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Composition operators between Bloch type spaces and Zygmund spaces in the unit ball

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Abstract. The boundedness and compactness of composition operators between Bloch type spaces and Zygmund spaces of holomorphic functions in the unit ball are characterized in the paper.

Keywords. Composition operator; Bloch type space; Zygmund space.

1. Introduction

Let

$$B^n = \{z = (z_1, \dots, z_n) : |z|^2 = |z_1|^2 + \cdots + |z_n|^2 < 1\}$$

be the unit ball of the complex Euclidean space of C^n $(n \geq 1)$. Define by $H(B^n)$ the class of all holomorphic functions on B^n and by H^∞ the set of all bounded holomorphic functions on B^n. Then H^∞ is the Banach algebra with the supremum norm $\|f\|_\infty = \sup_{z\in B^n}|f(z)|$. For $f \in H(B^n)$, its complex gradient and radial derivative are defined by

$$\begin{aligned}\nabla f(z) &= \nabla_z f = \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z)\right),\\ Rf(z) &= \langle \nabla f(z), \overline{z}\rangle = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z),\end{aligned}$$

respectively. For a positive integer k, denote $R^{(k+1)} f = R(R^{(k)} f)$ the higher radial derivative with order $k + 1$. We write $D_j f = \frac{\partial f}{\partial z_j}$ for convenience.

For $f \in H(B^n)$, we define

$$Q_f(z) = \sup\left\{\frac{|\langle \nabla f(z), \bar{u}\rangle}{\sqrt{H_z(u, u)}} : 0 \neq u \in C^n\right\},$$

where $H_z(u,u)$ is the Bergman metric on B^n defined by

$$H_z(u,u) = \frac{n+1}{2}\frac{(1-|z|^2)|u|^2 + |\langle u, z\rangle|^2}{(1-|z|^2)^2}.$$

The Bloch space is the space of all holomorphic functions f on B^n such that

$$\|f\| = |f(0)| + \sup\{Q_f(z) : z \in B^n\} < \infty. \tag{1}$$

Let M be the class of all positive continuous and decreasing functions $\mu(t)(0 \le t < 1)$ such that $\mu(t) \to 0$ as $t \to 1$, and $\frac{\mu(t)}{(1-t)^\delta}$ is increasing when t is close to 1 for some $\delta > 0$. A function $f \in H(B^n)$ is said to belong to the μ-Bloch space B_μ if

$$\|f\|_{B_\mu} = |f(0)| + \sup\{\mu(|z|)|Rf(z)| : z \in B^n\} < \infty. \tag{2}$$

As known, $f \in B_\mu$ if and only if

$$\sup\{\mu(|z|)|\nabla f(z)| : z \in B^n\} < \infty.$$

In particular, when $\mu(t) = (1-t^2)^\alpha$ with $\alpha > 0$, μ-Bloch space is also called Bloch type space, denoted by B_α. For $\alpha = 1$, the equivalence of these two norms (1) and (2) has been proved by Timoney [9]. Moreover, when $\alpha > 1/2$, a function $f \in B_\alpha$ if and only if

$$\sup\{(1-|z|^2)^{\alpha-1}Q_f(z) : z \in B^n\} < \infty. \tag{3}$$

However, when $0 < \alpha \le 1/2$, (2) and (3) are not equivalent (see [12]).

In view of the problem, Chen and Gauthier in [1] introduced a new metric $H_z^\mu(u,u)$ which generalizes the classic Bergman metric in a more general situation. Let

$$I_\mu = \int_0^1 \frac{\mathrm{d}t}{(1-t)^{1/2}\mu(t)}. \tag{4}$$

If $I_\mu = \infty$, let

$$\nu(t) = \left(\frac{1}{\mu(0)} + \int_0^t \frac{\mathrm{d}t}{(1-t)^{1/2}\mu(t)}\right)^{-1},$$

otherwise, let $\nu(t) \equiv \mu(0)$. The metric $H_z^\mu(u,u)$ corresponding to μ is defined by

$$H_z^\mu(u,u) = \frac{n+1}{2}\frac{1}{\mu^2(|z|)}\left\{\frac{\mu^2(|z|)}{\nu^2(|z|)}|u|^2 + \left(1 - \frac{\mu^2(|z|)}{\nu^2(|z|)}\right)\frac{|\langle u, z\rangle|^2}{|z|^2}\right\}$$

for $0 \ne z \in B^n$ and $u \in C^n$. For $z = 0$, let $H_0^\mu(u,u) = |u|^2/\mu^2(0)$. Put

$$Q_f^\mu(z) = \sup\left\{\frac{|\langle \nabla f(z), \bar{u}\rangle}{\sqrt{H_z^\mu(u,u)}} : 0 \ne u \in C^n\right\}.$$

In [1], the authors proved that when $\mu(t) = (1 - t^2)^\alpha$ with $\alpha > 1/2$, $H_z^\mu(u, u)$ is equivalent to

$$\left(\frac{1-|z|^2}{\mu(|z|)}\right)^2 H_z(u, u),$$

which implies that $Q_f^\mu(z)$ is equivalent to $(1-|z|^2)^{\alpha-1} Q_f(z)$. What is more, they proved in general conditions that a function $f \in B_\mu$ if and only if

$$\sup\{Q_f^\mu(z) : z \in B^n\} < \infty.$$

A function $f \in H(B^n)$ is said to belong to the Zygmund space, denoted by Z, if its radial derivative is in the Bloch space. That is,

$$\|f\|_Z = |f(0)| + \sup\{(1-|z|^2)|R^{(2)} f(z)| : z \in B^n\} < \infty.$$

It is easy to see that if $f \in Z$, then

$$|Rf(z)| \leq C\|f\|_Z \log \frac{e}{1-|z|^2}. \tag{5}$$

It is well-known that $f \in Z$ if and only if (see [12])

$$\sup\{(1-|z|^2)|D_j D_i f(z)| : z \in B^n\} < \infty, \qquad i, j = 1, \ldots, n.$$

Hence the left-hand side of inequality (5) can be replaced by $|D_j f(z)| (j = 1, \ldots, n)$. Moreover, we can see that $Z \subset B_\alpha$ from (2) and (5). Note that, when $0 < \alpha < 1$, B^α can be identified with the Lipschitz space $\mathrm{Lip}_{1-\alpha}$ consisting of holomorphic functions f on B^n such that

$$|f(z) - f(w)| \leq C|z-w|^{1-\alpha}$$

for all $z, w \in B^n$ (see [6]). Therefore, for a function $f \in Z$, it may extend continuously to the closed unit ball $\overline{B^n}$.

Suppose that $(X, \|\cdot\|_X)$ is a Banach space of holomorphic functions on B^n satisfying the following conditions.

(i) Every function in X extends continuously to the closed unit ball.
(ii) X contains all the polynomials.
(iii) Evaluation at each point of B^n is a bounded linear functional.
(iv) If φ is a conformal automorphism of B^n and $f \in X$, then $f \circ \varphi \in X$.

We refer to such space X as an automorphism invariant boundary regular small space (see [3] or [7]). The first and the third assumption guarantee convergence in the norm of X implies convergence in the sup norm: the identity map from $(X, \|\cdot\|_X)$ to $(X, \|\cdot\|_\infty)$ is continuous by the closed graph theorem. Moreover, another closed graph theorem argument using (iii) shows that (iv) implies that C_φ is bounded on X whenever φ is a conformal automorphism of B^n.

For any point $a \in B^n - \{0\}$, we define

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \qquad z \in B^n, \tag{6}$$

where $s_a = \sqrt{1-|a|^2}$, P_a is the orthogonal projection from C^n onto the one-dimensional subspace $[a]$ generated by a and $Q_a = I - P_a$ is the projection onto the orthogonal complement of $[a]$. That is

$$P_a(z) = \frac{\langle z, a\rangle}{|a|^2} a, \qquad Q_a(z) = z - \frac{\langle z, a\rangle}{|a|^2} a, \qquad z \in B^n.$$

When $a = 0$, we simply define $\varphi_a(z) = -z$. It is well known that each φ_a is a homeomorphism of the closed unit ball $\overline{B^n}$ onto $\overline{B_n}$ and every automorphism φ of B^n is the form $\varphi = \varphi_a U$, where U is a unitary transformation of C^n.

In the paper, let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a holomorphic self-map of B^n. Associated with φ is the composition operator $C_\varphi : X \to Y$ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(B^n)$ if X and Y are Banach spaces of holomorphic functions in the unit ball. The main subject in the study of composition operators is to describe operator theoretic properties of C_φ in terms of function theoretic properties of φ. Good references for the theory of composition operators can be found in [3] and [8]. As for the case of one complex variable, there exist many results in the literature. For instance, bounded and compact operators $C_\varphi : B \to B$ were characterized in [5] and the general case $C_\varphi : B_\alpha \to B_\beta$ was solved in [11]. Operators $C_\varphi : B_\alpha \rightleftarrows Z$ were studied in [4]. Compact operators on some small spaces were considered in [7] and [2]. However, in several complex variables case, it is so difficult to treat them that most known results are not satisfactory. Recently, in [1], Chen and Gauthier completely obtained necessary and sufficient conditions for the boundedness and compactness of composition operators on μ-Bloch spaces by using a smart technique (see Lemmas 2.2 and 2.3 in §2).

The main purpose of this paper is to study the boundedness and compactness of composition operators between Bloch type spaces and Zygmund spaces of holomorphic functions in the unit ball of C^n. More precisely, in §2, we give necessary and sufficient conditions for the boundedness and compactness of $C_\varphi : Z \to B_\alpha$ by means of the technique borrowed from [1], which has been modified to make it more efficient. Meanwhile, we find that these conditions are similar to that of $C_\varphi : B_\mu \to B_\alpha$ to be bounded and compact for $\mu(t) = \left(\log \frac{e}{1-t^2}\right)^{-1}$ respectively, although the Zygmund space is a subspace of B_μ. Section 3 is devoted to the study of bounded and compact composition operators $C_\varphi : B_\alpha \to Z$. Our results show that the boundedness of $C_\varphi : B_\alpha \to Z$ implies the compactness of the composition operator C_φ on the Zygmund space. Thus, we assert that the condition $\|\varphi\|_\infty < 1$ is necessary for $C_\varphi : B_\alpha \to Z$ to be bounded by an interesting result of Shapiro [7].

Throughout this paper, constants are denoted by C which they are positive and not necessarily the same in each occurrence. The expression $E \approx F$ means that there exists a positive constant C such that $C^{-1}E \le F \le CE$. For a holomorphic self-map $\varphi(z) = (\varphi_1(z), \dots, \varphi_n(z))$ of B^n, it will be convenient for us to write $R\varphi(z) = (R\varphi_1(z), \dots, R\varphi_n(z))$.

2. The boundedness and compactness of $C_\varphi : Z \to B_\alpha$

In this section, we characterize the boundedness and compactness of composition operators from the Zygmund space to the Bloch type space. Before doing them, we need the following lemmas.

Lemma 2.1. *Let* $\alpha > 0$ *and* $X, Y = Z$ *or* B_α. *The composition operator* $C_\varphi : X \to Y$ *is compact if and only if for any bounded sequence* $\{f_m\}$ *in* X *which converges to* 0 *uniformly on any compact subsets of* B^n *as* $m \to \infty$, *we have* $\|C_\varphi f_m\|_Y \to 0$ *as* $m \to \infty$.

Proof. The result can be obtained by standard arguments (see, for example, Lemma 3.8 of [10]). The details are omitted here.

Lemma 2.2 [1]. *Let* $\mu_1, \mu_2 \in M$. *Then* $C_\varphi : B_{\mu_1} \to B_{\mu_2}$ *is bounded if and only if*

$$\sup_{z\in B^n} \mu_2(|z|)\left\{H^{\mu_1}_{\varphi(z)}(R\varphi(z), R\varphi(z))\right\}^{\frac{1}{2}} < \infty.$$

Lemma 2.3 [1]. *Let* $\mu_1, \mu_2 \in M$ *and* $C_\varphi : B_{\mu_1} \to B_{\mu_2}$ *be bounded. If* $I_{\mu_1} = \infty$, *then* $C_\varphi : B_{\mu_1} \to B_{\mu_2}$ *is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \mu_2(|z|)\left\{H^{\mu_1}_{\varphi(z)}(R\varphi(z), R\varphi(z))\right\}^{\frac{1}{2}} = 0.$$

If $I_{\mu_1} < \infty$, *then* $C_\varphi : B_{\mu_1} \to B_{\mu_2}$ *is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_2(|z|)|\langle R\varphi(z), \varphi(z)\rangle|}{\mu_1(|\varphi(z)|)} = 0.$$

Lemma 2.4. *Let* $\mu(t) = \left(\log \frac{e}{1-t^2}\right)^{-1}$. *Then*

$$H^\mu_z(u,u) \approx \frac{1}{\mu^2(|z|)}\{\mu^2(|z|)|u|^2 + |\langle u, z\rangle|^2\}.$$

Proof. It is easy to see that $I_\mu < \infty$, therefore $\nu(t) \equiv \mu(0) = 1$. Note that

$$\frac{1-\mu^2(t)}{t^2} = \frac{\left(1 + \frac{1}{\log\frac{e}{1-t^2}}\right)\log\frac{1}{1-t^2}}{t^2 \log \frac{e}{1-t^2}} \to 2$$

as $t \to 0$. That is, $\frac{1-\mu^2(|z|)}{|z|^2} \approx C$ for $z \neq 0$. Thus the desired result follows from the definition of $H^\mu_z(u,u)$.

Lemma 2.5. *Let* $\mu(t) = \left(\log \frac{e}{1-t^2}\right)^{-1}$ *and* $f \in Z$. *Then* $f \in B_\mu$ *and* $\|f\|_{B_\mu} \le C\|f\|_Z$.

Proof. It immediately follows from the inequality (5).

Theorem 2.1. *Let* $\mu(t) = \left(\log \frac{e}{1-t^2}\right)^{-1}$. *Then the following statements are equivalent.*

(i) $C_\varphi : B_\mu \to B_\alpha$ *is bounded.*
(ii) $C_\varphi : Z \to B_\alpha$ *is bounded.*
(iii) $\sup_{z\in B^n}(1-|z|^2)^\alpha \left\{|R\varphi(z)|^2 + \left(\log \frac{e}{1-|\varphi(z)|^2}\right)^2 |\langle R\varphi(z), \varphi(z)\rangle|^2\right\}^{\frac{1}{2}} < \infty.$

Proof.

(i) $\Rightarrow$ (ii). It is clear from Lemma 2.5.
(ii) $\Rightarrow$ (iii). Suppose $C_\varphi : Z \to B_\alpha$ is bounded. Then we can easily obtain that $\varphi_i \in B_\alpha$ by taking $f(z) = z_i (i = 1, \dots, n)$. To show that (iii) holds, we have two cases to consider:

For any given $w \in B^n$, if $|\varphi(w)| \le 1/2$, it follows from $\varphi_i \in B_\alpha$ that

$$\begin{aligned}(1-|w|^2)^\alpha &\left\{|R\varphi(w)|^2 + \left(\log \frac{e}{1-|\varphi(w)|^2}\right)^2 |\langle R\varphi(w), \varphi(w)\rangle|^2\right\}^{\frac{1}{2}} \\ &\le C(1-|w|^2)^\alpha |R\varphi(w)| \\ &\le C(1-|w|^2)^\alpha \sum_{i=1}^n |R\varphi_i(w)| < \infty.\end{aligned}$$

In the following, we always assume that $|\varphi(w)| > 1/2$. Let $z_0 = \frac{\varphi(w)}{|\varphi(w)|}$. By the projection theorem, there exists a vector $\xi \in C^n$ such that $R\varphi(w) = v_1 z_0 + v_2\xi = e^{i\theta_1}|v_1|z_0 + e^{i\theta_2}|v_2|\xi$, where $\langle z_0, \xi\rangle = 0$, $|\xi| = 1$, $v_1 = \langle R\varphi(w), z_0\rangle$, $v_2 = \langle R\varphi(w), \xi\rangle$ and $|R\varphi(w)|^2 = |v_1|^2 + |v_2|^2$. Define

$$f_w(z) = e^{-i\theta_1}\int_0^{\langle z,\varphi(w)\rangle} \log \frac{e}{1-t}dt + e^{-i\theta_2}\langle z, \xi\rangle.$$

Then we get

$$\nabla f_w(z) = e^{-i\theta_1}\overline{\varphi(w)}\log \frac{e}{1-\langle z, \varphi(w)\rangle} + e^{-i\theta_2}\bar{\xi} \tag{7}$$

and

$$\begin{aligned}R^{(2)} f_w(z) &= e^{-i\theta_1}\langle z, \varphi(w)\rangle \log \frac{e}{1-\langle z, \varphi(w)\rangle} \\ &\quad + \frac{e^{-i\theta_1}\langle z, \varphi(w)\rangle^2}{1-\langle z, \varphi(w)\rangle} + e^{-i\theta_2}\langle z, \xi\rangle.\end{aligned} \tag{8}$$

From (8) it is easy to see that $\sup_{w\in B^n}\|f_w\|_Z \le C$. Note that there exists a positive constant $C(C<1)$ such that

$$\left(\log\frac{e}{1-x^2}\right)^2-\frac{1}{x^2}\ge C\left(\log\frac{e}{1-x^2}\right)^2. \tag{9}$$

Since $C_\varphi : Z \to B_\alpha$ is bounded, by (7) and (9) we have

$$\begin{aligned}
\infty &> \|C_\varphi\|\|f_w\|_Z \ge \|C_\varphi f_w\|_{B_\alpha} \ge (1-|w|^2)^\alpha |R(f_w\circ\varphi)(w)| \\
&= (1-|w|^2)^\alpha |\langle \nabla f_w(\varphi(w)), \overline{R\varphi(w)}\rangle| \\
&= (1-|w|^2)^\alpha \left\{|v_1||\varphi(w)|\log\frac{e}{1-|\varphi(w)|^2}+|v_2|\right\} \\
&\ge (1-|w|^2)^\alpha \left\{|v_1|^2|\varphi(w)|^2\left(\log\frac{e}{1-|\varphi(w)|^2}\right)^2+|v_2|^2\right\}^{\frac12} \\
&= (1-|w|^2)^\alpha \left\{|\langle R\varphi(w),\varphi(w)\rangle|^2\left(\log\frac{e}{1-|\varphi(w)|^2}\right)^2\right. \\
&\qquad\left. +|R\varphi(w)|^2-\frac{|\langle R\varphi(w),\varphi(w)\rangle|^2}{|\varphi(w)|^2}\right\}^{\frac12} \\
&= (1-|w|^2)^\alpha \left\{|R\varphi(w)|^2+|\langle R\varphi(w),\varphi(w)\rangle|^2\right. \\
&\qquad\left.\left[\left(\log\frac{e}{1-|\varphi(w)|^2}\right)^2-\frac{1}{|\varphi(w)|^2}\right]\right\}^{\frac12} \\
&\ge C(1-|w|^2)^\alpha \left\{|R\varphi(w)|^2+|\langle R\varphi(w),\varphi(w)\rangle|^2\left(\log\frac{e}{1-|\varphi(w)|^2}\right)^2\right\}^{\frac12}.
\end{aligned}$$

This shows that (iii) holds.
(iii) $\Leftrightarrow$ (i). It follows from Lemmas 2.2 and 2.4.

Theorem 2.2. *Suppose* $\mu(t)=\left(\log\frac{e}{1-t^2}\right)^{-1}$. *Then the following statements are equivalent.*

(i) $C_\varphi : B_\mu \to B_\alpha$ *is compact.*
(ii) $C_\varphi : Z \to B_\alpha$ *is compact.*
(iii) $C_\varphi : B_\mu \to B_\alpha$ *is bounded and* $\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha \log\frac{e}{1-|\varphi(z)|^2}|\langle R\varphi(z),\varphi(z)\rangle| = 0.$
(iv) $C_\varphi : Z \to B_\alpha$ *is bounded and* $\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha \log\frac{e}{1-|\varphi(z)|^2}|\langle R\varphi(z),\varphi(z)\rangle| = 0.$

Proof.

(i) $\Rightarrow$ (ii). It is obvious.
(i) $\Leftrightarrow$ (iii). Note that $I_\mu < \infty$, then the result follows from Lemma 2.3.
(iii) $\Leftrightarrow$ (iv). The equivalence follows from Theorem 2.1.

(ii) $\Rightarrow$ (iv). Suppose $C_\varphi : Z \to B_\alpha$ is compact. Then $C_\varphi : Z \to B_\alpha$ is bounded. Let $\{z_k\}$ be a sequence in B^n such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ and $z_{0,k} = \frac{\varphi(z_k)}{|\varphi(z_k)|}$. Then there exists $\xi_k \in C^n$ satisfying $R\varphi(z_k) = v_k z_{0,k} + w_k \xi_k$, where $\langle z_{0,k}, \xi_k\rangle = 0$, $|\xi_k| = 1$, $v_k = \langle R\varphi(z_k), z_{0,k}\rangle$ and $|R\varphi(z_k)|^2 = |v_k|^2 + |w_k|^2$. Define

$$f_k(z) = \left(\log \frac{e}{1-|\varphi(z_k)|^2}\right)^{-1} \int_0^{\langle z,\varphi(z_k)\rangle} \left(\log \frac{e}{1-t}\right)^2 \mathrm{d}t. \tag{10}$$

By a simple calculation, we obtain

$$\nabla f_k(z) = \overline{\varphi(z_k)} \left(\log \frac{e}{1-|\varphi(z_k)|^2}\right)^{-1} \left(\log \frac{e}{1-\langle z,\varphi(z_k)\rangle}\right)^2 \tag{11}$$

and

$$\begin{aligned} R^{(2)} f_k(z) &= \langle z,\varphi(z_k)\rangle \left(\log \frac{e}{1-|\varphi(z_k)|^2}\right)^{-1} \left(\log \frac{e}{1-\langle z,\varphi(z_k)\rangle}\right)^2 \\ &\quad + \frac{2\langle z,\varphi(z_k)\rangle^2}{1-\langle z,\varphi(z_k)\rangle} \left(\log \frac{e}{1-|\varphi(z_k)|^2}\right)^{-1} \log \frac{e}{1-\langle z,\varphi(z_k)\rangle}. \end{aligned} \tag{12}$$

It is easy to check that $\|f_k\|_Z \le C$ for any positive integer k from (12), and $\{f_k\}$ converges to 0 uniformly on any compact subsets of B^n from (10). Since $C_\varphi : Z \to B_\alpha$ is compact, it follows that $\lim_{k\to\infty} \|C_\varphi f_k\|_{B_\alpha} = 0$ from Lemma 2.1. On the other hand, by (11), we have

$$\begin{aligned} \|C_\varphi f_k\|_{B_\alpha} &\ge (1-|z_k|^2)^\alpha |R(f_k \circ \varphi)(z_k)| \\ &= (1-|z_k|^2)^\alpha |\langle \nabla f_k(\varphi(z_k)), \overline{R\varphi(z_k)}\rangle| \\ &= (1-|z_k|^2)^\alpha |v_k||\varphi(z_k)| \log \frac{e}{1-|\varphi(z_k)|^2} \\ &= (1-|z_k|^2)^\alpha \log \frac{e}{1-|\varphi(z_k)|^2} |\langle R\varphi(z_k), \varphi(z_k)\rangle|. \end{aligned}$$

That is, $\lim_{k\to\infty} (1-|z_k|^2)^\alpha \log \frac{e}{1-|\varphi(z_k)|^2} |\langle R\varphi(z_k), \varphi(z_k)\rangle| = 0$. Therefore the desired result follows and the proof is complete.

3. The boundedness and compactness of $C_\varphi : B_\alpha \to Z$

The following results completely characterize the boundedness and compactness of composition operators from the Bloch type space to the Zygmund space. For the purpose, we need some lemmas.

Lemma 3.1. If

$$\sup_{z\in B^n} \sum_{j=1}^n \frac{(1-|z|^2)|R\varphi_j(z)|^2}{1-|\varphi(z)|^2} < \infty \tag{13}$$

and

$$\sup_{z\in B^n}\sum_{j=1}^{n}(1-|z|^2)|R^{(2)}\varphi_j(z)|\log\frac{e}{1-|\varphi(z)|^2}<\infty. \tag{14}$$

Then $C_\varphi : Z \to Z$ is bounded.

Proof. Suppose that (13) and (14) hold. For any $f \in Z$, by the characterization of the Zygmund space, we have

$$\begin{aligned}
&(1-|z|^2)|R^{(2)}(f\circ\varphi)(z)|\\
&=(1-|z|^2)\left|\sum_{i,j=1}^{n}D_jD_if\circ\varphi(z)R\varphi_i(z)R\varphi_j(z)\right.\\
&\quad\left.+\sum_{j=1}^{n}D_jf\circ\varphi(z)R^{(2)}\varphi_j(z)\right|\\
&\le C(1-|z|^2)\left(\sum_{i,j=1}^{n}\frac{|R\varphi_i(z)R\varphi_j(z)|\|f\|_Z}{1-|\varphi(z)|^2}\right.\\
&\quad\left.+\sum_{j=1}^{n}|R^{(2)}\varphi_j(z)|\log\frac{e}{1-|\varphi(z)|^2}\|f\|_Z\right)\\
&\le C\|f\|_Z.
\end{aligned}$$

Clearly, $|(f\circ\varphi)(0)|\le C\|f\|_Z$. So it follows that $C_\varphi : Z \to Z$ is bounded.

PROPOSITION 3.1

Z is an automorphism invariant boundary regular small space.

Proof. We will show that the space Z has all the properties (i)–(iv) with automorphism invariant boundary regular small spaces. From the definitions it is easy to see that (i)–(iii) hold for Z. To show that (iv) holds, we need to prove that for any conformal automorphism $\varphi=\varphi_aU=(\varphi_1,\dots,\varphi_n)$ of B^n, if $f\in Z$ then $f\circ\varphi\in Z$, where a is a point of B^n and U is a unitary transformation of C^n. Note that $\varphi_j\in H(\overline{B^n})(j=1,\dots,n)$ from (6), which implies that $R^{(k)}\varphi_j\in H(\overline{B^n})$ and $R^{(k)}\varphi_j$ is bounded in the closed unit ball for any positive integer k. By the Schwarz-Pick lemma in the unit ball (see [6])

$$\frac{1-|z|^2}{1-|\varphi(z)|^2}\le\frac{1-|\varphi(0)|^2}{|1-\langle\varphi(z),\varphi(0)\rangle|^2}\le\frac{1+|\varphi(0)|}{1-|\varphi(0)|},$$

we have

$$\frac{(1-|z|^2)|R\varphi_j(z)|^2}{1-|\varphi(z)|^2}\le C\frac{1-|z|^2}{1-|\varphi(z)|^2}\le C$$

and

$$(1-|z|^2)|R^{(2)}\varphi_j(z)|\log\frac{e}{1-|\varphi(z)|^2}\le C(1-|z|^2)\log\frac{e}{1-|z|^2}\le C$$

for all $z\in B^n$. Thus it follows that $f\circ\varphi\in Z$ from Lemma 3.1. The proof is complete.

Lemma 3.2 [7]. *Suppose X is an automorphism invariant boundary regular small space. If $C_\varphi : X\to X$ is compact, then $\|\varphi\|_\infty<1$.*

Theorem 3.1. *Let $\alpha>0$. Then the following statements are equivalent.*

(i) *$C_\varphi : B_\alpha\to Z$ is compact.*
(ii) *$C_\varphi : B_\alpha\to Z$ is bounded.*
(iii) *$C_\varphi : Z\to Z$ is compact.*
(iv) *$\varphi_j\in Z(j=1,\ldots,n)$ and $\|\varphi\|_\infty<1$.*

Proof.

(i) ⇒ (ii). It is obvious.
(ii) ⇒ (iii). Suppose $\{f_m\}$ is a bounded sequence in Z and it converges to 0 uniformly on compact subsets of B^n. By the Weierstrass theorem the sequence $\{R^{(2)}f_m\}$ also converges to 0 uniformly on compact subsets of B^n. We now prove that $\|f_m\|_{B_\alpha}$ converges to 0. For any $\varepsilon>0$, there exists $\delta(0<\delta<1)$ such that $(1-|z|^2)^\alpha<\varepsilon$ whenever $\delta<|z|<1$. Meanwhile, there exists a positive integer N such that $|f_m(0)<\varepsilon$, $|R^{(2)}f_m(z)|<\varepsilon$ for all $|z|\le\delta$ and all $m\ge N$. Note that a function $f\in B_\alpha$ if and only if $Rf\in B_{\alpha+1}$ (see [12]). Hence we obtain

$$\begin{aligned}\|f_m\|_{B_\alpha} &\le |f_m(0)|+\sup_{z\in B_\alpha}(1-|z|^2)^{\alpha+1}|R^{(2)}f_m(z)|\\ &\le \varepsilon+\sup_{|z|\le\delta}(1-|z|^2)^{\alpha+1}|R^{(2)}f_m(z)|\\ &\quad+\sup_{\delta<|z|<1}(1-|z|^2)^{\alpha+1}|R^{(2)}f_m(z)|\\ &\le \varepsilon+\varepsilon+\varepsilon\sup_{\delta<|z|<1}(1-|z|^2)|R^{(2)}f_m(z)|\\ &\le 2\varepsilon+\varepsilon\|f_m\|_Z\\ &\le C\varepsilon\end{aligned}$$

for all $m\ge N$. Thus, it follows that $\|C_\varphi f_m\|_Z\le C\|f_m\|_{B_\alpha}\to 0$ as $m\to\infty$ from the boundedness of $C_\varphi : B_\alpha\to Z$. Therefore $C_\varphi : Z\to Z$ is compact by Lemma 2.1.
(iii) ⇒ (iv). Suppose $C_\varphi : Z\to Z$ is compact. Then it is clear that $\varphi_j\in Z$ $(j=1,\ldots,n)$. The necessary condition $\|\varphi\|_\infty<1$ follows from Proposition 3.1 and Lemma 3.2.
(iv) ⇒ (i). Suppose $\{f_m\}$ is a bounded sequence in B_α and it converges to 0 uniformly on compact subsets of B^n. Then there exists a positive integer N such that $|D_jf_m(z)|<\varepsilon$ and $|D_iD_jf_m(z)|<\varepsilon(i,j=1\ldots,n)$ for all $|z|\le\|\varphi\|_\infty$ and all $m\ge N$. Since $\varphi_j\in Z$, by (5) we have

$$(1-|z|^2)|R\varphi_j(z)|^2\le C(1-|z|^2)\left(\log\frac{e}{1-|z|^2}\right)^2\|\varphi_j\|_Z^2\le C.$$

Hence it follows that

$$\begin{aligned}
&(1-|z|^2)|R^{(2)}(f_m\circ\varphi)(z)| \\
&= (1-|z|^2)\left|\sum_{i,j=1}^{n} D_iD_jf_m\circ\varphi(z)R\varphi_i(z)R\varphi_j(z)\right. \\
&\qquad\left. +\sum_{j=1}^{n} D_jf_m\circ\varphi(z)R^{(2)}\varphi_j(z)\right| \\
&\le C\varepsilon(1-|z|^2)\left(\sum_{j=1}^{n}|R\varphi_j(z)|^2+\sum_{j=1}^{n}|R^{(2)}\varphi_j(z)|\right) \\
&\le C\varepsilon
\end{aligned}$$

for all $z \in B^n$ and $m \ge N$. Clearly, $f_m(\varphi(0)) \to 0$ as $m \to \infty$. Thus, $\|C_\varphi f_m\|_Z \to 0$ and so $C_\varphi : B_\alpha \to Z$ is compact by Lemma 2.1. The proof is complete.

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