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On conformal minimal 2-spheres in complex Grassmann manifold $G(2,n)$

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Abstract. For a harmonic map f from a Riemann surface into a complex Grassmann manifold, Chern and Wolfson [4] constructed new harmonic maps ∂f and $\bar{\partial} f$ through the fundamental collineations ∂ and $\bar{\partial}$ respectively. In this paper, we study the linearly full conformal minimal immersions from S^2 into complex Grassmannians $G(2, n)$, according to the relationships between the images of ∂f and $\bar{\partial} f$. We obtain various pinching theorems and existence theorems about the Gaussian curvature, Kähler angle associated to the given minimal immersions, and characterize some immersions under special conditions. Some examples are given to show that the hypotheses in our theorems are reasonable.

Keywords. Gaussian curvature; Kähler angle; function of analytic type.

1. Introduction

The complex Grassmannian manifold $G(k, n)$ is the set of all k-dimensional complex linear subspaces of $\mathbb{C}^n$, and $G(1, n)$ is the complex projective space $\mathbb{C}P^{n-1}$. When $k \geq 2$, the geometrical structure of $G(k, n)$ is much more complex than the complex projective space. For a harmonic map f from a Riemann surface into complex Grassmannians, Chern and Wolfson [4] defined two fundamental transforms ∂ and $\bar{\partial}$, through which one can get two new harmonic maps ∂f and $\bar{\partial} f$. If the given immersion is isometric, then harmonicity condition is equivalent to minimality. A minimal immersion from a Riemann surface into complex Grassmannians is obtained by holomorphic immersion through ∂-transforms and is called *pseudo-holomorphic*. In this paper, we are interested in studying the geometrical properties of minimal two-spheres immersed in $G(2, n)$, according to the two fundamental transforms ∂ and $\bar{\partial}$.

It is known that a harmonic map from S^2 into $\mathbb{C}P^n$ is determined by a holomorphic map from S^2 into $\mathbb{C}P^n$, which was first proved by Din and Zakrzewski [7] and also by Eells and Wood in [8]. However, this beautiful result is not true when ambient manifold is the general Grassmannian, which enhances the difficulty in studying minimal surfaces in the general Grassmannians. The geometrical properties of conformal minimal two-spheres immersed in complex projective space were studied by Bando and Ohnita [1] and by Bolton *et al* [2]. They classified the minimal two-spheres immersed in $\mathbb{C}P^n$ and proved the rigidity theorems of conformal minimal two-spheres in $\mathbb{C}P^n$, but some of these properties are not inherited when the ambient space is $G(k,n)$, $k \geq 2$. The pseudo-holomorphic two-spheres in $G(k,n)$ were studied by Jiao and Peng [10], Jiao [11] and Zheng [16]. They got various pinching theorems about the Gaussian curvature and Kähler angle.

Let f be a minimal isometric immersion from S^2 into $G(k,n)$. If f is pseudo-holomorphic with constant curvature K, then $K = \frac{4}{N}$ for some positive integer N (c.f. [10, 16]). Li [12] has studied the minimal constant curved two-spheres into $G(2,4)$. For $k \geq 2$ and $n > 4$, there is no more information about value distributions of Gaussian curvature of the constant curved minimal two-spheres in $G(k,n)$. So, studying the value distributions of Gaussian curvature of the minimal (non-pseudoholomorphic) constant curved two-spheres in $G(k,n)$ is an interesting problem. In our paper, the existence theorems (Theorem 4.3, Theorem 5.3, etc.) give an estimation of the upper-bound of the constant curvature.

Our method is moving frames, which is inspired from Chern and Wolfson's early paper [4]. The conjugate transformations ∂^*, $\bar{\partial}^*$ (see [4]) of the fundamental collineations ∂ and $\bar{\partial}$ play an important role in choosing a suitable frame. We will treat $G(2,4)$ and $G(2,5)$ separately, because their understanding is basic, and also the minimal maps of S^2 to $G(2,4)$, $G(2,5)$ exhibit many special features not present in the general case, which is explained in §3 and §4 respectively. In §5, we investigate the holomorphic and general minimal two-spheres in $G(2,n)$, where the various pinching theorems with respect to curvature and Kähler angle are obtained.

Throughout this paper we will use the following ranges of indices:

$$1 \leq A, B \leq \cdots \leq n; \quad 1 \leq i, j \leq \cdots \leq k; \quad k+1 \leq \alpha, \beta, \gamma \leq \cdots \leq n.$$

And also, we use the summation convention, and the convention $\bar{a}_{i\bar{\alpha}} = a_{\bar{i}\alpha}$, etc. Some of the notations used here are as follows:

$[Z_i] :=$ the space spanned by the vectors Z_1, Z_2, similarly, for $[Z_\alpha]$;

$\partial f \perp \bar{\partial} f := \partial f(x)$ and $\bar{\partial} f(x)$ are perpendicular under the standard Hermitian inner product of $\mathbb{C}^n$ for all $x \in S^2$. Similar understandings for $\bar{\partial} f \subset \partial f$, $\bar{\partial} f \not\subseteq \partial f$, etc.

2. Preliminaries

In this section we recall some basic formulas of minimal surfaces into complex Grassmannians, and prove some propositions with respect to the Frenet frame associated to a holomorphic two-sphere immersed into complex projective space $\mathbb{C}P^n$.

The complex Grassmannian manifold $G(k,n)$ is the set of all k-dimensional complex linear subspaces of $\mathbb{C}^n$, or equivalently, $G(k,n) \cong \frac{U(n)}{U(k)\times U(n-k)}$, here $U(n)$ is the unitary group. Particularly, $G(1, n+1)$ is the complex projective space $\mathbb{C}P^n$.

Let $Z = (Z_1, Z_2, \ldots, Z_n)$ be the elements of $U(n)$, with

$$dZ_A = \omega_{A\bar{B}} Z_B, \tag{2.1}$$

here $\omega_{A\bar{B}}$ are the Maurer–Cartan forms of $U(n)$. They are skew-Hermitian, i.e.

$$\omega_{A\bar{B}} + \omega_{\bar{B}A} = 0. \tag{2.2}$$

Taking the exterior derivative of (2.1), we get the Maurer–Cartan equations of $U(n)$ as

$$\mathrm{d}\omega_{A\bar{B}} = \sum_C \omega_{A\bar{C}} \wedge \omega_{C\bar{B}}, \tag{2.3}$$

which play an important role in the following computations. The form

$$\mathrm{d}s_G^2 = \sum_{i,\alpha} \omega_{i\bar{\alpha}}\omega_{\bar{i}\alpha}, \tag{2.4}$$

defines a positive definite Hermitian metric on $G(k,n)$, which is Kählerian.

Let $f : S^2 \longrightarrow G(2,n)$ be a conformal immersion. Locally, the metric $\mathrm{d}s^2$ on S^2 induced by f can be written as

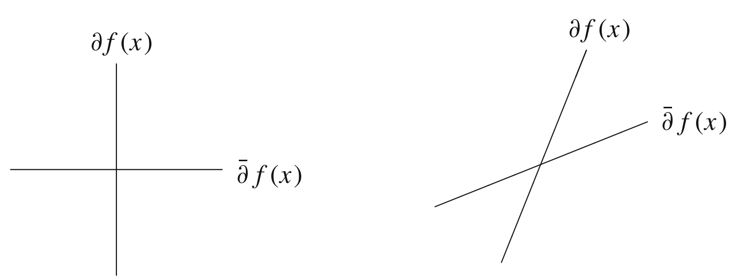
$$\mathrm{d}s^2 = f^*\mathrm{d}s_G^2 = \phi\bar{\phi}, \tag{2.5}$$

where ϕ is a local complex-valued one-form of type $(1,0)$ on S^2, which is defined up to a complex factor of absolute value 1.

To express the situation analytically we choose, locally, a field of unitary frame Z_A, such that Z_i span $f(x)$ and Z_α span $f^\perp(x)$ respectively. We set

$$f^*\omega_{A\bar{B}} = a_{A\bar{B}}\phi + b_{A\bar{B}}\bar{\phi}. \tag{2.6}$$

For convenience, we denote by $A := (a_{i\bar{\alpha}})_{2\times(n-2)}$, $B := (b_{i\bar{\alpha}})_{2\times(n-2)}$, $C := AB^*$, where B^* is the conjugate transpose of the matrix B. The geometric meanings of A and B are very clear, i.e., $\partial f(x) = [a_{1\bar{\alpha}}Z_\alpha, \ldots, a_{k\bar{\alpha}}Z_\alpha]$, $\bar{\partial} f(x) = [b_{1\bar{\alpha}}Z_\alpha, \ldots, b_{k\bar{\alpha}}Z_\alpha]$. By the well-known vanishing theorem 3.1 in [4] we know that $\mathrm{rank}(C) < k$. Clearly, the matrix C reflects the relationship between $\partial f(x)$ and $\bar{\partial} f(x)$ in $f^\perp(x)$ as shown below.



$\mathrm{rank}(C) = 0$ $\qquad$ $k > \mathrm{rank}(C) \geq 1$

It is known that f is holomorphic if and only if $b_{i\bar{\alpha}} = 0$ for all i and α. From (2.4), (2.5) and (2.6), one has

$$\sum_{i,\alpha} a_{i\bar{\alpha}} b_{\bar{i}\alpha} = 0, \tag{2.7}$$

$$\sum_{i,\alpha} a_{i\bar{\alpha}} a_{\bar{i}\alpha} + b_{i\bar{\alpha}} b_{\bar{i}\alpha} = 1. \tag{2.8}$$

If f is minimal, recalling the definition of Kähler angle $\theta \in [0,\pi]$ associated to a given immersion from a Riemann surface into Kähler manifold (c.f. [6]), and through direct computation we obtain

$$\sum_{i,\alpha} a_{i\bar{\alpha}} a_{\bar{i}\alpha} = \frac{1+\cos\theta}{2}, \quad \sum_{i,\alpha} b_{i\bar{\alpha}} b_{\bar{i}\alpha} = \frac{1-\cos\theta}{2}. \tag{2.9}$$

The structure equations of S^2 with respect to the induced metric are

$$\mathrm{d}\phi = -\rho \wedge \phi, \tag{2.10}$$

$$\mathrm{d}\rho = \frac{K}{2}\phi \wedge \bar{\phi}, \tag{2.11}$$

where the purely imaginary one-form ρ (i.e. $\bar{\rho} = -\rho$) is the connection form with respect to the co-frame ϕ, and K is the Gaussian curvature.

Taking the exterior derivatives of (2.6) – here we take $A = i$ and $B = \alpha$, together with (2.3) and (2.10), one gets

$$Da_{i\bar{\alpha}} \wedge \phi + Db_{i\bar{\alpha}} \wedge \bar{\phi} = 0, \tag{2.12}$$

where

$$Da_{i\bar{\alpha}} = \mathrm{d}a_{i\bar{\alpha}} - \omega_{i\bar{j}}a_{j\bar{\alpha}} + a_{i\bar{\beta}}\omega_{\beta\bar{\alpha}} - a_{i\bar{\alpha}}\rho, \tag{2.13}$$

$$Db_{i\bar{\alpha}} = \mathrm{d}b_{i\bar{\alpha}} - \omega_{i\bar{j}}b_{j\bar{\alpha}} + b_{i\bar{\beta}}\omega_{\beta\bar{\alpha}} + b_{i\bar{\alpha}}\rho. \tag{2.14}$$

Set

$$Da_{i\bar{\alpha}} = p_{i\bar{\alpha}}\phi + q_{i\bar{\alpha}}\bar{\phi}, \; Db_{i\bar{\alpha}} = q_{i\bar{\alpha}}\phi + r_{i\bar{\alpha}}\bar{\phi}. \tag{2.15}$$

Then the immersion $f : S^2 \longrightarrow G(k, n)$ is minimal if and only if $q_{i\bar{\alpha}} = 0$, equivalently

$$Da_{i\bar{\alpha}} \equiv 0 \bmod \phi, \quad \text{or} \quad Db_{i\bar{\alpha}} \equiv 0 \bmod \bar{\phi}. \tag{2.16}$$

The quadratic form

$$\Pi^{\mathbb{C}}_{i\alpha} = Da_{i\bar{\alpha}}\phi + Db_{i\bar{\alpha}}\bar{\phi} = p_{i\bar{\alpha}}\phi\phi + 2q_{i\bar{\alpha}}\phi\bar{\phi} + r_{i\bar{\alpha}}\bar{\phi}\,\bar{\phi}, \tag{2.17}$$

is called *complex second fundamental form* of the immersion f with respect to the co-frames $\omega_{i\bar{\alpha}}$.

Let $\varphi : S^2 \longrightarrow \mathbb{C}P^n$ be a *linearly full* holomorphic immersion. The phrase 'linearly full' means that the tautological bundle $\bigcup_{x\in S^2} f(x)$ is not contained in any trivial subbundle of $S^2 \times \mathbb{C}^{n+1}$. It is known that there exists a *Frenet frame* $\varphi_0, \varphi_1, \ldots, \varphi_n$ (see [15]) along φ such that φ_A defines a minimal map $\varphi_A : S^2 \longrightarrow \mathbb{C}P^n$, where $\varphi_0 = \varphi$, $\partial\varphi_A = \varphi_{A+1}$ if $A < n$ and $\partial\varphi_n = 0$.

There is a well-known example of Frenet frame, the so-called *Veronese sequence* $\varphi^n_0, \ldots, \varphi^n_n$, which is defined as follows:

$$\varphi^n_k : S^2 \longrightarrow \mathbb{C}P^n, \; (z_0, z_1) \longmapsto (\varphi^n_{k,0}, \ldots, \varphi^n_{k,n}),$$

in terms of homogeneous coordinate, and where

$$\varphi^n_{k,l} = \sqrt{\frac{l!(n-l)!}{k!(n-k)!}} \sum_{i+j=l} \binom{k}{i}\binom{n-k}{j} z_0^{n-k-j}\bar{z}_0^i z_1^j (-\bar{z}_1)^{k-i}.$$

For more details about the Veronese sequence one can refer to [1], [2].

The following two propositions and a lemma will be repeatedly used in the following proofs.

PROPOSITION 2.1

Let $\varphi_0, \varphi_1, \dots, \varphi_n$ be the Frenet frame along the linearly full holomorphic immersion $\varphi = \varphi_0 : S^2 \longrightarrow \mathbb{C}P^n$. If the immersion $\varphi_i \wedge \varphi_j : S^2 \longrightarrow G(2, n+1)$ has constant Gaussian curvature for any $0 \le i < j \le n$, then $\varphi_0, \varphi_1, \dots, \varphi_n$ is the Veronese sequence $\varphi_0^n, \varphi_1^n, \dots, \varphi_n^n$.

Proof. It is similar to the case $j = i + 1$, which is implied in Shen's paper [13]. □

PROPOSITION 2.2

Let $\varphi_1, \varphi_2 : S^2 \longrightarrow \mathbb{C}P^n$ be the non-constant holomorphic and anti-holomorphic immersion respectively, with φ_1 and φ_2 the orthogonals under the standard Hermitian inner product of $\mathbb{C}^{n+1}$. If $\varphi_1 \wedge \varphi_2 : S^2 \longrightarrow G(2, n)$ is minimal with constant curvature K and constant Kähler angle θ, then $\theta \in [0, \frac{\pi}{2})$ and there exist positive integers n_1 and n_2 such that $\varphi_1 = \varphi_0^{n_1}$, $\varphi_2 = \varphi_{n_2}^{n_2}$, up to $U(n+1)$-transformations.

Proof. We write φ_1 and φ_2 in the homogenous coordinate as follows:

$$\varphi_1 : S^2 \longrightarrow \mathbb{C}P^n,\ z = (z_0, z_1) \longmapsto \big(\varphi_0^1(z), \dots, \varphi_n^1(z)\big);$$

$$\varphi_2 : S^2 \longrightarrow \mathbb{C}P^n,\ z = (z_0, z_1) \longmapsto \big(\varphi_0^2(z), \dots, \varphi_n^2(z)\big).$$

It is well-known that $\varphi_i^1(z)$ (resp, $\varphi_i^2(z)$) are homogeneous polynomials of degree d_1 (resp. d_2) with respect to the variables z_0, z_1 (resp. $\bar{z}_0, \bar{z}_1$) since φ_1 (resp. φ_2) is holomorphic (resp. antiholomorphic).

Since the metric on S^2 induced by $\varphi_1 \wedge \varphi_2$ has constant curvature, there exists a positive number α such that the Kähler form of $G(2, n)$ restrict to S^2 satisfying

$$-\sqrt{-1}\partial\bar{\partial} \log |\varphi_1|^2 |\varphi_2|^2 = -\sqrt{-1}\alpha \cos\theta \partial\bar{\partial} \log |z|^2, \tag{2.18}$$

where θ is the Kähler angle and $|z|^2 = |z_0|^2 + |z_1|^2$. Therefore, $\log \frac{|\varphi_1|^2|\varphi_2|^2}{|z|^{2\alpha\cos\theta}}$ is a harmonic function on S^2, which is a constant by the maximum principle. Then we obtain $|\varphi_1|^2|\varphi_2|^2 = c|z|^{2\alpha\cos\theta}$, $c > 0$. It is clear that $\cos\theta > 0$, $|\varphi_1|^2 = c_1|z|^{2\alpha_1}$ and $|\varphi_2|^2 = c_2|z|^{2\alpha_2}$ for some positive constants c_1, c_2, α_1 and α_2, since $\varphi_i^1(z)$ (resp. $\varphi_i^2(z)$) are homogeneous polynomials with respect to the variables z_0, z_1 (resp. $\bar{z}_0, \bar{z}_1$) and the irreducibility of $|z|^2$. Then we know that φ_1 and φ_2 have constant curvature according to the fact that $|\varphi_1|^2 = c_1|z|^{2\alpha_1}$ and $|\varphi_2|^2 = c_2|z|^{2\alpha_2}$. So the result follows from the rigidity theorem of Calabi [3]. □

Remark. There exists a result which is similar to this proposition when φ_1 and φ_2 are holomorphic immersions.

Lemma 2.3. *Let U be an open subset of Riemannian surface M, and g be a complex-valued smooth function defined on U, and $\mathrm{d}s^2 = \phi\bar{\phi}$ on U. Suppose that g satisfies*

$$\mathrm{d}g \equiv g\psi, \qquad \mathrm{mod}\ \phi,$$

where ψ is a purely imaginary valued one-form ($i.e.$ $\bar{\psi} = -\psi$), *then*

$$\Delta_M \log|g|\, \phi \wedge \bar{\phi} = 2\mathrm{d}\psi$$

away from its zeros, and Δ_M is the Laplace–Beltrami operator with respect to $\mathrm{d}s^2$.

Proof. The proof can be found in [14]. □

3. Minimal immersions of S^2 into $G(2, 4)$

In this section, we study the minimal (not ±holomorphic) 2-spheres immersed in the complex Grassmann manifold $G(2,4)$. According to the known vanishing theorem 3.1 in [4], we have that one of $\mathrm{rank}(A)$, $\mathrm{rank}(B)$ is equal to 1 and $\mathrm{tr}(C) = 0$. Without loss of generality, assuming $\mathrm{rank}(A) = 1$, we can study the following cases.

(a) $\mathrm{rank}(\partial) = 1$ *and* $\mathrm{rank}(\bar{\partial}) = 2$. We choose a field of unitary frame Z_A such that $f(x) = [Z_i]$, $f^{\perp}(x) = [Z_\alpha]$, $\ker(\partial) = [Z_1]$ and $\partial[Z_2] = [Z_3]$. This field is defined up to $U(1) \times U(1) \times U(1) \times U(1)$, under which we have

$$a_{1\bar{\alpha}} = 0,\ a_{2\bar{4}} = 0,\ a_{2\bar{3}} \neq 0. \tag{3.1}$$

So, one has $b_{2\bar{3}} = 0$ by the fact that $\mathrm{tr}(C) = 0$ and $a_{2\bar{3}} \neq 0$. Since $\mathrm{rank}(B) = 2$, we have $b_{1\bar{3}} \neq 0$ and $b_{2\bar{4}} \neq 0$. It is easily seen that $|b_{1\bar{3}}|$ and $|b_{2\bar{4}}|$ are globally defined functions on S^2. The minimality of f gives $\omega_{1\bar{2}} = a_{1\bar{2}}\phi$ and $\omega_{3\bar{4}} = a_{3\bar{4}}\phi$ by the equations (2.16).

Through direct computation one gets the following relations:

$$\begin{aligned} \mathrm{d}b_{\bar{1}3} &\equiv b_{\bar{1}3}(\omega_{3\bar{3}} - \omega_{1\bar{1}} + \rho), && (3.2)\\ \mathrm{d}b_{\bar{2}4} &\equiv b_{\bar{2}4}(\omega_{4\bar{4}} - \omega_{2\bar{2}} + \rho), \quad \mod \phi, && (3.3)\end{aligned}$$

by (2.16), which give

$$\begin{aligned} \Delta_M \log|b_{1\bar{3}}| &= K + 2\big(|a_{2\bar{3}}|^2 - 2|b_{1\bar{3}}|^2 - |b_{1\bar{4}}|^2 + |a_{1\bar{2}}|^2 - |a_{3\bar{4}}|^2\big), && (3.4)\\ \Delta_M \log|b_{2\bar{4}}| &= K + 2\big(|a_{2\bar{3}}|^2 - 2|b_{2\bar{4}}|^2 - |b_{1\bar{4}}|^2 - |a_{1\bar{2}}|^2 + |a_{3\bar{4}}|^2\big), && (3.5)\end{aligned}$$

by Lemma 2.3. Here K is the Gaussian curvature. The summation of (3.4) and (3.5) is

$$\Delta_M \log|b_{1\bar{3}} b_{2\bar{4}}| = 2(K + 2\cos\theta). \tag{3.6}$$

Here we use the identities (2.9). Applying the E. Hopf's maximum principle to (3.6), we have proved the following theorem.

Theorem 3.1. *Let f be a linearly full minimal immersion from S^2 into $G(2,4)$ with* $\mathrm{rank}(\partial) = 1$ *and* $\mathrm{rank}(\bar{\partial}) = 2$, *$K$ and θ its curvature and Kähler angle respectively. Then $K = -2\cos\theta$ if $K \geq -2\cos\theta$ or $K \leq -2\cos\theta$.*

Example. The map

$$(z_0, z_1) \longmapsto \begin{pmatrix} -\sqrt{3}z_0^2\bar{z}_1 & z_0\big(|z_0|^2 - 2|z_1|^2\big) & z_1\big(2|z_0|^2 - |z_1|^2\big) & \sqrt{3}\bar{z}_0 z_1^2 \\ -\bar{z}_1^3 & \sqrt{3}\bar{z}_0\bar{z}_1^2 & -\sqrt{3}\bar{z}_0^2\bar{z}_1 & \bar{z}_0^3 \end{pmatrix},$$

is a minimal immersion from S^2 into $G(2,4)$ which satisfies the rank hypothesis in the theorem, with $\cos\theta = -\frac{1}{5}$ and $K = \frac{2}{5}$. Indeed, if we set $z = \frac{z_1}{z_0}$, in terms of local coordinate, the map is given by

$$z \longmapsto \begin{pmatrix} -\sqrt{3}\bar{z} & 1-2z\bar{z} & z(2-z\bar{z}) & \sqrt{3}z^2 \\ -\bar{z}^3 & \sqrt{3}\bar{z}^2 & -\sqrt{3}\bar{z} & 1 \end{pmatrix}.$$

Set

$$\begin{aligned} Z_1 &= u(z)\left(-\sqrt{3}\bar{z},\ 1-2z\bar{z},\ z(2-z\bar{z}),\ \sqrt{3}z^2 \right), \\ Z_2 &= u(z)\left(-\bar{z}^3,\ \sqrt{3}\bar{z}^2,\ -\sqrt{3}\bar{z},\ 1 \right), \\ Z_3 &= u(z)\left(\sqrt{3}\bar{z}^2,\ \bar{z}(z\bar{z}-2),\ 1-2z\bar{z},\ \sqrt{3}z \right), \\ Z_4 &= u(z)\left(1,\ \sqrt{3}z,\ \sqrt{3}z^2,\ z^3 \right), \end{aligned}$$

where $u(z) = \frac{1}{\sqrt{(1+|z|^2)(1+2|z|^2+|z|^4)}}$. It is clear that $f(x) = [Z_1, Z_2]$ and $f^{\perp}(x) = [Z_3, Z_4]$. Through direct calculations, we obtain

$$\begin{aligned} \omega_{13} &= (\mathrm{d}Z_1, Z_3) = \frac{2}{1+|z|^2}\mathrm{d}z, \quad \omega_{14} = (\mathrm{d}Z_1, Z_4) = -\frac{\sqrt{3}}{1+|z|^2}\mathrm{d}\bar{z}, \\ \omega_{23} &= (\mathrm{d}Z_2, Z_3) = -\frac{\sqrt{3}}{1+|z|^2}\mathrm{d}\bar{z},\ \omega_{24} = (\mathrm{d}Z_2, Z_4) = 0. \end{aligned}$$

By (2.5), the induced metric is $\mathrm{d}s^2 = \frac{10}{(1+|z|^2)^2}|\mathrm{d}z|^2$, which implies the Gaussian curvature $K = \frac{2}{5}$. Note that $\phi = \frac{\sqrt{10}}{1+|z|^2}\mathrm{d}z$, so we have

$$A = \begin{pmatrix} \frac{\sqrt{10}}{5} & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\frac{\sqrt{30}}{10} \\ -\frac{\sqrt{30}}{10} & 0 \end{pmatrix},$$

from which we obtain $\cos\theta = -\frac{1}{5}$ by (2.9).

(b) $\operatorname{rank}(\partial) = \operatorname{rank}(\bar{\partial}) = 1$. One can show that f is not linearly full when $\partial f = \bar{\partial} f$, so we just need to consider following cases.

(b.I) $\partial f \perp \bar{\partial} f$. We choose a field of unitary frame Z_A such that $f(x) = [Z_i]$, $f^{\perp}(x) = [Z_\alpha]$, $\ker(\partial) = [Z_1]$, $\partial f = [Z_3]$ and $\bar{\partial} f = [Z_4]$, so this field is defined up to $U(1) \times U(1) \times U(1) \times U(1)$. Under such a frame, we obtain

$$a_{i\bar{4}} = 0, \quad a_{1\bar{3}} = 0, \quad b_{i\bar{3}} = 0, \quad a_{2\bar{3}} \neq 0. \tag{3.7}$$

The minimality equation (2.16) implies $\omega_{1\bar{2}} = a_{1\bar{2}}\phi$ and $\omega_{3\bar{4}} = a_{3\bar{4}}\phi$.

Lemma 3.2. *$P = \omega_{2\bar{3}}\omega_{3\bar{4}}\omega_{4\bar{2}}$ is a holomorphic symmetric $(3,0)$-form on S^2, and $P = 0$.*

Proof. Since Z_A is defined up to a transformation of group $U(1) \times U(1) \times U(1) \times U(1)$, P is globally defined. To show P is holomorphic, we choose a complex coordinate ζ on S^2, and write

$$\omega_{2\bar{3}} = x\mathrm{d}\zeta, \quad \omega_{3\bar{4}} = y\mathrm{d}\zeta, \quad \omega_{4\bar{2}} = z\mathrm{d}\zeta, \tag{3.8}$$

so that $P = xyz\mathrm{d}\zeta^3$. By differentiating (3.8), we obtain

$$\begin{aligned} \mathrm{d}x &\equiv x(\omega_{2\bar{2}} - \omega_{3\bar{3}}), \\ \mathrm{d}y &\equiv y(\omega_{3\bar{3}} - \omega_{4\bar{4}}), \\ \mathrm{d}z &\equiv z(\omega_{4\bar{4}} - \omega_{2\bar{2}}), \quad \mod \mathrm{d}\zeta, \end{aligned}$$

from which we conclude $\mathrm{d}(xyz) \equiv 0$, mod $\mathrm{d}\zeta$, i.e. xyz is a holomorphic function. It is known that there are non-zero holomorphic forms on S^2, so the statement holds. □

(b.I.I) $\partial f \perp \bar{\partial} f$ *and* $\ker(\partial) = \ker(\bar{\partial})$. Since $\ker(\partial) = \ker(\bar{\partial})$, we have $b_{1\bar{4}} = 0$, $b_{2\bar{4}} \neq 0$. Thus

$$\omega_{1\bar{2}} = b_{1\bar{2}}\bar{\phi}, \tag{3.9}$$

by (2.16), however, we have known that $\omega_{1\bar{2}} = a_{1\bar{2}}\phi$, so $\omega_{1\bar{2}} = 0$. Therefore, Z_1 is a constant vector in $\mathbb{C}^4$, up to a rigid motion, for $\omega_{1\bar{2}} = \omega_{1\bar{\alpha}} = 0$. In other words, f is not linearly full and we discard this situation.

(b.I.II) $\partial f \perp \bar{\partial} f$ *and* $\ker(\partial) \perp \ker(\bar{\partial})$. Since $\ker(\partial) \perp \ker(\bar{\partial})$, we have $b_{2\bar{4}} = 0$, $b_{1\bar{4}} \neq 0$. Equation (2.16) gives

$$\begin{aligned} \mathrm{d}a_{2\bar{3}} &\equiv a_{2\bar{3}}(\omega_{2\bar{2}} - \omega_{3\bar{3}} + \rho), \\ \mathrm{d}b_{1\bar{4}} &\equiv b_{\bar{1}4}(\omega_{4\bar{4}} - \omega_{1\bar{1}} + \rho), \quad \mod \phi, \end{aligned}$$

which imply

$$\Delta_M \log |a_{2\bar{3}}| = K + 2\big(|a_{1\bar{2}}|^2 + |a_{3\bar{4}}|^2 - 2|a_{2\bar{3}}|^2\big), \tag{3.10}$$
$$\Delta_M \log |b_{1\bar{4}}| = K + 2\big(|a_{1\bar{2}}|^2 + |a_{3\bar{4}}|^2 - 2|b_{1\bar{4}}|^2\big), \tag{3.11}$$

by Lemma 2.3. Subtracting equation (3.11) from (3.10), we get

$$\Delta_M \log |b_{1\bar{4}}||a_{2\bar{3}}|^{-1} = 4\cos\theta, \tag{3.12}$$

by (2.9).

Lemma 3.3. $P = \omega_{1\bar{2}}\omega_{2\bar{3}}\omega_{3\bar{4}}\omega_{4\bar{1}}$ *is zero on* S^2.

Proof. The proof is similar to Lemma 3.2. □

It is easily seen that $a_{1\bar{2}}, a_{3\bar{4}}$ are *functions of analytic type* [5, 15], which are either identically zero or with isolated zeros. Therefore, at least, one of $a_{1\bar{2}}, a_{3\bar{4}}$ is zero by Lemma 3.3.

Firstly, we assume that one of $a_{1\bar{2}}, a_{3\bar{4}}$ is zero but the other is nonzero. Without loss of generality, $a_{1\bar{2}} = 0$, $a_{3\bar{4}} \neq 0$. Computating as before, one can obtain

$$\Delta_M \log |a_{3\bar{4}}| = K + 2\big(|a_{2\bar{3}}|^2 - 2|a_{3\bar{4}}|^2 + |b_{1\bar{4}}|^2\big). \tag{3.13}$$

The summation of (3.10), (3.11) and (3.13), together with $a_{1\bar{2}} = 0$ gives

$$\Delta_M \log |a_{2\bar{3}}a_{3\bar{4}}b_{1\bar{4}}| = 3\left(K - \frac{2}{3}\right), \tag{3.14}$$

away from some isolated zeros of $|a_{3\bar{4}}|$.

Secondly, assuming that both $a_{1\bar{2}}$ and $a_{3\bar{4}}$ are zeros, the identities (3.10) and (3.11) become

$$\Delta_M \log |a_{2\bar{3}}| = K - 2(1 + \cos\theta), \tag{3.15}$$
$$\Delta_M \log |b_{1\bar{4}}| = K - 2(1 - \cos\theta). \tag{3.16}$$

Theorem 3.4. *Let f be a linearly full minimal immersion from S^2 into $G(2,4)$ with* $\mathrm{rank}(\partial) = \mathrm{rank}(\bar\partial) = 1$, $\partial f \perp \bar\partial f$ *and* $\ker(\partial) \perp \ker(\bar\partial)$, *$K$ and θ be its curvature and Kähler angle respectively. Then $f = \varphi_1 \wedge \varphi_2$ or $(\varphi_1 \wedge \varphi_2)^\perp$, here φ_1, φ_2 are antiholomorphic and holomorphic immersions from S^2 into $\mathbb{C}P^3$ respectively; f is totally real if the Kähler angle $\theta \in [0, \frac{\pi}{2}]$ or $[\frac{\pi}{2}, \pi]$ everywhere on S^2. If f has constant curvature and constant Kähler angle, then there exists positive integers $n_1, n_2 \le 3$ such that $f_1 = \varphi_{n_1}^{n_1}$ and $f_2 = \varphi_0^{n_2}$, up to a rigid motion. And also, f satisfies one of the following:*
(1) *The curvature $K = \frac{2}{3}$ if $K \ge \frac{2}{3}$ everywhere on S^2;*
(2) *$K = 2(1+\cos\theta)$ (resp. $2(1-\cos\theta)$) if $K \ge 2(1+\cos\theta)$ or $K \le 2(1+\cos\theta)$ (resp. $K \ge 2(1-\cos\theta)$ or $K \le 2(1-\cos\theta)$) everywhere on S^2.*

Proof. The first part of this theorem follows from the fact that one of $|a_{1\bar{2}}|$, $|a_{3\bar{4}}|$ is zero by Lemma 3.3 and identity (3.12). The statement (1), (2) are implied in (3.14), (3.15) and (3.16) respectively. □

Examples. The maps

$$(z_0, z_1) \longmapsto \begin{pmatrix} z_0^3 & \sqrt{3}z_0^2 z_1 & \sqrt{3}z_0 z_1^2 & z_1^3 \\ -\bar z_1^3 & \sqrt{3}\bar z_0 \bar z_1^2 & -\sqrt{3}\bar z_0^2 \bar z_1 & \bar z_0^3 \end{pmatrix},$$

$$(z_0, z_1) \longmapsto \begin{pmatrix} z_0 & 0 & z_1 & 0 \\ 0 & \bar z_0 & 0 & \bar z_1 \end{pmatrix},$$

satisfy the conclusions (1), (2) in Theorem 3.4 respectively, which are both totally real (i.e. $\cos\theta = 0$). Their curvatures are $\frac{2}{3}$ and 2 respectively.

(b.I.III) $\partial f \perp \bar\partial f$, $\ker(\partial) \ne \ker(\bar\partial)$, $\ker(\partial)$ *and* $\ker(\bar\partial)$ *are not perpendicular*. Since $\ker(\partial)$ and $\ker(\bar\partial)$ are not perpendicular, $b_{1\bar{4}} \ne 0$ and $b_{2\bar{4}} \ne 0$. Similarly, one can get

$$\Delta_M \log |a_{2\bar{3}} b_{2\bar{4}}| = 2(K-1) + 4|a_{3\bar{4}}|^2. \tag{3.17}$$

Here $|a_{2\bar{3}} b_{2\bar{4}}|$ and $|a_{3\bar{4}}|$ are globally defined functions on S^2.

(b.II) ∂f *and* $\bar\partial f$ *are not perpendicular in* $f^\perp(x)$. It is clear that $\mathrm{rank}(C) = 1$ when ∂f and $\bar\partial f$ are not perpendicular in $f^\perp(x)$. Note that $\bar\partial^*|_{\partial f(x)} : \partial f(x) \longrightarrow f(x)$, so one can choose Z_1 in $f(x)$ such that $c_{i\bar{2}} = 0$, then $c_{2\bar{1}} \ne 0$ and $c_{1\bar{1}} = 0$ for $\mathrm{tr}(C) = 0$. Taking $[Z_3] = \partial f(x)$, $[Z_4]$ is the orthogonal complement of $[Z_3]$ in $\partial f(x) \wedge \bar\partial f(x)$, and we have

$$a_{i\bar{4}} = 0,\ a_{2\bar{3}} \ne 0.$$

Then $c_{2\bar{2}} = 0$, $c_{2\bar{1}} \ne 0$ and $\mathrm{rank}(\partial^*|_{\bar\partial f(x)}) = 1$ imply that $b_{2\bar\alpha} = 0$, $b_{1\bar{3}} \ne 0$ and also $a_{1\bar{3}} = 0$ by $c_{1\bar{1}} = 0$, $b_{1\bar{3}} \ne 0$. Under such a field of unitary frame, one can get

$$\Delta_M \log |a_{2\bar{3}} b_{1\bar{3}}| = 2(K-1) + 4|a_{1\bar{2}}|^2. \tag{3.18}$$

Theorem 3.5. *Let f be a linearly full minimal immersion from S^2 into $G(2,4)$ with* $\mathrm{rank}(\partial) = \mathrm{rank}(\bar\partial) = 1$, *$\partial f$ and $\bar\partial f$ are not perpendicular in $f^{\perp}(x)$ (or $\partial f \perp \bar\partial f$,* $\ker(\partial)$ *and* $\ker(\bar\partial)$ *are not perpendicular). If the Gaussian curvature $K \geq 1$ everywhere on S^2, then $K = 1$.*

Proof. Applying the maximum principle of subharmonic functions to (3.17), (3.18). □

4. Minimal immersions of S^2 into $G(2, 5)$

In this section, we study the minimal (not ±holomorphic) 2-spheres immersed in the complex Grassmann manifold $G(2,5)$ by the method of moving frames.

(a) $\mathrm{rank}(\partial) = \mathrm{rank}(\bar\partial) = 2$. In this case, we choose the unitary frame Z_A as Chern and Wolfson did in p. 316 of [4]. Thus we have

$$\begin{aligned} a_{i\bar5} &= 0,\ a_{1\bar4} = 0,\ a_{1\bar3} \neq 0,\ a_{2\bar4} \neq 0, \\ b_{i\bar3} &= 0,\ b_{2\bar4} = 0,\ b_{1\bar4} \neq 0,\ b_{2\bar5} \neq 0, \\ \omega_{1\bar2} &= a_{1\bar2}\phi,\ \omega_{3\bar4} = a_{3\bar4}\phi,\ \omega_{4\bar5} = a_{4\bar5}\phi,\ \omega_{3\bar5} = 0. \end{aligned}$$

The minimality of f, i.e. eqs (2.16), gives

$$\begin{aligned} \mathrm{d}a_{1\bar3} &\equiv a_{1\bar3}(\omega_{1\bar1} - \omega_{3\bar3} + \rho), \\ \mathrm{d}a_{2\bar4} &\equiv a_{2\bar4}(\omega_{2\bar2} - \omega_{4\bar4} + \rho), \\ \mathrm{d}b_{\bar14} &\equiv b_{\bar14}(\omega_{4\bar4} - \omega_{1\bar1} + \rho), \\ \mathrm{d}b_{\bar25} &\equiv b_{\bar25}(\omega_{5\bar5} - \omega_{2\bar2} + \rho), \quad \mod \phi, \end{aligned}$$

and therefore

$$\Delta_M \log |b_{1\bar4}||b_{2\bar5}|^2|a_{1\bar3}|^{-2}|a_{2\bar4}|^{-1} = 10\cos\theta, \tag{4.1}$$

by Lemma 2.2, in which, $|b_{1\bar4}|$, $|b_{2\bar5}|$, $|a_{1\bar3}|$ and $|a_{2\bar4}|$ are globally defined on S^2, since the frames we choose is defined up to $U(1) \times \cdots \times U(1)$.

Theorem 4.1. *Let f be a linearly full minimal immersion from S^2 into $G(2,5)$ with* $\mathrm{rank}(\partial) = \mathrm{rank}(\bar\partial) = 2$. *Then f is totally real if its Kähler angle $\theta \in [0, \frac{\pi}{2}]$ or $[\frac{\pi}{2}, \pi]$ everywhere on S^2.*

Proof. Applying the E. Hopf's maximum principle to identity (4.1). □

Example. The map $\varphi_1^4 \wedge \varphi_3^4$ (see the definition of Veronese sequence) is totally real with $\mathrm{rank}(\partial) = \mathrm{rank}(\bar\partial) = 2$. To calculate the curvature and the Kähler angle of $\varphi_i^n \wedge \varphi_j^n$ for $i \neq j$ is not simple, so one can refer to [9].

(b) $\mathrm{rank}(\partial) = 2$ and $\mathrm{rank}(\bar\partial) = 1$.

(b.I) $\bar\partial f \subset \partial f$. In this case, we choose frames Z_A so that $f(x) = [Z_i]$, $f^{\perp}(x) = [Z_\alpha]$, $\partial f = [Z_3, Z_4]$, $\bar\partial f = [Z_4]$ and $\ker(\bar\partial) = [Z_2]$. Thus, we have

$$a_{i\bar 5} = 0,\ a_{1\bar 3} \neq 0,\ a_{2\bar 4} \neq 0, \tag{4.2}$$

$$b_{2\bar\alpha} = 0,\ b_{1\bar 3} = 0,\ b_{1\bar 5} = 0,\ b_{1\bar 4} \neq 0, \tag{4.3}$$

which imply $a_{1\bar 4} = 0$ for $c_{2\bar 2} = 0$ and $\mathrm{tr}(C) = 0$. Using (4.2), (4.3) and (2.16), we obtain

$$\omega_{1\bar 2} = a_{1\bar 2}\phi,\ \omega_{3\bar 4} = a_{3\bar 4}\phi,\ \omega_{3\bar 5} = a_{3\bar 5}\phi,\ \omega_{4\bar 5} = 0. \tag{4.4}$$

and

$$\mathrm{d}a_{1\bar 3} \equiv a_{1\bar 3}(\omega_{1\bar 1} - \omega_{3\bar 3} + \rho), \tag{4.5}$$

$$\mathrm{d}a_{2\bar 4} \equiv a_{2\bar 4}(\omega_{2\bar 2} - \omega_{4\bar 4} + \rho), \tag{4.6}$$

$$\mathrm{d}b_{\bar 1 4} \equiv b_{\bar 1 4}(\omega_{4\bar 4} - \omega_{1\bar 1} + \rho), \quad \mod \phi, \tag{4.7}$$

Taking the exterior derivative of $\omega_{3\bar 5} = a_{3\bar 5}\phi$, we have

$$\mathrm{d}a_{3\bar 5} \equiv a_{3\bar 5}(\omega_{3\bar 3} - \omega_{5\bar 5} + \rho), \quad \mod \phi. \tag{4.8}$$

Thus $a_{3\bar 5}$ is a function of analytic type (c.f. [5, 15]) by (4.8). Since f is a linearly full immersion, we conclude that $a_{3\bar 5} \neq 0$ except for some isolated zeros, by reading the pull back of the Maurer–Cartan forms (see the definition below (4.13)) and since $a_{3\bar 5}$ is of analytic type.

According to Lemma 2.3, equations (4.5), (4.6), (4.7) and (4.8) give

$$\Delta_M \log |a_{1\bar 3}|^2 |a_{2\bar 4}|^{\frac{3}{2}} |a_{3\bar 5}| |b_{1\bar 4}|^{\frac{1}{2}} = 5(K - \cos\theta), \tag{4.9}$$

away from the zeros of $|a_{3\bar 5}|$. Applying the maximum principle of subharmonic function to (4.9), we have proved the following.

Theorem 4.2. *Let f be a linearly full minimal immersion from S^2 into $G(2,5)$ with* $\mathrm{rank}(\partial) = 2$, $\mathrm{rank}(\bar\partial) = 1$ *and $\bar\partial f \subset \partial f$, K and θ be its curvature and Kähler angle respectively. Then $K = \cos\theta$ if $K \geq \cos\theta$ everywhere on S^2.*

Example. The minimal immersion $\varphi_0^4 \wedge \varphi_2^4$ from S^2 into $G(2,5)$ has $K = \cos\theta = \frac{1}{4}$.

(b.II) $\partial f \perp \bar\partial f$. Choosing the unitary frames Z_A so that $f(x) = [Z_i]$, $f^{\perp}(x) = [Z_\alpha]$, $\partial f = [Z_3, Z_4]$, $\ker(\bar\partial) = [Z_1]$ and $\partial[Z_1] = [Z_3]$, these frames are defined up to $U(1) \times \cdots \times U(1)$. Thus we have

$$a_{i\bar 5} = 0,\ a_{1\bar 4} = 0,\ a_{1\bar 3} \neq 0,\ a_{2\bar 4} \neq 0, \tag{4.10}$$

$$b_{i\bar 3} = 0,\ b_{i\bar 4} = 0,\ b_{1\bar 5} = 0,\ b_{2\bar 5} \neq 0, \tag{4.11}$$

Using eq. (2.16), together with (4.10) and (4.11), we obtain

$$\omega_{1\bar 2} = b_{1\bar 2}\bar\phi,\ \omega_{3\bar 4} = a_{3\bar 4}\phi,\ \omega_{3\bar 5} = a_{3\bar 5}\phi,\ \omega_{4\bar 5} = a_{4\bar 5}\phi. \tag{4.12}$$

From (4.10), (4.11) and (4.12), we have the equations

$$\mathrm{d}\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_4 \end{pmatrix} = \begin{pmatrix} \omega_{1\bar{1}} & b_{1\bar{2}}\bar{\phi} & a_{1\bar{3}}\phi & 0 & 0 \\ -b_{\bar{1}2}\phi & \omega_{2\bar{2}} & a_{2\bar{3}}\phi & a_{2\bar{4}}\phi & b_{2\bar{5}}\bar{\phi} \\ -a_{\bar{1}3}\bar{\phi} & -a_{\bar{2}3}\bar{\phi} & \omega_{3\bar{3}} & a_{3\bar{4}}\phi & a_{3\bar{5}}\phi \\ 0 & -a_{\bar{2}4}\bar{\phi} & -a_{\bar{3}4}\bar{\phi} & \omega_{4\bar{4}} & a_{4\bar{5}}\phi \\ 0 & -b_{\bar{2}5}\phi & -a_{\bar{3}5}\bar{\phi} & -a_{\bar{4}5}\bar{\phi} & \omega_{5\bar{5}} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \end{pmatrix}, \tag{4.13}$$

in which, the matric of forms is called *the pull-back of Maurer–Cartan forms.*

Therefore, through direct computation as before, we obtain

$$\Delta_M \log |a_{1\bar{3}}a_{2\bar{4}}b_{2\bar{5}}| = 3\left(K - \frac{3+\cos\theta}{3}\right) + \delta^2, \tag{4.14}$$

by eq. (2.16) and Lemma 2.3, where $\delta^2 = |b_{1\bar{2}}|^2 + |a_{2\bar{3}}|^2 + |a_{2\bar{4}}|^2 + 2\big(|a_{3\bar{5}}|^2 + |a_{4\bar{5}}|^2\big)$.

Theorem 4.3. *Let f be a linearly full minimal immersion from S^2 into $G(2,5)$ with* $\mathrm{rank}(\partial) = 2$, $\mathrm{rank}(\bar\partial) = 1$ *and $\partial f \perp \bar\partial f$, K and θ be its curvature and Kähler angle respectively. Then there exists a point $x \in S^2$ such that $K(x) < \frac{3+\cos\theta(x)}{3}$.*

Proof. If not, we assume that $K \geq \frac{3+\cos\theta}{3}$ on S^2. Then the identity (4.14) becomes $3(K - \frac{3+\cos\theta}{3}) + \delta^2 = 0$ by the maximum principle of subharmonic functions. Hence, $a_{2\bar{3}} = a_{2\bar{4}} = 0$, which implies that $\mathrm{rank}(\partial) = 1$ by reading the pull-back of Maurer–Cartan forms in (4.13). It is a contradiction. So, the statement is true. □

Example. The map $\varphi_0^4 \wedge \varphi_3^4$ has $K = \frac{2}{7}$ and $\cos\theta = \frac{1}{7}$, which satisfies the inequality in Theorem 4.3.

(b.III) $\bar\partial f \nsubseteq \partial f$ *and* $\bar\partial f$ *is not orthogonal to* ∂f *in* $f^{\perp}(x)$. Choosing the unitary frames Z_A such that $f(x) = [Z_i]$, $f^{\perp}(x) = [Z_\alpha]$. Since $\bar\partial f$ and ∂f are not perpendicular we know that $\mathrm{rank}(C) = 1$. Note that $\partial^*|_{\bar\partial f(x)} : \bar\partial f(x) \longrightarrow f(x)$, we can choose Z_1 in $f(x)$ such that $c_{2\bar{i}} = 0$, so $c_{1\bar{1}} = 0$ for $\mathrm{tr}(C) = 0$. We can further specify the frame by demanding that $\partial[Z_2] = [Z_3]$, $\bar\partial f = [Z_4]$. Under these unitary frames, we have

$$a_{2\bar{4}} = a_{2\bar{5}} = 0,\ a_{2\bar{3}} \neq 0,\ b_{i\bar{3}} = b_{i\bar{5}} = 0. \tag{4.15}$$

Since $\mathrm{rank}(\partial^*|_{\bar\partial f(x)}) = 1$ and $\dim \bar\partial f(x) = 1$, we have

$$b_{1\bar{4}} = 0, \tag{4.16}$$

thus $b_{2\bar{4}} \neq 0$, $a_{1\bar{4}} \neq 0$ by $c_{1\bar{2}} \neq 0$.

According to the minimality equation (2.16), together with (4.15) and (4.16), we obtain

$$\omega_{1\bar{2}} = b_{1\bar{2}}\bar{\phi}, \quad \omega_{3\bar{4}} = a_{3\bar{4}}\phi, \quad \omega_{3\bar{5}} = a_{3\bar{5}}\phi, \quad \omega_{4\bar{5}} = b_{4\bar{5}}\bar{\phi},$$

and

$$\begin{aligned} \mathrm{d}a_{1\bar{4}} &\equiv a_{1\bar{4}}(\omega_{1\bar{1}} - \omega_{4\bar{4}} + \rho), \\ \mathrm{d}a_{2\bar{3}} &\equiv a_{2\bar{3}}(\omega_{2\bar{2}} - \omega_{3\bar{3}} + \rho), \\ \mathrm{d}b_{\bar{2}4} &\equiv b_{\bar{2}4}(\omega_{4\bar{4}} - \omega_{2\bar{2}} + \rho), \quad \mod \phi, \end{aligned}$$

which imply

$$\Delta_M \log |a_{1\bar{4}} a_{2\bar{3}} b_{2\bar{4}}| = 3\left(K - \frac{2}{3}(1+\cos\theta)\right) + \delta^2, \tag{4.17}$$

where $\delta^2 = |b_{1\bar{2}}|^2 + |a_{1\bar{4}}|^2 + |a_{1\bar{5}}|^2 + |a_{2\bar{3}}|^2 + |a_{3\bar{4}}|^2 + |a_{3\bar{5}}|^2$.

Theorem 4.4. *Let f be a linearly full minimal immersion from S^2 into $G(2,5)$ with* $\mathrm{rank}(\partial) = 2$, $\mathrm{rank}(\bar{\partial}) = 1$, $\bar{\partial} f \not\subseteq \partial f$ *and $\bar{\partial} f$ is not orthogonal to ∂f in $f^{\perp}(x)$. Then there exists a point $x \in S^2$ such that the Gaussian curvature $K(x) < \frac{2}{3}(1+\cos\theta(x))$.*

Proof. It is similar to Theorem 4.3 by identity (4.17). □

(c) $\mathrm{rank}(\partial) = \mathrm{rank}(\bar{\partial}) = 1$. This case, like the case of minimal maps of S^2 to $G(2,n)$ with $n \geq 5$, exhibits many features present in the general case. One can find the results in Theorem 5.5.

5. Minimal immersions of S^2 into $G(2, n)$

In this section, we firstly study the holomorphic immersions from S^2 into $G(2,n)$. Later, we investigate the minimal immersions of S^2 into $G(2,n)$, which are neither holomorphic nor antiholomorphic.

5.1 *Holomorphic 2-spheres into $G(2,n)$*

Let f be a linearly full holomorphic immersion from S^2 into $G(2, n)$. According to the theory of harmonic sequence (c.f. [4, 16]), f can generate an orthogonal harmonic sequence $f_1, f_2, \ldots, f_m$ through ∂-transforms, where $f_i : S^2 \longrightarrow G(k_i, n)$ is minimal with $k_i \leq 2$, and $f_1 = f$, $\partial f_{i+1} = f_i$ for $i < m$, $\partial f_m = 0$, and $\sum_{i=1}^m k_i = n$. If $k_1 = k_2 = \cdots = k_r = 2$ and $k_i = 1$ for $r < i \leq m$, we say that f *degenerates at position r*. In special cases, f degenerates at position 1 means that $\mathrm{rank}(\partial|_{f(x)}) = 1$, and f degenerates at position m means that $k_i = 2$ for all $i = 1, \ldots, m$.

Since any member of $f_1, f_2, \ldots, f_m$ are orthogonal, we can choose the unitary frames Z_A so that $f_i = [Z_{2i-1}, Z_{2i}]$ for $i \leq r$, $f_i = [Z_{r+i}]$ for $i > r$, $\partial[Z_{2r-1}] = 0$ and $\partial[Z_{2r}] = f_{r+1}$. Under these frames, the pull back of the Maurer–Cartan forms are

$$\begin{pmatrix}
\Omega_{1\bar{1}} & A_1\phi & & & & & & \\
-A_1^*\bar{\phi} & \Omega_{2\bar{2}} & A_2\phi & & & & & \\
 & -A_2^*\bar{\phi} & \Omega_{3\bar{3}} & & & & & \\
 & & & \ddots & & & & \\
 & & & & \Omega_{r\bar{r}} & A_r\phi & & \\
 & & & & -A_r^*\bar{\phi} & \Omega_{r+1\,\overline{r+1}} & & \\
 & & & & & & \ddots & \\
 & & & & & & & \omega_{n-1\,\overline{n-1}} \quad a_{n-1\,\bar{n}}\phi \\
 & & & & & & & -a_{\overline{n-1}\,n}\bar{\phi} \quad \omega_{n\bar{n}}
\end{pmatrix}, \tag{5.1}$$

where $\Omega_{i\bar{i}} = \begin{pmatrix} \omega_{2i-1\,\overline{2i-1}} & \omega_{2i-1\,\overline{2i}} \\ -\omega_{\overline{2i-1}\,2i} & \omega_{2i\,\overline{2i}} \end{pmatrix}$ for $i \leq r$, $A_i = \begin{pmatrix} a_{2i-1\,\overline{2i+1}} & a_{2i-1\,\overline{2i+2}} \\ a_{2i\,\overline{2i+1}} & a_{2i\,\overline{2i+2}} \end{pmatrix}$ for $i < r$, $A_r = \begin{pmatrix} 0 & 0 \\ a_{2r\,\overline{2r+1}} & 0 \end{pmatrix}$ and $\Omega_{r+1\,\overline{r+1}} = \begin{pmatrix} \omega_{2r+1\,\overline{2r+1}} & a_{2r+1\,\overline{2r+2}}\phi \\ -a_{\overline{2r+1}\,2r+2}\bar{\phi} & \omega_{2r+2\,\overline{2r+2}} \end{pmatrix}$. Using the Maurer–Cartan equations (2.3), and taking the exterior derivative of $\omega_{2r-1\,\overline{2r+1}} = 0$, we obtain $\omega_{2r-1\,\overline{2r}} = a_{2r-1\,\overline{2r}}\phi$.

Note that the frames we choose is determined to a transformation of the group $\underbrace{U(2)\times\cdots\times U(2)}_{r-1}\times\underbrace{U(1)\times\cdots\times U(1)}_{n-2(r-1)}$, so $|\det A_i|$ ($i \leq r-1$) and $|a_{p\,\overline{p+1}}|$ ($2r \leq p \leq n-1$) are globally defined functions on S^2.

By the Maurer–Cartan forms (5.1) and through direct computations, one has

$$\mathrm{d}\det A_1 = \det A_1(\omega_{1\bar{1}} + \omega_{2\bar{2}} - \omega_{3\bar{3}} - \omega_{4\bar{4}} + 2\rho), \quad \mod \phi,$$

which implies

$$\Delta_M \log|\det A_1| = 2K + 2(\delta_2 - 2\delta_1), \tag{5.2}$$

by Lemma 2.3.

Similarly,

$$\Delta_M \log|\det A_i| = 2K + 2(\delta_{i-1} - 2\delta_i + \delta_{i+1}), \quad 2 \leq i < r, \tag{5.3}$$

$$\Delta_M \log|a_{2r\,\overline{2r+1}}| = K + 2\big(\delta - 2|a_{2r\,\overline{2r+1}}|^2 + |a_{2r+1\,\overline{2r+2}}|^2\big), \tag{5.4}$$

$$\Delta_M \log|a_{p\,p+1}| = K + 2\big(|a_{p-1\,\bar{p}}|^2 - 2|a_{p\,\overline{p+1}}|^2 + |a_{p+1\,\overline{p+2}}|^2\big), \tag{5.5}$$

for $2r+1 \leq p < n-1$, and

$$\Delta_M \log|a_{n-1\bar{n}}| = K + 2\big(|a_{n-2\,\overline{n-1}}|^2 - 2|a_{n-1\,\bar{n}}|^2\big), \tag{5.6}$$

where $\delta_i := \mathrm{tr}(A_i A_i^*)$ and $\delta := |a_{2r-3\,\overline{2r}}|^2 + |a_{2r-2\,\overline{2r}}|^2 + |a_{2r-1\,\overline{2r}}|^2$. The identities (5.2)–(5.6) are called the *Plücker formulas* in [2].

Theorem 5.1. *Let f be a linearly full holomorphic immersion from S^2 into $G(2,n)$, K be its Gaussian curvature. Then*

(1) *If f degenerates at position 1 and has constant curvature, then $K = \frac{2}{n-2}$ and $f = \varphi_0^{n-1} \wedge \varphi_1^{n-1}$ up to rigid motion;*

(2) *If f degenerates at position r with $1 < r$ and $n - 2r \geq 1$, then there exists a point $x \in S^2$ such that $K(x) < \frac{4r}{n+2r^2-4r}$;*

(3) *If $n \equiv 0 \mod 2$, f degenerates at position $\frac{n}{2}$ and K is a constant, then $K = \frac{4}{n-2}$.*

Proof.

(1) Choosing the frames as we do at the beginning of this subsection, essentially means that f is spanned by the first and second elements in a Frenet frame. So, the results follow from Proposition 2.1.

(2) From the Plücker formulas (5.2)–(5.6) and the fact that $\delta_1 = 1$, we obtain

$$\Delta_M \log\left(\prod_{p=1}^{n-2r} |a_{n-p\,\overline{n-p+1}}|^p \prod_{q=1}^{r-1} |\det A_{r-q}|^q\right) = c\left(K - \frac{4r}{n+2r^2-4r}\right) + \delta, \tag{5.7}$$

where $c = \frac{(n-2r+1)(n+2r^2-4r)}{2}$ and δ is the positive functions in (5.4).

We assume that $K \geq \frac{4r}{n+2r^2-4r}$ everywhere on S^2. Applying the maximum principle of subharmonic functions to (5.7), we get $\delta = 0$, which implies that $|a_{2r-3\,\overline{2r}}| = |a_{2r-2\,\overline{2r}}| = 0$. In other words, f degenerates at position $r-1$ by reading the pull back of Maurer–Cartan forms in (5.1). It is a contradiction, so the statements is true.
(3) For this case, utilizing the corresponding Plücker formulas, one has

$$\Delta_M \log\left(\prod_{i=1}^{\frac{n-4}{2}} |\det A_{\frac{n-2}{2}-i}|^i\right) = \frac{n^2-6n-8}{4}\left(K - \frac{4}{n-2}\right),$$

which implies the result. □

Remark. Some of the functions in $\log(\cdot)$ probably have isolated zeros, however, we assume them to have no zeros in the proof. The result (3) was proved by Jiao in his recent paper [11], and also by Zheng in [16] using different methods.

5.2 *Minimal 2-spheres into $G(2,n)$*

In this subsection, we agree on the following ranges of indices:

$$\lambda,\ \mu = 3,4; \quad \sigma,\tau = 5,6; \quad \xi,\ \eta = 7,8,\ldots,n.$$

(a) $\mathrm{rank}(\partial) = \mathrm{rank}(\bar\partial) = 2$. Firstly, we investigate a special case that ∂f and $\bar\partial f$ are perpendicular. In this case, choosing the unitary frames Z_A so that $f(x) = [Z_i]$, $\partial f(x) = [Z_\lambda]$, $\bar\partial f(x) = [Z_\sigma]$, $[Z_\xi]$ is orthogonal to ∂f and $\bar\partial f$ in $f^\perp(x)$. Therefore, one has

$$a_{i\bar\sigma} = a_{i\bar\xi} = 0, \quad b_{i\bar\lambda} = b_{i\bar\xi} = 0. \tag{5.8}$$

Since $\mathrm{rank}(\partial) = \mathrm{rank}(\bar\partial) = 2$, we have $\det(a_{i\bar\lambda}) \neq 0$, $\det(b_{i\bar\sigma}) \neq 0$.

Utilizing the minimality of f, i.e. eq. (2.16), together with (5.8), we obtain

$$a_{i\bar\lambda}\omega_{\lambda\bar\sigma} \equiv 0, \quad \mod \phi, \tag{5.9}$$

and therefore

$$\omega_{\lambda\bar\sigma} = a_{\lambda\bar\sigma}\phi, \tag{5.10}$$

by the fact that $\det(a_{i\bar\lambda}) \neq 0$. Similarly,

$$\omega_{\lambda\bar\xi} = a_{\lambda\bar\xi}\phi,\ \omega_{\sigma\bar\xi} = b_{\sigma\bar\xi}\bar\phi. \tag{5.11}$$

Thus, under such a frame, the pull back of the Maurer–Cartan forms are

$$\begin{pmatrix} \Omega_{1\bar1} & A_1\phi & B_1\bar\phi & 0 \\ -A_1^*\bar\phi & \Omega_{2\bar2} & A_2\phi & A_3\phi \\ -B_1^*\phi & -A_2^*\bar\phi & \Omega_{3\bar3} & B_2\bar\phi \\ 0 & -A_3^*\bar\phi & -B_2^*\phi & \Omega_{4\bar4} \end{pmatrix}, \tag{5.12}$$

where $A_1 = (a_{i\bar\lambda})$, $A_2 = (a_{\lambda\bar\sigma})$, $A_3 = (a_{\lambda\bar\xi})$, $B_1 = (b_{i\bar\sigma})$, $B_2 = (b_{\sigma\bar\xi})$, $\Omega_{1\bar1} = (\omega_{i\bar j})$, $\Omega_{2\bar2} = (\omega_{\lambda\bar\mu})$, $\Omega_{3\bar3} = (\omega_{\sigma\bar\tau})$ and $\Omega_{4\bar4} = (\omega_{\xi\bar\eta})$.

By the Maurer–Cartan equation (2.3), we have

$$\begin{aligned} \mathrm{d}\det A_1 &\equiv \det A_1(\omega_{1\bar1} + \omega_{2\bar2} - \omega_{3\bar3} - \omega_{4\bar4} + 2\rho), \\ \mathrm{d}\det \bar B_1 &\equiv \det \bar B_1(\omega_{5\bar5} + \omega_{6\bar6} - \omega_{1\bar1} - \omega_{2\bar2} + 2\rho), \quad \mod \phi, \end{aligned}$$

which implies

$$\Delta_M \log|\det A_1| = 2K + 2(\delta_{B_1} - 2\delta_{A_1} + \delta_{A_2} + \delta_{A_3}), \tag{5.13}$$

$$\Delta_M \log|\det B_1| = 2K + 2(\delta_{A_1} - 2\delta_{B_1} + \delta_{A_2} + \delta_{B_2}), \tag{5.14}$$

where $\delta_{A_i} = \mathrm{tr}(A_i A_i^*)$, $\delta_{B_i} = \mathrm{tr}(B_i B_i^*)$. The summation of (5.13) and (5.14) is

$$\Delta_M \log|\det A_1 \det B_1| = 4\left(K - \frac{1}{2}\right) + 2\delta_{A_2} + \delta_{A_3} + \delta_{B_2}, \tag{5.15}$$

by the fact that $\delta_{A_1} + \delta_{B_1} = 1$.

Theorem 5.2. *Let f be a linearly full minimal immersion from S^2 into $G(2,n)$ with* $\mathrm{rank}(\partial) = \mathrm{rank}(\bar{\partial}) = 2$ *and $\partial f \perp \bar{\partial} f$. Then*

(1) *If $n > 6$, then there exists a point $x \in S^2$ such that $K(x) < \frac{1}{2}$;*
(2) *For $n = 6$, f is totally real if the Kähler angle $\theta \in [0, \frac{\pi}{2}]$ or $[\frac{\pi}{2}, \pi]$ everywhere on S^2. The curvature $K = \frac{1}{2}$ if $K \geq \frac{1}{2}$ everywhere on S^2, at this time, f is generated by a holomorphic immersion through the ∂-transform.*

Proof.

(1) By (5.15), if $K \geq \frac{1}{2}$ everywhere on S^2, then we have $\delta_{A_3} = \delta_{B_2} = 0$, i.e. $A_3 = B_2 = 0$, according to maximum principle of subharmonic functions. Its contradiction to f is linearly full by reading the pull back of the Maurer–Cartan forms (5.12). Hence, our statement is valid.
(2) For $n = 6$, there are no terms $\delta_{A_3}, \delta_{B_2}$ in the identities (5.13), (5.14) and (5.15), so we have

$$\Delta_M \log|\det B_1||\det A_1|^{-1} = 6\cos\theta, \tag{5.16}$$

$$\Delta_M \log|\det A_1 \det B_1| = 4\left(K - \frac{1}{2}\right) + 2\delta_{A_2} \tag{5.17}$$

by (2.9). The statements follow from (5.16) and (5.17) respectively. Moreover, $f = \partial[Z_5 \wedge Z_6]$ if the curvature $K = \frac{1}{2}$. □

Remark. For the case where ∂f and $\bar{\partial} f$ are not perpendicular, we have the same statement (1) as in Theorem 5.2.

Example. The map $\varphi_1^{n-1} \wedge \varphi_4^{n-1}$ has constant curvature $K = \frac{2}{6n-23} < \frac{1}{2}$ when $n > 6$. For $n = 6$, the map $\varphi_1^5 \wedge \varphi_4^5$ is totally real. Both of them satisfy the rank condition in the theorem.

(b) $\mathrm{rank}(\partial) = 2$ and $\mathrm{rank}(\bar{\partial}) = 1$. Studying this case is similar to the corresponding case in §4. We write down these results in the following.

Theorem 5.3. *Let f be a linearly full minimal immersion from S^2 into $G(2,n)$ with $n \geq 6$,* $\mathrm{rank}(\partial) = 2$ *and* $\mathrm{rank}(\bar{\partial}) = 1$, *$K$ and θ be its curvature and Kähler angle respectively. Then there exists a point $x \in S^2$ such that*

(1) $K(x) < \frac{3+\cos\theta(x)}{3}$ *if* $\partial f \perp \bar{\partial} f$;
(2) $K(x) < \frac{2(1+\cos\theta(x))}{3}$ *if* $\bar{\partial} f \not\subseteq \partial f$, ∂f *and* $\bar{\partial} f$ *are not perpendicular*;
(3) $K(x) < \frac{2(1+\cos\theta(x))}{3}$ *if* $\bar{\partial} f \subset \partial f$.

Proof. The proof of (1) and (2) are the same as Theorems 4.3 and 4.4 respectively, we only need to prove (3).

If $\bar{\partial} f \subset \partial f$, we choose the unitary frames Z_A so that $f(x) = [Z_i]$, $\partial f(x) = [Z_3, Z_4]$, $\ker(\bar{\partial}) = [Z_2]$ and $\bar{\partial} f = [Z_4]$. Thus, under such frames we have

$$a_{1\bar{4}} = a_{1\bar{\sigma}} = a_{1\bar{\xi}} = 0, \quad a_{2\bar{\sigma}} = a_{2\bar{\xi}} = 0, \quad a_{1\bar{3}} \neq 0, \tag{5.18}$$

$$a_{2\bar{4}} \neq 0, \quad b_{1\bar{\sigma}} = b_{1\bar{\xi}} = 0, \quad b_{2\bar{\alpha}} = 0, \quad b_{1\bar{4}} \neq 0, \tag{5.19}$$

which gives $b_{1\bar{3}} = 0$ for $c_{1\bar{1}} = 0$.

By the minimality equation (2.16), and together with (5.18) and (5.19), we have

$$\omega_{1\bar{2}} = a_{1\bar{2}}\phi, \quad \omega_{3\bar{4}} = a_{3\bar{4}}\phi, \quad \omega_{3\bar{\sigma}} = a_{3\bar{\sigma}}\phi, \tag{5.20}$$

$$\omega_{3\bar{\xi}} = a_{3\bar{\xi}}\phi, \quad \omega_{4\bar{\sigma}} = \omega_{4\bar{\xi}} = 0. \tag{5.21}$$

Using the identities (5.18), (5.19), (5.20) and (5.21) by direct computation as we have done before, we obtain

$$\Delta_M \log |a_{1\bar{3}} a_{2\bar{4}} b_{1\bar{4}}| = 3\left(K - \frac{2(1+\cos\theta)}{3}\right) + \delta^2,$$

where $\delta^2 = 2\{|a_{1\bar{2}}|^2 + |a_{1\bar{3}}|^2 + |a_{2\bar{4}}|^2 + \sum_{p\geq 4} |a_{3\bar{p}}|^2\}$. If $K \geq \frac{2(1+\cos\theta)}{3}$ everywhere on S^2, we conclude that $|a_{1\bar{3}}| = |a_{2\bar{4}}| = 0$, i.e., $\text{rank}(\partial) \leq 1$, which is a contradiction. Hence, our statement is valid. □

Example. The map $\varphi_0^{n-1} \wedge \varphi_3^{n-1}$ (resp. $\varphi_0^{n-1} \wedge \varphi_2^{n-1}$) satisfies the conditions in (1) (resp. (3)), whose curvature and Kähler angle are $\frac{2}{4n-13}$ (resp. $\frac{2}{3n-7}$) and $\frac{n-4}{4n-13}$ (resp. $\frac{n-3}{3n-7}$) respectively. Both of them satisfy the corresponding inequality.

The results in Theorem 5.3 give an estimation of the upper-bound of the curvature if f has constant curvature. Hence, we have

COROLLARY 5.4

Let f be a linearly full minimal immersion from S^2 into $G(2,n)$ which satisfies the conditions in Theorem 5.3, *if f has constant curvature K, then $K < \frac{4}{3}$.*

(c) $\text{rank}(\partial) = \text{rank}(\bar{\partial}) = 1$. In this case, the method of choosing unitary frames is similar to the corresponding case (b.I) in §3. Economically, we write down the results without explicit proofs.

Theorem 5.5. *Let f be a linearly full minimal immersion from S^2 into $G(2,n)$ with* $\text{rank}(\partial) = \text{rank}(\bar{\partial}) = 1$, *$n \geq 5$, K and θ be its curvature and Kähler angle respectively.*

(1) *If $\partial f \perp \bar{\partial} f$*, $\ker(\partial)$ *and* $\ker(\bar{\partial})$ *are not perpendicular, then there exists a point $x \in S^2$ such that $K(x) < 1$.*
(2) *If $\partial f \perp \bar{\partial} f$ and* $\ker(\partial) \perp \ker(\bar{\partial})$, *then f satisfies one of the following*:

(i) *There exists a point $x \in S^2$ such that $K(x) < \frac{2}{3}$.*

(ii) *The curvature $K = \frac{2}{3}$ if $K \geq \frac{2}{3}$ on S^2, or there exists a point $x \in S^2$ such that $K(x) < 2$. The immersion $f = \varphi_1 \wedge \varphi_2$, where φ_1, φ_2 are antiholomorphic and holomorphic immersions from S^2 into $\mathbb{C}P^{n-1}$ respectively. Furthermore, if f has constant curvature and constant Kähler angle, then there exist positive integers $n_1, n_2 \leq n-1$ such that $\varphi_1 = \varphi_{n_1}^{n_1}$ and $\varphi_2 = \varphi_0^{n_2}$, up to rigid motion.*

(3) *If ∂f and $\bar{\partial} f$ are not perpendicular in $f^{\perp}(x)$ and $K \geq 1$ on S^2, then the curvature $K = 1$ and $f = \varphi_1 \wedge \varphi_2$, where φ_1, φ_2 are antiholomorphic and holomorphic immersions from S^2 into $\mathbb{C}P^{n-1}$ respectively. At this time, if f has constant Kähler angle, then there exist positive integers $n_1, n_2 \leq n-1$ such that $\varphi_1 = \varphi_{n_1}^{n_1}$ and $\varphi_2 = \varphi_0^{n_2}$, up to rigid motion.*

Proof. The existence arguments are similar to Theorem 4.3. One also can construct corresponding examples to satisfy the conditions and conclusions in Theorem 5.5, from the Veronese sequence. □

Finally, we give some comments on this paper. Our method is moving frames and we study the given immersion according to the relationships between the images of ∂f and $\bar{\partial} f$. Due to limitation of the method some conditions of the theorems are technically necessary, which probably will be reduced or replaced by equivalent geometric conditions. However, to our knowledge, so far, there is no better method to study the geometry of general minimal 2-spheres in $G(k, n)$. In this paper, the results we obtained reflect the fact that the geometric properties of minimal 2-spheres in general Grassmannians are restricted by the relative position of $\partial f(x)$ and $\bar{\partial} f(x)$ in $f^{\perp}(x)$, which make one believe that it is better to study the general minimal 2-spheres in $G(k, n)$ from the viewpoint of algebraic geometry. We wish to focus on this subject in our later study.

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