest sizes of the non-central G-conjugacy classes of N with $m > n$ and $(m, n) = 1$. If n is square-free, then either $N/(Z(G) \cap N)$ is a prime power order group or

(i) *$C_N(a)$ and $C_N(b)$ are abelian and $C_N(a) \cap C_N(b) = Z(G) \cap N$;*
(ii) *the G-conjugacy class sizes of elements in N are 1, m and n;*
(iii) *$N/Z(N)$ is a Frobenius group with the kernel $C_N(a)/Z(N)$ and complement $C_N(b)/Z(N)$.*

Proof. Suppose that $N/(Z(G) \cap N)$ is not a prime power order group, by examination of the proof in Theorem A, we find that Steps 1–9 still hold. This implies that

$\pi(|a^N|) = \pi(|a^G|)$. By the hypothesis that n is square-free, we have that $|a^N| = |a^G| = n$. Since $|a^N|$ divides $|N/(Z(G) \cap N)|$, we get that n divides $|N/(Z(G) \cap N)|$. The result now follows from Theorem A. □

Now, in the following, we can see that theorem in [3] is a corollary of Theorem A.

COROLLARY 2

Let G be a finite group and $n < m$ be the two longest sizes of the non-central conjugacy classes of G. Let $a, b \in G$ whose G-conjugacy classes have sizes n, m respectively. If $(m, n) = 1$, then

(i) *$C_G(a)$, $C_G(b)$ are abelian and $C_G(a) \cap C_G(b) = Z(G)$.*
(ii) *$G/Z(G)$ is a Frobenius group with kernal $C_G(a)/Z(G)$ and complement $C_G(b)/Z(G)$.*

Proof. Obviously, n divides $|G/Z(G)|$. The corollary follows by taking $N = G$ in Theorem A. □

COROLLARY 3

Let N be a normal subgroup of a group G. Suppose that $1 < k_1 < k_2 < \cdots < k_r$ are the G-conjugacy class sizes of N, where $r \geq 3$. If $N/N \cap Z(G)$ is not a prime power order group and k_{r-1} divides $|N/N \cap Z(G)|$, then $(k_{r-1}, k_r) \neq 1$.

Proof. It follows straightforward from Theorem A. □

Remark. It is an interesting topic to discuss if the restriction 'n divides $|G/Z(G)|$' in Theorem A can be removed.

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