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Uniqueness and zeros of *q*-shift difference polynomials

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Abstract. In this paper, we consider the zero distributions of q-shift difference polynomials of meromorphic functions with zero order, and obtain two theorems that extend the classical Hayman results on the zeros of differential polynomials to q-shift difference polynomials. We also investigate the uniqueness problem of q-shift difference polynomials that share a common value.

Keywords. Meromorphic function; q-shift; difference polynomial; uniqueness; zero order.

1. Introduction and main results

A meromorphic function $f(z)$ means meromorphic in the complex plane. If no poles occur, then $f(z)$ reduces to an entire function. Given a meromorphic function $f(z)$, recall that $\alpha(z) \not\equiv 0, \infty$ is a small function with respect to $f(z)$, if $T(r, \alpha) = S(r, f)$, where $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$, and $r \to \infty$ outside a possible exceptional set of logarithmic density 0. In addition, if $f - \alpha$ and $g - \alpha$ have the same zeros with the same multiplicities, then we say that f and g share the small function α CM (counting multiplicities). The order $\sigma(f)$ is defined by

$$\sigma(f) := \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r}.$$

The logarithmic density of set F_n is defined as follows:

$$\overline{\lim_{r\to\infty}} \frac{1}{\log r} \int_{[1,r]\cap F_n} \frac{1}{t}\mathrm{dt}.$$

We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory [10, 11, 20].

The classical problem of value distributions of differential polynomials is Hayman conjecture [9], i.e. if f is a transcendental meromorphic function and n is a positive integer, then $f^n f'$ takes every finite nonzero value infinitely often. This conjecture has been considered by many authors. For example, Hayman [8] proved that if f is a transcendental meromorphic function and $n \geq 3$, then $f^n f'$ takes every finite nonzero complex value infinitely often. The case $n = 2$ was settled by Mues [17]. Bergweiler and Eremenko [2] proved the case of $n = 1$, which was also considered by Chen and Fang [3] and Chen [4]. For an analogue of the above results in difference polynomials, Laine and Yang (Theorem 2 of [12]) proved:

Theorem A. *Let f be a transcendental entire function of finite order and c be a nonzero complex constant. Then for $n \geq 2$, $f(z)^n f(z+c)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

Theorem A implies that $f(z)^n f(z+c)$ has no nonzero finite Picard exceptional value. Recently, Liu and Yang [13] and Liu [14] have made some improvements of Theorem A to more general difference products of meromorphic functions. Zhang and Korhonen [21] and Zhang [22] also obtained some results on the value distributions of q-difference polynomials of different types, such as:

Theorem B (Theorems 4.1 and 4.3 of [21]). *Let f be a transcendental meromorphic (resp. entire) function of zero order and q be a nonzero complex constant. Then for $n \geq 6$ (resp. $n \geq 2$), both $f(z)^n f(qz)$ and $f(z)^n(f(z)-1)f(qz)$ assume every nonzero value $a \in \mathbb{C}$ infinitely often.*

Here, q-shift of $f(z)$ is defined by $f(qz+c)$. In the following, we assume that q, a are nonzero complex constants and m, n are positive integers. Some results on the zeros of $f(z)^n f(qz+c)$ can be found in [15]. In this paper, we mainly consider the zeros distributions of q-shift difference polynomials $f(z)^n(f(z)^m - a)f(qz+c)$ and $f(z)^n(f(z)^m - a)[f(qz+c) - f(z)]$, and we obtain the following results.

Theorem 1.1. *Let f be a transcendental meromorphic (resp. entire) function with zero order. If $n \geq 6$ (resp. $n \geq 2$), then $f(z)^n(f(z)^m - a)f(qz+c) - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a nonzero small function with respect to f. In particular, if f is a transcendental entire function and $\alpha(z)$ is a nonzero rational function, then m, n can be any positive integers.*

COROLLARY 1.2

The q-shift difference equation $f(z)^n(f(z)^m - a)f(qz+c) - R(z) = 0$ has no transcendental meromorphic solutions of zero order when $n \geq 6$, and it has no transcendental entire solutions of zero order for any positive integers m, n.

Remark 1. From the conditions of Theorem B and Theorem 1.1, it is interesting to show that m can be any positive integer. Furthermore, from the proof of Theorem 1.1 below, it follows that we can replace the constant a with a small function with respect to $f(z)$.

Remark 2. The condition of zero order can not be removed, which can be seen by function $f(z) = \mathrm{e}^z$, $q = -n$, $\alpha(z) = -a\mathrm{e}^c$. Thus

$$f(z)^n(f(z)^m - a)f(qz+c) - \alpha(z) = \mathrm{e}^{mz+c},$$

but e^{mz+c} has no zeros.

Theorem 1.3. *Let f be a transcendental meromorphic (resp. entire) function with zero order. If $n \geq 7$ (resp. $n \geq 3$), then $f(z)^n(f(z)^m - a)[f(qz+c) - f(z)] - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a nonzero small function with respect to f.*

COROLLARY 1.4

The q-shift difference equation $f(z)^n(f(z)^m - a)[f(qz+c) - f(z)] - R(z) = 0$ has no transcendental meromorphic solutions of zero order when $n \geq 7$, and it has no transcendental entire solutions of zero order when $n \geq 3$.

Yang and Hua [19] studied the uniqueness of the differential monomials $f^n f'$ and proved the uniqueness theorem of meromorphic functions (Theorem 1 of [19]).

Theorem C. *Let f and g be nonconstant entire functions, a be a nonzero finite constant, and let $n \geq 7$ be a positive integer. If $f^n f'$ and $g^n g'$ share a CM, then either $f = tg$ for $t^{n+1} = 1$ or $f = c_1 \mathrm{e}^{cz}$ and $g = c_2 \mathrm{e}^{-cz}$ for nonzero constants c, c_1 and c_2 with $(c_1 c_2)^{n+1} c^2 = -a^2$.*

Similarly, we can investigate the uniqueness problem of difference polynomials sharing a common value. From this point of view, Qi (Theorem 2 of [18]) considered the case of difference polynomials $f(z)^n(f(z) - 1)f(z+c)$, and Zhang and Korhonen (Theorem 5.2 of [21]) considered the case of q-difference polynomials and obtained the following theorem.

Theorem D. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order. If $n \geq 6$, $f(z)^n(f(z) - 1)f(qz)$ and $g(z)^n(g(z) - 1)g(qz)$ share 1 CM, then $f(z) \equiv g(z)$.*

In this paper, we consider the case of q-shift difference polynomials and extend Theorem D as follows:

Theorem 1.5. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order. If $n \geq m+5$, $f(z)^n(f(z)^m - a)f(qz+c)$ and $g(z)^n(g(z)^m - a)g(qz+c)$ share a nonzero polynomial $P(z)$ CM, then $f(z) \equiv g(z)$.*

2. Some lemmas

The difference logarithmic derivative lemma, given by Chiang and Feng (Corollary 2.5 of [5]), Halburd and Korhonen (Theorem 2.1 of [6], Theorem 5.6 of [7]) respectively, plays an important part in considering the value distributions of difference polynomials. For q-difference, a similar result can be found in (Theorem 1.1 of [1]). The following q-shift difference logarithmic derivative lemma is very important in considering the zero distributions of q-shift difference polynomials.

Lemma 2.1 (*Theorem* 2.1 *of* [15]). *Let f be a meromorphic function of zero order, and let $c \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(qz+c)}{f(z)}\right) = S(r, f) \tag{2.1}$$

on a set of logarithmic density 1.

Lemma 2.2 (*Theorem* 1.1 *of* [21]). *Let $f(z)$ be a transcendental meromorphic function of zero order. Then*

$$T(r, f(qz)) = T(r, f(z)) + S(r, f)$$

on a set of logarithmic density 1.

The following lemma has little modifications of the original version (Theorem 2.1 of [5]).

Lemma 2.3. *Let $f(z)$ be a transcendental meromorphic function of finite order. Then,*

$$T(r, f(z+c)) = T(r, f) + S(r, f). \tag{2.2}$$

Combining Lemma 2.2 with Lemma 2.3, we get the following result easily.

Lemma 2.4. *Let $f(z)$ be a transcendental meromorphic function of zero order. Then*

$$T(r, f(qz+c)) = T(r, f(z)) + S(r, f)$$

on a set of logarithmic density 1.

For the proof of Theorem 1.1, we need the following lemma.

Lemma 2.5. *Let $f(z)$ be a transcendental entire function of zero order, $F = f(z)^n(f(z)^m - a)f(qz+c)$. Then*

$$T(r, F) = (n+m+1)T(r, f) + S(r, f) \tag{2.3}$$

on a set of logarithmic density 1. *If $f(z)$ is a transcendental meromorphic function of zero order. Then*

$$(n+m-1)T(r, f)+S(r, f) \leq T(r, F) \leq (n+m+1)T(r, f)+S(r, f) \tag{2.4}$$

on a set of logarithmic density 1.

Proof. From Lemma 2.1 and the standard Valiron–Mohon'ko theorem [16], if f is a transcendental entire function, then

$$\begin{aligned}
(n+m+1)T(r, f) &= T(r, f^{n+1}(f^m - a)) + S(r, f) \\
&\leq m(r, f^{n+1}(f^m - a)) + S(r, f) \\
&\leq m\left(r, \frac{f^{n+1}(f^m - a)}{f^n(f^m - a)f(qz+c)}\right) + m\,(r, F) + S(r, f) \\
&\leq m\left(r, \frac{f(z)}{f(qz+c)}\right) + m(r, F) + S(r, f) \\
&\leq T(r, F) + S(r, f),
\end{aligned} \tag{2.5}$$

on a set of logarithmic density 1. On the other hand, combining Lemma 2.4 by the fact that f is a transcendental function of zero order, we have

$$\begin{aligned} T(r,F) &\leq (n+m)T(r,f) + T(r, f(qz+c)) + S(r,f) \\ &\leq (n+m+1)T(r,f) + S(r,f), \end{aligned} \tag{2.6}$$

on a set of logarithmic density 1. Thus, (2.3) follows from (2.5) and (2.6).

If $f(z)$ is a transcendental meromorphic function of zero order, then

$$\begin{aligned} (n+m+1)T(r,f) &= T(r, f^{n+1}(f^m - a)) + S(r,f) \\ &= m(r, f^{n+1}(f^m - a)) + N(r, f^{n+1}(f^m - a)) + S(r,f) \\ &\leq m\left(r, F(z)\frac{f(z)}{f(qz+c)}\right) \\ &\quad + N\left(r, F(z)\frac{f(z)}{f(qz+c)}\right) + S(r,f) \\ &\leq T(r,F) + m\left(r, \frac{f(z)}{f(qz+c)}\right) \\ &\quad + N\left(r, \frac{f(z)}{f(qz+c)}\right) + S(r,f) \\ &\leq T(r,F) + 2T(r,f) + S(r,f), \end{aligned} \tag{2.7}$$

on a set of logarithmic density 1. Thus, (2.4) follows from (2.6) and (2.7). □

The following result will be used in the proof of Theorem 1.3, i.e. similar to the proof of Theorem 1.1. Hence, we will not give the details of the proof of Theorem 1.3 below.

Lemma 2.6. *Let $f(z)$ be a transcendental meromorphic function of zero order. Then,*

$$T(r, f(z)^n(f(z)^m - a)[f(qz+c) - f(z)]) \geq (n+m-1)T(r,f) + S(r,f). \tag{2.8}$$

If $f(z)$ is a transcendental entire function of zero order, then

$$T(r, f(z)^n(f(z)^m - a)[f(qz+c) - f(z)]) \geq (n+m)T(r,f) + S(r,f). \tag{2.9}$$

Proof. Assume that $G(z) = f(z)^n(f(z)^m - a)[f(qz+c) - f(z)]$. Then

$$\frac{1}{f(z)^{n+1}(f(z)^m - a)} = \frac{1}{G}\frac{f(qz+c) - f(z)}{f(z)}. \tag{2.10}$$

Using the first main theorem of Nevanlinna theory, Lemmas 2.1 and 2.4, we get

$$\begin{aligned}(n+m+1)T(r,f) &\leq T(r,G(z))+T\left(r,\frac{f(qz+c)-f(z)}{f(z)}\right)+S(r,f)\\ &\leq T(r,G(z))+m\left(r,\frac{f(qz+c)-f(z)}{f(z)}\right)\\ &+ N\left(r,\frac{f(qz+c)-f(z)}{f(z)}\right)+S(r,f)\\ &\leq T(r,G(z))+N\left(r,\frac{f(qz+c)}{f(z)}\right)+S(r,f)\\ &\leq T(r,G(z))+2T(r,f)+S(r,f), \qquad (2.11)\end{aligned}$$

thus, we get (2.8). From the above proof, it is easy to get (2.9) if f is an entire function. □

For the proof of Theorem 1.5, we need the following lemma. The main idea is from [18]. For the complete proof, we give the details.

Lemma 2.7. *Let f and g be transcendental entire functions of zero order. If $n \geq m+5$, m, n are positive integers and*

$$f^n(f^m-1)f(qz+c)=g^n(g^m-1)g(qz+c), \qquad (2.12)$$

then $f=tg$, and $t^{n+1}=t^m=1$.

Proof. Let $G(z)=\frac{f(z)}{g(z)}$. In the following, we will prove $G(z)$ is a constant. Otherwise, from (2.12), we have

$$g(z)^m=\frac{G(z)^nG(qz+c)-1}{G(z)^{n+m}G(qz+c)-1}. \qquad (2.13)$$

If 1 is a Picard exceptional value of $G(z)^{n+m}G(qz+c)$, applying the Nevanlinna second main theorem with Lemma 2.3, we get

$$\begin{aligned}T(r,G^{n+m}G(qz+c)) &\leq \bar{N}(r,G^{n+m}G(qz+c))+\bar{N}\left(r,\frac{1}{G^{n+m}G(qz+c)}\right)\\ &+\bar{N}\left(r,\frac{1}{G^{n+m}G(qz+c)-1}\right)+S(r,G)\\ &\leq 2T(r,G(z))+2T(r,G(qz+c))+S(r,G)\\ &\leq 4T(r,G(z))+S(r,G), \qquad (2.14)\end{aligned}$$

on a set of logarithmic density 1. On the other hand, combining the standard Valiron–Mohon'ko theorem [16] with (2.14) and Lemma 2.3, we get

$$\begin{aligned}(n+m)T(r,G) &= T(r,G^{n+m})+S(r,f)\\ &\leq T(r,G^{n+m}G(qz+c))+T(r,G(qz+c))+O(1)\\ &\leq 5T(r,G(z))+S(r,G), \qquad (2.15)\end{aligned}$$

which is a contradiction with $n \geq m+5$. Therefore, 1 is not a Picard exceptional value of $G(z)^{n+m}G(qz+c)$. Thus, there exists z_0 such that $G(z_0)^{n+m}G(qz_0+c) = 1$. Then, we will have two cases.

Case 1. $G(z)^{n+m}G(qz+c) \not\equiv 1$. From (2.13), $g(z)$ is an entire function, then we get $G(z_0)^n G(qz_0+c) = 1$, thus $G(z_0)^m = 1$. Therefore,

$$\bar{N}\left(r, \frac{1}{G^{n+m}G(qz+c)-1}\right) \leq \bar{N}\left(r, \frac{1}{G^m-1}\right) \leq mT(r,G) + S(r,G). \tag{2.16}$$

By (2.16) and Lemma 2.4, applying the Nevanlinna second main theorem, we get

$$\begin{aligned} T(r, G^{n+m}G(qz+c)) &\leq \bar{N}(r, G^{n+m}G(qz+c)) \\ &\quad + \bar{N}\left(r, \frac{1}{G^{n+m}G(qz+c)}\right) \\ &\quad + \bar{N}\left(r, \frac{1}{G^{n+m}G(qz+c)-1}\right) + S(r,G) \\ &\leq (m+2)T(r, G(z)) + 2T(r, G(qz+c)) \\ &\quad + S(r,G) \\ &\leq (m+4)T(r, G(z)) + S(r,G). \end{aligned} \tag{2.17}$$

On the other hand, we have

$$\begin{aligned} (n+m)T(r,G) &= T(r, G^{n+m}) \\ &\leq T(r, G^{n+m}G(qz+c)) \\ &\quad + T(r, G(qz+c)) + O(1) \\ &\leq (m+5)T(r, G(z)) + S(r,G), \end{aligned} \tag{2.18}$$

it implies that $n \leq 5$ which is a contradiction with $n \geq m+5$, where m is a positive integer.

Case 2. $G(z)^{n+m}G(qz+c) \equiv 1$. Thus,

$$\begin{aligned} (n+m)T(r,G) &= T(r, G(qz+c)) + S(r,G) \\ &= T(r, G(z)) + S(r,G), \end{aligned} \tag{2.19}$$

which also is a contradiction with $n \geq m+5$. Thus, G must be a constant. Then $f(z) = tg(z)$, where t is a nonzero constant. From (2.12), we know that $t^m = 1$ and $t^{n+1} = 1$, n, m are positive integers. □

3. Proof of Theorem 1.1

Denote $F(z) = f(z)^n(f(z)^m - a)f(qz+c)$. We claim that $F(z)$ is not a rational function. Suppose for the contrary $F(z)$ is a rational function, then $f(z)^n(f(z)^m - a) = R(z)/f(qz+c)$. Thus

$$\begin{aligned} (n+m)T(r, f(z)) &= T(r, R(z)/f(qz+c)) \\ &= T(r, f(z)) + S(r,f), \end{aligned} \tag{3.1}$$

which contradicts that n, m are positive integers. Therefore, $F(z)$ is not a rational function. We discuss the following two cases.

Case 1. Suppose that f is a transcendental meromorphic function. Using the Nevanlinna second main theorem for three small functions (Theorem 2.5 of [10]) and (2.4), we obtain

$$\begin{aligned}(n+m-1)T(r,f) &\leq T(r,F(z))+S(r,f)\\ &\leq \bar{N}(r,F)+\bar{N}\left(r,\frac{1}{F}\right)+\bar{N}\left(r,\frac{1}{F-\alpha(z)}\right)+S(r,f)\\ &\leq \bar{N}(r,f)+\bar{N}(r,f(qz+c))+\bar{N}\left(r,\frac{1}{f}\right)\\ &\quad +\bar{N}\left(r,\frac{1}{f^m-a}\right)\\ &\quad +\bar{N}\left(r,\frac{1}{f(qz+c)}\right)+\bar{N}\left(r,\frac{1}{F-\alpha(z)}\right)+S(r,f)\\ &\leq (m+4)T(r,f)+\bar{N}\left(r,\frac{1}{F-\alpha(z)}\right)+S(r,f),\end{aligned} \tag{3.2}$$

which implies that $F-\alpha(z)$ has infinitely many zeros when $n \geq 6$.

Case 2. Suppose that f is an entire function and $\alpha(z)$ is a small function with respect to f. From Lemma 2.5 and the Nevanlinna second main theorem,

$$\begin{aligned}(n+m+1)T(r,f) &= T(r,F)+S(r,f)\\ &\leq \bar{N}\left(r,\frac{1}{F}\right)+\bar{N}\left(r,\frac{1}{F-\alpha(z)}\right)+S(r,f)\\ &\leq \bar{N}\left(r,\frac{1}{f}\right)+\bar{N}\left(r,\frac{1}{f^m-a}\right)\\ &\quad +\bar{N}\left(r,\frac{1}{f(qz+c)}\right)+\bar{N}\left(r,\frac{1}{F-\alpha(z)}\right)+S(r,f)\\ &\leq (m+2)T(r,f)+\bar{N}\left(r,\frac{1}{F-\alpha(z)}\right)+S(r,f),\end{aligned} \tag{3.3}$$

so if $n \geq 2$, then $F-\alpha(z)$ has infinitely many zeros.

In particular, if $\alpha(z)$ is a nonzero rational function, and we assume that $F(z)$ has only finitely many zeros, since $F(z)$ is a function with zero order, then we get

$$f(z)^n(f(z)^m-1)f(qz+c)-\alpha(z)=R(z). \tag{3.4}$$

Then, if for any n, m are positive integers, we get $T(r,f)=S(r,f)$ from the proof of the beginning of this part, which is a contradiction. Thus, we have completed the proof of Theorem 1.1.

4. Proof of Theorem 1.5

Let $F(z) = \frac{f(z)^n(f(z)^m-a)f(qz+c)}{P(z)}$ and $G(z) = \frac{g(z)^n(g(z)^m-a)g(qz+c)}{P(z)}$. Thus, F and G share the value 1 CM. Since f and g are transcendental entire functions with zero order, then we get

$$F - 1 = A(G-1). \tag{4.1}$$

It implies that $F = AG + (1-A)$, where A is a nonzero constant. The following, we will prove $A = 1$. Otherwise, assume that $A \neq 1$. Using the second main theorem and (2.3), we obtain

$$\begin{aligned}(n+m+1)T(r,f) &= T(r,F) + S(r,f)\\ &\leq \bar{N}(r,F) + \bar{N}(r,\frac{1}{F}) + \bar{N}\left(r,\frac{1}{F-(1-A)}\right) + S(r,f)\\ &\leq \bar{N}\left(r,\frac{1}{f}\right) + \bar{N}\left(r,\frac{1}{f^m-1}\right) + \bar{N}\left(r,\frac{1}{f(qz+c)}\right)\\ &\quad + \bar{N}\left(r,\frac{1}{G}\right) + S(r,f)\\ &\leq (m+2)T(r,f) + (m+2)T(r,g) + S(r,f) + S(r,g).\end{aligned} \tag{4.2}$$

Thus, we get

$$(n-1)T(r,f) \leq (m+2)T(r,g) + S(r,f) + S(r,g). \tag{4.3}$$

Similarly,

$$(n-1)T(r,g) \leq (m+2)T(r,f) + S(r,f) + S(r,g). \tag{4.4}$$

Hence, we get

$$(n-m-3)[T(r,f) + T(r,g)] \leq S(r,f) + S(r,g), \tag{4.5}$$

which is a contradiction with $n \geq m+5$.

Thus, we get $A = 1$, the conclusion of Theorem 1.5 followed by Lemma 2.7.

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