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# Test rank of an abelian product of a free Lie algebra and a free abelian Lie algebra

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**Abstract.** Let $F$ be a free Lie algebra of rank $n \geq 2$ and $A$ be a free abelian Lie algebra of rank $m \geq 2$. We prove that the test rank of the abelian product $F \times A$ is $m$. Morever we compute the test rank of the algebra $F/\gamma_k(F)'$.

**Keywords.** Automorphism; free Lie algebras; test elements; test sets; test rank.

## 1. Introduction

The notions test set, test rank and test elements are interesting for groups and Lie algebras. Examples of test elements of free Lie algebras of rank two were given by Mikhalev and Yu [10]. Other examples of test elements were considered by Mikhalev, Umirbaev and Yu [11], Temizyurek and Ekici [13] and Esmerligil [2].

Existence of test elements in groups and Lie algebras is almost likely an exception rather than a rule. Usually a single element is not enough to test the automorphisms of a Lie algebra. Therefore a natural generalization appeared: The notion of a test set. In [1] and [14] the test rank of a free metabelian Lie algebra was computed independently. As a further step in the study of the test ranks of relatively free Lie algebras we can consider the result of [3]. It turns out that the test rank of a free polynilpotent Lie algebra is either equal to the rank of the Lie algebra or is one less than the rank of it. Interest in the test ranks of free soluble Lie algebras is explained in [13] and [15]. Temizyurek and Ekici [13] showed that a free soluble Lie algebra of rank 2 with solvability class 3 has test rank 1 and pointed out the particular test element for such algebras. The test rank for free soluble Lie algebras of solvability class greater than 3 is calculated by Timoshenko and Shevelin in [15] and it is shown that the test rank of a free soluble Lie algebra of rank $n \geq 2$ is equal to $n - 1$.

In this paper we study the following two problems.

First, we compute the test rank of an abelian product of Lie algebras. The notion of the test rank is interesting for $M$-product of Lie algebras, where $M$ is an arbitrary variety of Lie algebras. The test rank of the metabelian product of free abelian groups was calculated in [6]. Morever in [8] it was proved that the test rank of soluble product of free abelian groups is one less than the number of factors. In [7] it was shown that for given integers $n$ and $k$ there exists a group of rank $n$ and test rank $k$. In the present paper, using the methods developed in [7], we calculated the test rank of the abelian product of a free Lie algebra and a free abelian Lie algebra of finite rank.

Second, we prove that an endomorphism of the algebra $F/R'$, which acts identically on $R/R'$, is an inner automorphism induced by some element of $R/R'$, where $F$ is a free Lie algebra of finite rank $n \geq 2$ , $R$ be an ideal of $F$ and $R'$ is the commutator subalgebra $[R, R]$ . Based on this result, we show that the test rank of the Lie algebra $F/\gamma_k(F)'$ is $n-1$, where $\gamma_k(F)$ is the $k$-th term of the lower central series of $F$.

## 2. Preliminaries

Let $L$ be a free Lie algebra of rank $n$ over a field $K$.

DEFINITION 1

A subset $\{g_1, \ldots, g_m\}$, $m \leq n$ of $L$, is said to be test set if for every endomorphisim $\varphi$ of $L$, the conditions $\varphi(g_i) = g_i$, $i = 1, 2, \ldots, m$ imply that $\varphi$ is an automorphism.

DEFINITION 2

The least length of a test set of $L$ is called the test rank of $L$.

By $U(L)$ we denote the universal enveloping algebra of $L$. It is well-known that $U(L)$ is an integral domain. Denote by a commutator the product in a Lie algebra $L$ over a field $K$. We consider $L$ as a Lie subalgebra of $U(L)$ with respect to Lie operation $[u, v] = uv - vu$, $u, v \in U(L)$.

Let $F$ be a free Lie algebra with free generators $x_1, x_2, \ldots, x_n$ over a field $K$. We denote by $\frac{\partial}{\partial x_i}$, $1 \leq i \leq n$ the left Fox derivatives [4]. The operators $\frac{\partial}{\partial x_i} : U(F) \to U(F)$ are linear mappings such that $\frac{\partial x_j}{\partial x_i} = \delta_{ij}$ (Kronecker delta), $\frac{\partial(u+v)}{\partial x_i} = \frac{\partial u}{\partial x_i} + \frac{\partial v}{\partial x_i}$, $\frac{\partial(uv)}{\partial x_i} = \frac{\partial u}{\partial x_i}\varepsilon(v) + u\frac{\partial v}{\partial x_i}$, where $\varepsilon : U(F) \to K$ is the homomorphism defined as $\varepsilon(x_i) = 0$ for all $i = 1, \ldots, n$. By $\Delta$ we denote the augmentation ideal of $U(F)$. The ideal $\Delta$ is a free left $U(F)$-module with basis $\{x_1, x_2, \ldots, x_n\}$ . Thus any element $u \in \Delta$ can be uniquely written in the form $u = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} x_i$.

Let $R$ be an ideal of $F$. By $I_R$ we denote the right ideal of $U(F)$ generated by $R$.

The proof of the following lemma can be found in [16].

*Lemma* 3. *Let $R$ be an ideal of $F$. Then $F \cap I_R = R$.*

Throughout this paper we need the following technical lemmas. The first lemma is an immediate consequence of the definitions.

*Lemma* 4. *Let $R$ be an ideal of $F$ and $u \in \Delta$. Then $u \in I_R\Delta$ if and only if $\frac{\partial u}{\partial x_i} \in I_R$ for each $i$, $1 \leq i \leq n$.*

Proof of the next lemma can be found in [17].

*Lemma* 5. *Let $R$ be an ideal of $F$ and $u \in F$. Then $u \in I_R\Delta$ if and only if $u \in R'$.*

Let $A$ be a free abelian Lie algebra and $F * A$ be the free product of the free Lie algebra $F$ and $A$. If $D$ is the cartesian subalgebra of $F * A$ then the algebra $(F * A)/D$ is the abelian product of $F$ and $A$. We denote this product as $F \times A$.

## 3. Test rank of a free abelian product

Let $F$ be a free Lie algebra with free generators $x_1, x_2, \ldots, x_n$ and $A$ be a free abelian Lie algebra of rank $m$ over a field $K$ of characteristic zero. It is well known that the test rank of $F$ is 1.

Consider any set $B = \{b_1, b_2, \ldots, b_k\}$, $k \leq m-1$ in the free abelian Lie algebra $A$. It is clear that if $B$ is linearly dependent then it can not be a test set. Assume that the set $B$ is linearly independent then it can be completed to a basis $\{b_1, b_2, \ldots, b_k, b_{k+1}, \ldots, b_m\}$ of $A$ as a vector space. The endomorphism $\varphi$ of $A$ defined as $\varphi(b_i) = b_i$, $1 \leq i \leq k$, $\varphi\left(b_j\right) = 0$, $k+1 \leq j \leq m$ keeps the set $B$ invariant but it is not an automorphism. Thus any set consisting less than $m$ element can not be a test set of $A$. On the other hand, any free generating set of $A$ is a test set. This shows that the minimal cardinality of a test set of $A$ is $m$.

In this section we will prove that for the given integers $l$ and $k$ there exists a free Lie algebra of rank $l$ and test rank $k$. Gupta, Romankov and Timoshenko [7] showed this result for groups. We will use this technique in computing the test rank of the abelian product $F \times A$. In our proof we use two special test elements of the free Lie algebra $F$. Shpilrain [12] described the rank of an element $u$ of $F$ to be the least number of free generators on which the image of $u$ under an arbitrary automorphism of $F$ can depend. Then he proved that homogeneous elements of maximal rank of a free Lie algebra of finite rank are test elements. He also found an algorithm for finding the rank of a homogeneous element. Using Shpilrain's algorithm we can show that the commutators

$$w_1 = [[\ldots[[x_1, x_2], x_3], \ldots], x_n], \quad w_2 = [[\ldots[[[x_1, x_2], x_1], x_3], \ldots], x_n]$$

have maximal rank. Therefore $w_1$ and $w_2$ are test elements of $F$. The property of being a test element of $w_1$ was also proved by Mikhalev *et al* [11].

**Theorem 6.** *Let $F$ be a free Lie algebra generated by the free generators $x_1, x_2, \ldots, x_n$, $n \geq 2$ and $A$ be a free abelian Lie algebra with free generators $a_1, a_2, \ldots, a_m$, $m \geq 2$. Then the test rank of the free abelian product $L = F \times A$ is $m$.*

*Proof.* Consider the test elements

$$w_1 = [[\ldots[[x_1, x_2], x_3], \ldots], x_n], \quad w_2 = [[\ldots[[[x_1, x_2], x_1], x_3], \ldots], x_n]$$

of $F$. We prove that the set $\{h_1, h_2, \ldots, h_m\}$ is a test set for $L$, where $h_1 = w_1 + a_1$, $h_i = w_2 + a_i$, $2 \leq i \leq m$. Let $\varphi$ be an endomorphism of $L$ such that $\varphi\left(h_j\right) = h_j$ for all $j = 1, \ldots, m$. Suppose that the endomorphism $\varphi$ of $L$ is defined by the rule

$$\varphi : x_i \to u_i + b_i, \ \varphi : a_j \to v_j + c_j,$$

where $u_i, v_j \in F$, $b_i, c_j \in A$, $1 \leq i \leq n$, $1 \leq j \leq m$. First we verify that $v_j = 0$ for all $j = 1, \ldots, m$. Assume that for some $j$, $v_j \neq 0$. Note that $\left[x_i, a_j\right] = 0$ and $\left[a_k, a_j\right] = 0$ in the abelian product $L = F \times A$, $1 \leq i \leq n$, $1 \leq j, k \leq m$. Then

$$\left[\varphi(x_i), \varphi\left(a_j\right)\right] = 0 \quad \text{and} \quad \left[\varphi(a_k), \varphi\left(a_j\right)\right] = 0.$$

Thus

$$\left[u_i, v_j\right] = 0 \quad \text{and} \quad \left[v_k, v_j\right] = 0, \quad 1 \leq i \leq n, 1 \leq j, k \leq m.$$

It follows that

$$u_i = \alpha_i v_j \quad \text{and} \quad v_k = \beta_k v_j, \tag{1}$$

where $\alpha_i, \beta_k \in K$. Therefore $\varphi(w_1) = 0$ and $\varphi(w_2) = 0$. Now consider the equality $\varphi(h_1) = h_1$. This yields

$$v_1 + c_1 = w_1 + a_1$$

and

$$v_1 = w_1, c_1 = a_1.$$

Likewise from the equality $\varphi(h_2) = h_2$ we get

$$v_2 = w_2, c_2 = a_2.$$

Taking into account (1) we get

$$w_2 = v_2 = \beta_2 v_j = \beta_2 \beta_1^{-1} w_1,$$

which is impossible. Thus $v_j = 0$ for all $j = 1, \ldots, m$. Then $\varphi\left(a_j\right) = a_j$, $j = 1, \ldots, m$ and

$$\begin{aligned} w_1 + a_1 &= \varphi(w_1 + a_1) = \varphi(w_1) + a_1, \\ w_2 + a_2 &= \varphi(w_2 + a_2) = \varphi(w_2) + a_2. \end{aligned}$$

That is $\varphi(w_1) = w_1$ and $\varphi(w_2) = w_2$. It follows that $w_1(u_1, u_2, \ldots, u_n) = w_1$ and $w_2(u_1, u_2, \ldots, u_n) = w_2$. These show that the endomorphism

$$\Psi : x_i \to u_i, \qquad i = 1, \ldots, n$$

of $F$ fixes the elements $w_1, w_2$. So $\Psi$ is an automorphism of $F$. On the other hand, the algebra $\varphi(L)$ is generated by the set

$$\{u_1 + b_1, u_2 + b_2, \ldots, u_n + b_n, a_1, a_2, \ldots, a_m\}.$$

Since $b_1, \ldots, b_n \in A$, the set

$$\{u_1, u_2, \ldots, u_n, a_1, a_2, \ldots, a_m\}$$

is also a generating set of $\varphi(L)$. Thus $\varphi(L)$ is generated by the set

$$\{\Psi(x_1), \ldots, \Psi(x_n), a_1, a_2, \ldots, a_m\}.$$

Hence $\varphi(L) = L$. Since $L$ is Hopfian $\varphi$ is an automorphism.

Now we are going to prove that the cardinality $m$ is minimum for the test sets.

Assume that $f_1 = y_1 + \alpha_1, \ldots, f_k = y_k + \alpha_k$, $k \leq m - 1$ is a test set for the algebra $L$, where $y_1, \ldots, y_k \in F$, $\alpha_1, \ldots, \alpha_k \in A$. Since test rank of the algebra $A$ is $m$ there exists an endomorphism $\Psi : a_i \to z_i$, $1 \leq i \leq m$ of $A$ such that it fixes all elements $\alpha_1, \ldots, \alpha_k$, but is not an automorphism. Now we define the endomorphism

$$\gamma : x_1 \to x_1, \ldots, x_n \to x_n, a_1 \to z_1, \ldots, a_m \to z_m$$

of $L$. Then

$$\gamma(f_i) = \gamma(y_i + \alpha_i) = y_i + \gamma(\alpha_i) = y_i + \Psi(\alpha_i) = y_i + \alpha_i, \quad 1 \le i \le k,$$

i.e., $\gamma$ fixes all elements $f_1, \ldots, f_k$. This means $\gamma$ is an automorphism. But on the other hand, the restriction of $\gamma$ to $A$ is not an automorphism. That is $\gamma$ is noninvertible. This contradiction completes the proof. □

## 4. Test rank of $F/\gamma_k(F)'$

Let $F$ be a free Lie algebra generated by the free generating set $\{x_1, x_2, \ldots, x_n\}$ and $R$ be an ideal of $F$. For any element $u$ of $F$ we denote by $\bar{u}$ and $\hat{u}$ the images of $u$ under the natural homomorphisms $F \to F/R$ and $F \to F/R'$ respectively, where $R'$ is the derived subalgebra $[R, R]$. On the universal enveloping algebra $U\left(F/R'\right)$, the left Fox derivatives $\frac{\partial}{\partial x_i}$ are defined so that their values are in $U(F/R)$. For every element $\hat{u}$ of $F/R'$ the equality

$$\sum_{i=1}^{n} \frac{\partial \hat{u}}{\partial x_i} x_i = \bar{u}$$

is satisfied in $U(F/R)$. Therefore, if

$$\bar{u} = \sum_{i=1}^{n} \overline{c_i x_i}$$

for some $\overline{c_i} \in U(F/R)$, then there exists an element $\hat{v} \in F/R'$ such that $\bar{u} = \bar{v}$ and $\overline{c_i} = \frac{\partial \hat{v}}{\partial x_i}$, $1 \le i \le n$.

The subalgebra $R/R'$ of $F/R'$ is endowed with a left $U(F/R)$-module structure. The module action on the elements of $R/R'$ is defined as

$$v_1 \ldots v_t \cdot \hat{u} = \left[v_1, \left[\ldots, \left[v_t, \hat{u}\right], \ldots\right]\right]$$

where $v_1, \ldots, v_t \in F/R$, $\hat{u} \in R/R'$.

PROPOSITION 7

*Let $R$ be a verbal ideal of $F$ such that $F/R$ is a free polynilpotent Lie algebra, $\hat{u}$ be an element of $F/R'$ and $\hat{r} \in R/R'$. If $[\hat{u}, \hat{r}] = R'$, then $\hat{u} \in R/R'$.*

*Proof.* Let $\hat{u} = u + R'$, $\hat{r} = r + R'$, where $u \in F \backslash R'$, $r \in R \backslash R'$. Since $[\hat{u}, \hat{r}] = [u, r] + R' = R'$, $[u, r] \in R'$. Thus $ur - ru \in I_R \Delta$ and $ur \in I_R \Delta$. Then by Lemma 4, $\frac{\partial (ur)}{\partial x_i} = u \frac{\partial r}{\partial x_i} \in I_R$ for each $i$, $1 \le i \le n$. Since $r \in R \backslash R'$ then in view of Lemmas 4, 5 for at least one $i$, $1 \le i \le n$ $\frac{\partial r}{\partial x_i} \notin I_R$. Hence $\frac{\partial \hat{r}}{\partial x_i} = \frac{\partial r}{\partial x_i} + I_R \ne I_R$. Computing Fox derivatives of $[\hat{u}, \hat{r}]$ in $U(F/R)$, we obtain $\hat{u} \frac{\partial \hat{r}}{\partial x_i} = I_R$. The absence of the zero divisors in $U(F/R)$ leads to $\hat{u} = I_R$. Hence $u \in F \cap I_R$. By Lemma 3 we obtain $u \in R$ and $\hat{u} = u + R' \in R/R'$. □

PROPOSITION 8

*Let $R$ be a verbal ideal of $F$ such that $F/R$ is a free polynilpotent Lie algebra. If an endomorphism $\varphi$ of $F/R'$ acts identically on $R/R'$ then $\varphi$ induces the identity on $F/R$.*

*Proof.* Let $\varphi$ be an endomorphism of $F/R'$ which is acting identically on $R/R'$, $\hat{r} \in R/R'$ and $\hat{u} \in F/R'$. Then

$$[\hat{u}, \hat{r}] = \varphi([\hat{u}, \hat{r}]) = [\varphi(\hat{u}), \hat{r}].$$

That is $[\varphi(\hat{u}) - \hat{u}, \hat{r}] = R'$. By proposition 7, $\varphi(\hat{u}) - \hat{u} \in R/R'$. Then $\varphi(\hat{u}) = \hat{u} + \hat{v}$ for some $\hat{v} \in R/R'$ . This completes the proof. □

The following theorem is an extension of Lemma 3 of [15].

*Lemma* 9. *Let $R$ be a verbal ideal of $F$ such that $F/R$ is a free polynilpotent Lie algebra. If an endomorphism $\varphi$ of $F/R'$ acts identically on $R/R'$ then $\varphi$ is an automorphism.*

*Proof.* Let $\varphi$ be an endomorphism of $F/R'$ acting identically on $R/R'$. By Proposition 8 we may assume that $\varphi$ is defined by the rule $\varphi\left(\widehat{x_i}\right) = \widehat{x_i} + \widehat{r_i}$ where $\widehat{r_i} \in R/R'$, $1 \le i \le n$. Put $\widehat{y_i} = \widehat{x_i} + \widehat{r_i}$. Since $\varphi\left(\widehat{r_i}\right) = \widehat{r_i}$, we have $\widehat{x_i} = \widehat{y_i} - \varphi\left(\widehat{r_i}\right)$. Therefore $\widehat{x_i}$ belongs to the subalgebra generated by $\widehat{y_1}, \ldots, \widehat{y_n}$. Consequently $\varphi$ is surjective.

Let $\hat{u} \in F/R'$ such that $\varphi(\hat{u}) = 0$. Since $\varphi$ induces the identity on $F/R$, $\varphi(\hat{u}) = \hat{u} \left(\operatorname{mod} R/R'\right)$. Thus $\hat{u} = 0 \left(\operatorname{mod} R/R'\right)$. This shows that $\hat{u} \in R/R'$. By hypothesis $\hat{u} = \varphi(\hat{u}) = 0$. Therefore $\varphi$ is injective. □

PROPOSITION 10

*Let $R$ be an ideal of $F$ and $\hat{u}(\widehat{x_1}, \ldots, \widehat{x_n}) \in F/R'$. Then*

$$\hat{u}\left(\widehat{x_1} + \widehat{r_1}, \ldots, \widehat{x_n} + \widehat{r_n}\right) = \hat{u}\left(\widehat{x_1}, \ldots, \widehat{x_n}\right) + \sum_{i=1}^{n} \frac{\partial \hat{u}\left(\widehat{x_1}, \ldots, \widehat{x_n}\right)}{\partial x_i} \cdot \widehat{r_i} \qquad (2)$$

*where $\widehat{r_i} \in R/R'$, $1 \le i \le n$.*

*Proof.* Without loss of generality we may assume that $\hat{u}\left(\widehat{x_1}, \ldots, \widehat{x_n}\right)$ is a monomial. Now (2) may be obtained by induction on the length of $\hat{u}\left(\widehat{x_1}, \ldots, \widehat{x_n}\right)$. □

*Remark* 11. The associative algebra $U(F/R)$ is an Ore domain and it can be embeddable in a skew field $Q(F/R)$ of fractions [9]. Thus the tensor product $V = R/R' \otimes_{U(F/R)} Q(F/R)$ is a vector space over $Q(F/R)$ and $\dim V = n - 1$. Consider the nonzero elements $\widehat{g_j} = \widehat{g_j}\left(\widehat{x_1}, \widehat{x_j}\right)$, $j = 2, \ldots, n$ in $R/R'$ such that $\widehat{g_j}$ is a monomial of $\widehat{x_1}$ and $\widehat{x_j}$. The vector space $V$ is generated as a module over $Q(F/R)$ by the set $\{g_2, \ldots, g_n\}$. For more details, see [9].

In computing test rank of a free polynilpotent Lie algebra the following theorem plays an important role. The analog of this result for groups was obtained in [5].

**Theorem 12.** *Let $R$ be a verbal ideal of $F$ such that $F/R$ is a free polynilpotent Lie algebra. If an endomorphism $\varphi$ of $F/R'$ acts identically on $R/R'$ then $\varphi$ is an inner automorphism of $F/R'$ induced by some element of $R/R'$.*

*Proof.* By Proposition 8 and Lemma 9 an endomorphism $\varphi$ of $F/R'$ which is acting identically on $R/R'$ is an automorphism of the form $\varphi(\widehat{x_i}) = \widehat{x_i} + \widehat{u_i}$ , where $\widehat{u_i} \in R/R'$, $1 \le i \le n$. We will prove that $\varphi$ is an inner automorphism. We choose a nonzero element $\widehat{r_j} \in R/R'$ depending on $\widehat{x_1}$ and $\widehat{x_j}$. Then by Proposition 10,

$$\begin{aligned} \widehat{r_j} &= \varphi\left(\widehat{r_j}\right) \\ &= \widehat{r_j}\left(\varphi\left(\widehat{x_1}\right), \varphi\left(\widehat{x_j}\right)\right) \\ &= \widehat{r_j}\left(\widehat{x_1}, \widehat{x_j}\right) + \frac{\partial \widehat{r_j}}{\partial x_1} \cdot \widehat{u_1} + \frac{\partial \widehat{r_j}}{\partial x_j} \cdot \widehat{u_j}. \end{aligned}$$

That is

$$\frac{\partial \widehat{r_j}}{\partial x_1} \cdot \widehat{u_1} + \frac{\partial \widehat{r_j}}{\partial x_j} \cdot \widehat{u_j} = 0.$$

Now we compute Fox derivatives of the left and right parts of this equality, in $U(F/R)$ we obtain:

$$\frac{\partial \widehat{r_j}}{\partial x_1} \cdot \frac{\partial \widehat{u_1}}{\partial x_i} + \frac{\partial \widehat{r_j}}{\partial x_j} \cdot \frac{\partial \widehat{u_j}}{\partial x_i} = 0, \quad i = 1, \ldots, n. \tag{3}$$

Since $\widehat{r_j} \in R/R'$ and $\overline{r_j} = \frac{\partial \widehat{r_j}}{\partial x_1} \cdot \overline{x_1} + \frac{\partial \widehat{r_j}}{\partial x_j} \cdot \overline{x_j}$, we have $\frac{\partial \widehat{r_j}}{\partial x_1} \cdot \overline{x_1} + \frac{\partial \widehat{r_j}}{\partial x_j} \cdot \overline{x_j} = 0$ in $U(F/R)$. By embedding the algebra $U(F/R)$ into a skew field $Q(F/R)$ of fractions we get $\frac{\partial \widehat{r_j}}{\partial x_j} = -\frac{\partial \widehat{r_j}}{\partial x_1} \cdot \overline{x_1} \cdot \overline{x_j}^{-1}$. We substitute this in (3):

$$\begin{aligned} &\frac{\partial \widehat{r_j}}{\partial x_1} \cdot \frac{\partial \widehat{u_1}}{\partial x_i} + \left(-\frac{\partial \widehat{r_j}}{\partial x_1} \cdot \overline{x_1} \cdot \overline{x_j}^{-1}\right) \cdot \frac{\partial \widehat{u_j}}{\partial x_i} = 0, \quad i = 1, \ldots, n, \\ &\frac{\partial \widehat{r_j}}{\partial x_1}\left(\frac{\partial \widehat{u_1}}{\partial x_i} - \overline{x_1} \cdot \overline{x_j}^{-1} \frac{\partial \widehat{u_j}}{\partial x_i}\right) = 0. \end{aligned}$$

This yields

$$\frac{\partial \widehat{u_1}}{\partial x_i} = \overline{x_1} \cdot \overline{x_j}^{-1} \frac{\partial \widehat{u_j}}{\partial x_i}. \tag{4}$$

Thus $\overline{x_1} \cdot \overline{x_j}^{-1} \frac{\partial \widehat{u_j}}{\partial x_i} \in U(F/R)$. Since $x_j$ can not be equal to unity in $Q(F/R)$ and $\overline{x_1} \cdot \overline{x_j}^{-1} \frac{\partial \widehat{u_j}}{\partial x_i} \in U(F/R)$, $\frac{\partial \widehat{u_j}}{\partial x_i} = \overline{x_j \alpha_i}$ for some $\overline{\alpha_i} \in U(F/R)$. Substituting this in (4) yields $\frac{\partial \widehat{u_1}}{\partial x_i} = \overline{x_1 \alpha_i}$, $i = 1, \ldots, n$. This shows that

$$0 = \overline{u_1} = \sum_{i=1}^{n} \frac{\partial \widehat{u_1}}{\partial x_i} \overline{x_i} = \sum_{i=1}^{n} \overline{x_1 \alpha_i x_i},$$

and consequently

$$\sum_{i=1}^{n} \overline{\alpha_i x_i} = 0,$$

since $\widehat{u_1} \in R/R'$. Hence we conclude that there exists an element $\hat{v} \in R/R'$ such that $\bar{v} = \overline{u_1}$ and $\overline{\alpha_i} = \frac{\partial \hat{v}}{\partial x_i}$, $i = 1, \ldots, n$. Since

$$\frac{\partial \widehat{u_1}}{\partial x_i} = \overline{x_1 \alpha_i} = \overline{x_1} \frac{\partial \hat{v}}{\partial x_i} = \frac{\partial \left[\widehat{x_1}, \hat{v}\right]}{\partial x_i},$$

$\widehat{u_1} = [\widehat{x_1}, \hat{v}]$. Likewise we obtain $\widehat{u_j} = \left[\widehat{x_j}, \hat{v}\right]$. Considering the elements $\widehat{r_j}$ for all $j = 2, \ldots, n$. We conclude that $\widehat{u_i} = [\widehat{x_i}, \hat{v}]$ for all $i = 1, \ldots, n$. Therefore,

$$\varphi\left(\widehat{x_i}\right) = \widehat{x_i} + \widehat{u_i} = \widehat{x_i} + [\hat{x_i}, \hat{v}], \quad i = 1, \ldots, n.$$

Hence $\varphi$ is an inner automorphism of $F/R'$ induced by an element $\hat{v}$ of $R/R'$. □

**Theorem 13.** *Let $F$ be a free Lie algebra generated by the set $\{x_1, x_2, \ldots, x_n\}$, $n \geq 3$, $R = \gamma_k(F)$ and*

$$\widehat{w_{jm}} = [[\ldots[[\widehat{x_1}, \underbrace{\widehat{x_j}], \widehat{x_j}], \ldots, ], \widehat{x_j}}_{(m-1)\text{-times}}], \quad j = 2, \ldots, n, m > k > 1.$$

*If an endomorphism $\varphi$ of $F/R'$ fixes the elements $\widehat{w_{jm}}$ for a fixed $m$ and for all $j = 2, \ldots, n$ then $\varphi$ is an inner automorphism induced by an element of $R/R'$.*

*Proof.* Consider the commutators $\widehat{w_{jm}} = [[\ldots\left[\left[\widehat{x_1}, \widehat{x_j}\right], \widehat{x_j}\right], \ldots, ], \widehat{x_j}]$ of weight $m$ for $j = 2, \ldots, n$, $m > k$. We verify that $\widehat{w_{2m}}, \ldots, \widehat{w_{nm}}$ form a test set. Let $\varphi$ be an endomorphism of $F/R'$ such that $\varphi\left(\widehat{x_i}\right) = \widehat{y_i}$. Assume that $\varphi(\widehat{w_{jm}}) = \widehat{w_{jm}}$ for a fixed $m$ and all $j$. We have to prove that $\varphi$ is an inner automorphism induced by an element of $R/R'$.

We prove the theorem by induction on $k$.

Let $k = 2$. By assumption,

$$[[\ldots[[\widehat{x_1}, \underbrace{\widehat{x_j}], \widehat{x_j}], \ldots, ], \widehat{x_j}}_{(m-1)\text{-times}}] = [[\ldots\left[\left[\widehat{y_1}, \widehat{y_j}\right], \widehat{y_j}\right], \ldots, ], \widehat{y_j}] \tag{5}$$

for $j = 2, \ldots, n$. Computing Fox derivatives of the left and right parts of this equality in $U(F/F^{'})$, we obtain

$$\left(-\overline{x_j}\right)^{m-1} = \left(-\overline{y_j}\right)^{m-2} . \frac{\partial\left[\widehat{y_1}, \widehat{y_j}\right]}{\partial x_1}. \tag{6}$$

This equality shows that $\widehat{y_j} \notin F'/F''$. Hence we may assume that the elements $\widehat{y_j}$ have the form $\widehat{y_j} = \sum_{1 \leq i \leq n} \alpha_i x_i + h + F''$, where $\alpha_i \in K$, $h \in F' \backslash F''$. Using (6) we conclude that the elements $\overline{y_j}$ have the form $\overline{y_j} = \left(\overline{x_j}\right)^p$, $1 \leq p \leq m-1$. Hence we have the equality

$$\overline{y_j} = \sum_{1 \leq i \leq n} \alpha_i \overline{x_i} = \left(\overline{x_j}\right)^p .$$

This leads $\alpha_j = 1$, $p = 1$ and $\alpha_i = 0$ for all $i \neq j$. Therefore we have $\widehat{y_j} = \widehat{x_j} (\text{mod}\, F'/F'')$ for all $j = 2, \ldots, n$.

We prove that $\widehat{y_1} = \widehat{x_1} (\text{mod}\, F'/F'')$.

Let $\widehat{y_1} = \sum_{1 \leq k \leq n} \beta_k \widehat{x_k} + \widehat{u_1}$ and $\widehat{y_j} = \widehat{x_j} + \widehat{u_j}$, where $\widehat{u_1}, \widehat{u_j} \in F'/F''$, $2 \leq j \leq n$. Substituting these in (5), we get

$$\begin{aligned}&[[\ldots[[\widehat{x_1}, \widehat{x_j}], \widehat{x_j}], \ldots, ], \widehat{x_j}] \\ &\quad = [[\ldots \left[\left[ \sum_{1 \leq k \leq n} \beta_k \widehat{x_k} + \widehat{u_1}, \widehat{x_j} + \widehat{u_j} \right], \widehat{x_j} + \widehat{u_j} \right], \ldots, ], \widehat{x_j} + \widehat{u_j}].\end{aligned}$$

Comparing the degrees of the left and right parts of the above equality we obtain

$$[[\ldots[[\widehat{x_1}, \widehat{x_j}], \widehat{x_j}], \ldots, ], \widehat{x_j}] = [[\ldots[[ \sum_{1 \leq k \leq n} \beta_k \widehat{x_k}, \widehat{x_j}], \widehat{x_j}], ..., ], \widehat{x_j}].$$

This implies $\beta_1 = 1$ and $\beta_j = 0$ for all $j \neq 1$ and consequently we see that $\widehat{y_1} = \widehat{x_1} (\text{mod} F'/F'')$. Hence the endomorphism $\varphi$ induces the identity modulo $F'/F''$ and keeps the elements $\widehat{w_{2m}}, \ldots, \widehat{w_{nm}}$ fixed. By Remark 11 these elements generate $F'/F''$ as $Q\left(F/F'\right)$-module. Therefore, for every element $\hat{r}$ of $F'/F''$ we can find $\overline{c_1}, \overline{c_2}, \ldots, \overline{c_n} \in U(F/F')$ such that $\overline{c_1}\hat{r} = \overline{c_2}\widehat{w_{2m}} + \cdots + \overline{c_n}\widehat{w_{nm}}$. Since $\varphi$ induces the identity over $F/F'$, it can be extended to an endomorphism of the $Q\left(F/F'\right)$-module $F'/F''$. This extension is the identity and consequently $\varphi$ induces the identical endomorphism over $F'/F''$. By Theorem 12, $\varphi$ is an inner automorphism induced by an element of $F'/F''$.

Let $k > 2$. Put $T = \gamma_{k-1}(F)$. Since $R \subseteq T$, the endomorphism $\varphi$ induces an endomorphism $\Psi : F/T' \to F/T'$ under the natural homomorphism $F/R' \to F/T'$. Since $\Psi$ keeps the elements $\widehat{w_{2m}}, \ldots, \widehat{w_{nm}}$ fixed, in view of the induction hypothesis $\Psi$ is an inner automorphism induced by an element $\tilde{u}$ of $T/T'$. Hence $\Psi$ has the form $\Psi(\widetilde{x_i}) = \widetilde{x_i} + [\widetilde{x_i}, \tilde{u}]$. Therefore

$$\varphi(x_i) \equiv x_i + [x_i, u] (\text{mod}\, T') \equiv x_i (\text{mod}\, R).$$

That is, $\varphi$ acts identically over $F/R$. On the other hand the elements $\widehat{w_{2m}}, \ldots, \widehat{w_{nm}}$ generates $R/R'$ as $Q(F/R)$-module. Consequently, $\varphi$ induces the identity over $R/R'$. By theorem 12, $\varphi$ is an inner automorphism induced by an element of $R/R'$.

COROLLARY 14

*Let $F$ be a free Lie algebra of rank $n$ and $R = \gamma_k(F)$, $k \geq 2$, $n \geq 3$. Then the test rank of the Lie algebra $F/R'$ is $n-1$.*

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