Proc. Indian Acad. Sci. (Math. Sci.) Vol. 121, No. 2, May 2011, pp. 143–154.

Approximation of the inverse G-frame operator

M R ABDOLLAHPOUR and A NAJATI

Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran
E-mail: mrabdollahpour@yahoo.com; a.nejati@yahoo.com

MS received 16 June 2010; revised 2 March 2011

Abstract. In this paper, we introduce the concept of (strong) projection method for g-frames which works for all conditional g-Riesz frames. We also derive a method for approximation of the inverse g-frame operator which is efficient for all g-frames. We show how the inverse of g-frame operator can be approximated as close as we like using finite-dimensional linear algebra.

Keywords. Frame; g-frame; g-frame operator; projection method.

1. Introduction and preliminaries

In 1952, Duffin and Schaeffer [8] introduced frames. They used frames as a tool in the study of nonharmonic Fourier analysis. Various generalizations of frames in Hilbert spaces have been proposed and studied recently. For example, frames of subspaces [1], pseudo frames for subspaces [10], bounded quasi-projectors [9] and oblique frames [7]. Wenchang Sun [12] introduced the concept of g-frames which includes all the mentioned generalizations. Members of ordinary frames are vectors of a Hilbert space, while members of g-frames are bounded operators.

In all that follows, $\mathcal{H}$ denotes a separable Hilbert space with the inner product $\langle .\,,.\rangle$ linear in the first entry. A family $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a *frame* for $\mathcal{H}$, if there exist two positive constants A, B such that

$$A\|f\|^2 \le \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \le B\|f\|^2 \tag{1.1}$$

for all $f \in \mathcal{H}$. The numbers A, B are called *frame bounds*. Observe, that if $\{f_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$ and $n \in \mathbb{N}$, then $\{f_i\}_{i=1}^{n}$ is a frame for $E_n = \operatorname{span}\{f_i\}_{i=1}^{n}$ (see [3]). We say that $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a *Riesz frame* if every subfamily of $\{f_i\}_{i=1}^{\infty}$ is a frame for its closed linear span, with the same frame bounds A, B for each subfamily. A frame $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is called a *conditional Riesz frame* for $\mathcal{H}$ if there exist common bounds $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{i=1}^{n} |\langle f, f_i\rangle|^2 \le B\|f\|^2$$

for every $n \in \mathbb{N}$ and all $f \in \operatorname{span}\{f_i\}_{i=1}^{n}$.

If $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a frame for $\mathcal{H}$, the *frame operator* $S : \mathcal{H} \to \mathcal{H}$ is defined by $Sf = \sum_{i=1}^{\infty} \langle f, f_i \rangle f_i$. The series $\sum_{i=1}^{\infty} \langle f, f_i \rangle f_i$ converges unconditionally for each $f \in \mathcal{H}$, and S is a bounded, invertible and positive operator. Using these properties of S, we have the following representations

$$f = \sum_{i=1}^{\infty} \langle f, S^{-1} f_i \rangle f_i = \sum_{i=1}^{\infty} \langle f, f_i \rangle S^{-1} f_i \tag{1.2}$$

for all $f \in \mathcal{H}$. Since $\mathcal{H}$ is usually an infinite dimensional Hilbert space, in practice it can be very difficult to apply the frame decomposition formula (1.2) directly. In case we can not find S^{-1} explicitly, we need to approximate S^{-1} or at least approximate the frame coefficients $\{\langle f, S^{-1} f_i \rangle\}_{i=1}^{\infty}$. In [5], Christensen introduced the projection method to approximate the frame coefficients $\{\langle f, S^{-1} f_i \rangle\}_{i=1}^{\infty}$. Here we summarize some results about the approximation of inverse frame operator.

Let $\{f_i\}_{i=1}^{\infty}$ be a frame for $\mathcal{H}$ with the frame operator S. For every $n \in \mathbb{N}$, $\{f_i\}_{i=1}^{n}$ is a frame for $E_n = \mathrm{span}\{f_i\}_{i=1}^{n}$; denote its frame operator by

$$S_n : E_n \to E_n, \quad S_n f = \sum_{i=1}^{n} \langle f, f_i \rangle f_i.$$

Since E_n is finite dimensional, we can find S_n^{-1} using linear algebra. We say that the *projection method* works if

$$\langle f, S_n^{-1} f_i \rangle \to \langle f, S^{-1} f_i \rangle \quad \text{as} \quad n \to \infty \tag{1.3}$$

for all $f \in \mathcal{H}$ and all $i \in \mathbb{N}$. Christensen [5] proved that (1.3) holds if and only if for all $j \in \mathbb{N}$, there exists $c_j \in \mathbb{R}$ such that $\|S_n^{-1} f_j\| \le c_j$ for all $n \ge j$. It is natural to ask under which conditions $\{\langle f, S_n^{-1} f_i \rangle\}_{i=1}^{n}$ converges to $\{\langle f, S^{-1} f_i \rangle\}_{i=1}^{\infty}$ in the ℓ^2-sense, i.e.,

$$\sum_{i=1}^{n} |\langle f, S_n^{-1} f_i \rangle - \langle f, S^{-1} f_i \rangle|^2 + \sum_{i=n+1}^{\infty} |\langle f, S^{-1} f_i \rangle| \to 0 \tag{1.4}$$

for all $f \in \mathcal{H}$. If (1.4) is satisfied, we say that the *strong projection method* works. It is clear that $\sum_{i=n+1}^{\infty} |\langle f, S^{-1} f_i \rangle| \to 0$ for all $f \in \mathcal{H}$ when $\{f_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$. So we only need to show that $\sum_{i=1}^{n} |\langle f, S_n^{-1} f_i \rangle - \langle f, S^{-1} f_i \rangle|^2 \to 0$ for all $f \in \mathcal{H}$. It is shown in [2] that the strong projection method works for all conditional Riesz frames.

Throughout this paper, $\{\mathcal{H}_i\}_{i \in J}$ is a sequence of Hilbert spaces, where J is a subset of $\mathbb{N}$, and $\mathcal{B}(\mathcal{H}, \mathcal{H}_i)$ is the collection of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}_i$.

DEFINITION 1.1

We call a sequence $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ a *g-frame* for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in J}$, if there exist two positive constants A and B such that

$$A\|f\|^2 \le \sum_{i \in J} \|\Lambda_i f\|^2 \le B\|f\|^2 \tag{1.5}$$

for all $f \in \mathcal{H}$. We call A and B the lower and upper *g-frame bounds*, respectively. $\{\Lambda_i\}_{i\in J}$ is called a *tight g-frame* if $A = B$ and a *Parseval g-frame* if $A = B = 1$.

Let $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ be given. Let us define

$$\left(\sum_{i\in J} \oplus \mathcal{H}_i\right)_{\ell_2} = \left\{\{f_i\}_{i\in J} : f_i \in \mathcal{H}_i \text{ and } \sum_{i\in J} \|f_i\|^2 < \infty\right\}$$

with inner product given by $\langle \{f_i\}_{i\in J}, \{g_i\}_{i\in J}\rangle = \sum_{i\in J}\langle f_i, g_i\rangle$. It is clear that $\left(\sum_{i\in J} \oplus\mathcal{H}_i\right)_{\ell_2}$ is a Hilbert space with respect to the pointwise operations. If $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-frame for $\mathcal{H}$, then the operator

$$T : \left(\sum_{i\in J} \oplus \mathcal{H}_i\right)_{\ell_2} \to \mathcal{H}, \quad T(\{f_i\}) = \sum_{i\in J} \Lambda_i^*(f_i) \tag{1.6}$$

is well defined, bounded and its adjoint is $T^* f = \{\Lambda_i f\}_{i\in J}$. A sequence $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-frame if and only if the operator T defined by (1.6) is bounded and onto (see [11]). The operators T and T^* are called synthesis and analysis operators for $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$, respectively.

PROPOSITION 1.2 [12]

Let $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ be a g-frame for $\mathcal{H}$. Then

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{i\in J} \Lambda_i^* \Lambda_i f$$

is a positive, bounded and invertible operator.

Proposition 1.2 implies that every $f \in \mathcal{H}$ can be represented as

$$f = SS^{-1} f = \sum_{i\in J} \Lambda_i^* \Lambda_i S^{-1} f, \qquad f = S^{-1} S f = \sum_{i\in J} S^{-1}\Lambda_i^* \Lambda_i f. \tag{1.7}$$

The operator S is called the *g-frame operator* of $\{\Lambda_i\}_{i\in J}$.

DEFINITION 1.3

A g-frame $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ for $\mathcal{H}$ is called a *conditional g-Riesz frame* if there exist common bounds A and B such that for every finite set $E \subseteq J$, $\{\Lambda_i\}_{i\in E}$ is a g-frame for $\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i\in E}$ with the bounds A and B.

2. Approximation of the inverse g-frame operator by the projection method

In this section, $\{\mathcal{H}_i\}_{i=1}^{\infty}$ is a sequence of finite dimensional Hilbert spaces. We start with the following lemma.

Lemma 2.1. *Let* $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^n$ *be a finite family of bounded operators. Then* $\{\Lambda_i\}_{i=1}^n$ *is a g-frame for* $\mathcal{K}_n = \operatorname{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i=1}^n$ *with respect to* $\{\mathcal{H}_i\}_{i=1}^n$.

Proof. We have

$$\sum_{i=1}^n \|\Lambda_i f\|^2 \leq \sum_{i=1}^n \|\Lambda_i\|^2 \|f\|^2$$

for all $f \in \mathcal{K}_n$. For the lower g-frame bound, we consider the continuous mapping

$$\Psi : \mathcal{K}_n \to \mathbb{R}, \quad \Psi(f) = \sum_{i=1}^n \|\Lambda_i f\|^2.$$

Since $\mathcal{K}_n$ is finite dimensional, $B_n = \{f \in \mathcal{K}_n : \|f\| = 1\}$ is compact. So we can find $g \in B_n$ such that $\Psi(g) = \inf_{f \in B_n} \Psi(f)$. We claim that $A = \Psi(g) > 0$. If $A = \sum_{i=1}^n \|\Lambda_i g\|^2 = 0$, then $\Lambda_i g = 0$ for $i = 1, 2, \ldots, n$. Therefore $\langle g, \Lambda_i^* h\rangle = 0$ for all $h \in \mathcal{H}_i$ and $i = 1, 2, \ldots, n$. Hence we get $g = 0$, which is impossible. Let $f \in \mathcal{H}$ and $f \neq 0$. We have

$$\sum_{i=1}^n \|\Lambda_i f\|^2 = \sum_{i=1}^n \left\|\Lambda_i\Big(\frac{f}{\|f\|}\Big)\right\|^2 \cdot \|f\|^2 \geq A\|f\|^2.$$

□

Let $\{\Lambda_i\}_{i=1}^\infty$ be a g-frame for $\mathcal{H}$ with g-frame operator S and $n \in \mathbb{N}$. Lemma 2.1 implies that $\{\Lambda_i\}_{i=1}^n$ is a g-frame for $\mathcal{K}_n = \operatorname{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i=1}^n$. Let

$$S_n : \mathcal{K}_n \to \mathcal{K}_n, \quad S_n f = \sum_{i=1}^n \Lambda_i^* \Lambda_i f$$

be the g-frame operator of $\{\Lambda_i\}_{i=1}^n$. We want to know how we can approximate S^{-1} by operators S_n^{-1}. Since $\mathcal{K}_n = \operatorname{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i=1}^n$ is finite dimensional, we can find S_n^{-1} using the linear algebra.

Theorem 2.2. *Let* $\{\Lambda_i\}_{i=1}^\infty$ *be a g-frame for* $\mathcal{H}$ *with bounds* A *and* B. *For a given* $n \in \mathbb{N}$, *let* S_n *be the g-frame operator of* $\{\Lambda_i\}_{i=1}^n$. *Then*

$$\langle g, S_n^{-1}\Lambda_i^*\Lambda_i f\rangle \to \langle g, S^{-1}\Lambda_i^*\Lambda_i f\rangle \quad \textit{as} \quad n \to \infty \tag{2.1}$$

for all $f, g \in \mathcal{H}$, *if and only if*

$$\forall i \in \mathbb{N}\, \forall f \in \mathcal{H}\, \exists c_{i,f} > 0 : \ \|S_n^{-1}\Lambda_i^*\Lambda_i f\| \leq c_{i,f} \tag{2.2}$$

for all $n \geq i$.

Proof. First, we assume that (2.2) is satisfied. Let $f \in \mathcal{H}$. Fix $i \in \mathbb{N}$, and define

$$\Phi_n = S_n^{-1}\Lambda_i^*\Lambda_i f - S^{-1}\Lambda_i^*\Lambda_i f, \quad n \geq i.$$

Then

$$S\Phi_n = SS_n^{-1}\Lambda_i^*\Lambda_i f - \Lambda_i^*\Lambda_i f. \tag{2.3}$$

Since $Sh = S_n h + \sum_{k=n+1}^{\infty} \Lambda_k^* \Lambda_k h$ for all $h \in \mathcal{K}_n$, we get from (2.3) that

$$\begin{aligned} S\Phi_n &= S_n S_n^{-1} \Lambda_i^* \Lambda_i f + \sum_{k=n+1}^{\infty} \Lambda_k^* \Lambda_k S_n^{-1} \Lambda_i^* \Lambda_i f - \Lambda_i^* \Lambda_i f \\ &= \sum_{k=n+1}^{\infty} \Lambda_k^* \Lambda_k S_n^{-1} \Lambda_i^* \Lambda_i f. \end{aligned}$$

Hence

$$\Phi_n = \sum_{k=n+1}^{\infty} S^{-1} \Lambda_k^* \Lambda_k S_n^{-1} \Lambda_i^* \Lambda_i f.$$

Now for $g \in \mathcal{H}$, we have

$$\begin{aligned} |\langle g, \Phi_n \rangle| &= \left| \left\langle g, \sum_{k=n+1}^{\infty} S^{-1} \Lambda_k^* \Lambda_k S_n^{-1} \Lambda_i^* \Lambda_i f \right\rangle \right| \\ &= \left| \sum_{k=n+1}^{\infty} \langle \Lambda_k S^{-1} g, \Lambda_k S_n^{-1} \Lambda_i^* \Lambda_i f \rangle \right| \\ &\le \left(\sum_{k=n+1}^{\infty} \| \Lambda_k S^{-1} g \|^2 \right)^{\frac{1}{2}} \left(\sum_{k=n+1}^{\infty} \| \Lambda_k S_n^{-1} \Lambda_i^* \Lambda_i f \|^2 \right)^{\frac{1}{2}} \\ &\le B^{\frac{1}{2}} \| S_n^{-1} \Lambda_i^* \Lambda_i f \| \left(\sum_{k=n+1}^{\infty} \| \Lambda_k S^{-1} g \|^2 \right)^{\frac{1}{2}} \\ &\le B^{\frac{1}{2}} c_{i,f} \left(\sum_{k=n+1}^{\infty} \| \Lambda_k S^{-1} g \|^2 \right). \end{aligned}$$

Since $\{\Lambda_i\}_{i=1}^{\infty}$ is a g-frame, $\sum_{k=n+1}^{\infty} \| \Lambda_k S^{-1} g \|^2 \to 0$ as $n \to \infty$. Therefore $|\langle g, \Phi_n \rangle| \to 0$ as $n \to \infty$. Conversely, if (2.1) holds, we can fix $f \in \mathcal{H}$, $i \in \mathbb{N}$ and define

$$\Psi_{n,f} : \mathcal{H} \to \mathbb{C}, \quad \Psi_{n,f}(g) = \langle g, S_n^{-1} \Lambda_i^* \Lambda_i f \rangle, \quad n \ge i.$$

It is clear that $\Psi_{n,f}$ is linear and bounded for $n \ge i$. Also by (2.1) the sequence $\{\Psi_{n,f}\}_{n \ge i}$ converges pointwise. By Banach–Steinhaus theorem there is a constant $c_{i,f} > 0$ such that $\| \Psi_{n,f} \| = \| S_n^{-1} \Lambda_i^* \Lambda_i f \| \le c_{i,f}$ for all $n \ge i$. □

Following Christensen [5], we say that the projection method works if (2.1) is satisfied for every $f, g \in \mathcal{H}$.

COROLLARY 2.3

The projection method works for any conditional g-Riesz frame.

If $S_n : \mathcal{K}_n \to \mathcal{K}_n$ is the g-frame operator of $\{\Lambda_i\}_{i=1}^{n}$, then S_n is invertible and every $h \in \mathcal{K}_n$ can be written as

$$h = \sum_{i=1}^{n} S_n^{-1} \Lambda_i^* \Lambda_i h = \sum_{i=1}^{n} \Lambda_i^* \Lambda_i S_n^{-1} h.$$

If $P_n : \mathcal{H} \to \mathcal{K}_n$ is the orthogonal projection, then $P_n f = \sum_{i=1}^{n} S_n^{-1}\Lambda_i^*\Lambda_i P_n f$ for all $f \in \mathcal{H}$. Let g be an arbitrary element of $\mathcal{K}_n$. Then

$$\begin{aligned}\langle P_n f, g\rangle = \left\langle \sum_{i=1}^{n} S_n^{-1}\Lambda_i^*\Lambda_i P_n f, g\right\rangle &= \sum_{i=1}^{n}\langle f, P_n\Lambda_i^*\Lambda_i S_n^{-1} g\rangle \\ &= \sum_{i=1}^{n}\langle f, \Lambda_i^*\Lambda_i S_n^{-1} g\rangle \\ &= \left\langle \sum_{i=1}^{n} S_n^{-1}\Lambda_i^*\Lambda_i f, g\right\rangle.\end{aligned}$$

Therefore

$$P_n f = \sum_{i=1}^{n} S_n^{-1}\Lambda_i^*\Lambda_i f$$

for all $f \in \mathcal{H}$.

Remark 2.4 Let $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ be a g-frame for $\mathcal{H}$, $\mathcal{K}_n = \operatorname{span}\{\Lambda_i(\mathcal{H}_i)\}_{i=1}^{n}$ and let $P_n : \mathcal{H} \to \mathcal{K}_n$ be the orthogonal projection. Since $\{P_n\}_{n=1}^{\infty}$ is increasing and $\overline{(\cup_{n=1}^{\infty}\mathcal{K}_n)} = \mathcal{H}$, we have $P_n f \to f$ as $n \to \infty$.

Following Christensen [6], we say that the strong projection method works if

$$\sum_{i=1}^{n} |\langle f, S_n^{-1}\Lambda_i^*\Lambda_i f - S^{-1}\Lambda_i^*\Lambda_i f\rangle|^2 \to 0 \ \text{ as } \ n \to \infty$$

satisfies for all $f \in \mathcal{H}$. It is clear that the projection method works if the strong projection method works.

Theorem 2.5. *Let $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}_{i\in J}$ be a g-frame for $\mathcal{H}$ with the upper bound B. The following conditions are equivalent*:

(a) $S_n^{-1}P_n f \to S^{-1} f$ *as* $n \to \infty$ *for all* $f \in \mathcal{H}$.
(b) $\|(S - S_n)S_n^{-1}P_n f\| \to 0$ *as* $n \to \infty$ *for all* $f \in \mathcal{H}$.
(c) $\sum_{i=n+1}^{\infty}\|\Lambda_i S_n^{-1}P_n f\|^2 \to 0$ *as* $n \to \infty$ *for all* $f \in \mathcal{H}$.

Moreover, *the* (*strong*) *projection method works in case the equivalent conditions are satisfied.*

Proof.

(a) $\Leftrightarrow$ (b). Let $f \in \mathcal{H}$. Since $P_n f \to f$ as $n \to \infty$, using

$$\begin{aligned}S_n^{-1}P_n f - S^{-1} f &= S^{-1}(P_n f - f) + S^{-1}(S - S_n)S_n^{-1}P_n f,\\ (S - S_n)S_n^{-1}P_n f &= S(S_n^{-1}P_n f - S^{-1} f) - (P_n f - f),\end{aligned}$$

we get that (a) and (b) are equivalent.

(a) $\Rightarrow$ (c). We have

$$
\begin{aligned}
&\left(\sum_{i=n+1}^{\infty} \|\Lambda_i S_n^{-1} P_n f\|^2\right)^{\frac{1}{2}} \\
&\quad = \left(\sum_{i=n+1}^{\infty} \|\Lambda_i (S_n^{-1} P_n f - S^{-1} f) + \Lambda_i S^{-1} f\|^2\right)^{\frac{1}{2}} \\
&\quad \leq \left(\sum_{i=n+1}^{\infty} \|\Lambda_i (S_n^{-1} P_n f - S^{-1} f)\|^2\right)^{\frac{1}{2}} + \left(\sum_{i=n+1}^{\infty} \|\Lambda_i S^{-1} f\|^2\right)^{\frac{1}{2}} \\
&\quad \leq \sqrt{B}\|S_n^{-1} P_n f - S^{-1} f\| + \left(\sum_{i=n+1}^{\infty} \|\Lambda_i S^{-1} f\|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

for all $f \in \mathcal{H}$. Since $\sum_{i=n+1}^{\infty} \|\Lambda_i S^{-1} f\|^2 \to 0$ as $n \to \infty$, the result follows.

(c) $\Rightarrow$ (b). For each $f \in \mathcal{H}$, we have

$$
\begin{aligned}
\|(S - S_n) S_n^{-1} P_n f\|^2 &= \sup_{\|g\|=1} |\langle (S - S_n) S_n^{-1} P_n f, g\rangle|^2 \\
&= \sup_{\|g\|=1} \left|\left\langle \sum_{i=n+1}^{\infty} \Lambda_i^* \Lambda_i S_n^{-1} P_n f, g \right\rangle\right|^2 \\
&\leq \sup_{\|g\|=1} \sum_{i=n+1}^{\infty} \|\Lambda_i S_n^{-1} P_n f\|^2 . \sum_{i=n+1}^{\infty} \|\Lambda_i g\|^2 \\
&\leq B \sum_{i=n+1}^{\infty} \|\Lambda_i S_n^{-1} P_n f\|^2 \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Moreover, let $f \in \mathcal{H}$ and $\Theta_i = \Lambda_i^* \Lambda_i$ for all i. We have

$$
\begin{aligned}
\sum_{i=1}^{n} |\langle f, S_n^{-1} \Theta_i f - S^{-1} \Theta_i f\rangle|^2 &= \sum_{i=1}^{n} |\langle P_n f, S_n^{-1} \Theta_i f\rangle - \langle f, S^{-1} \Theta_i f\rangle|^2 \\
&= \sum_{i=1}^{n} |\langle S_n^{-1} P_n f - S^{-1} f, \Theta_i f\rangle|^2 \\
&\leq \left(\sum_{i=1}^{n} |\langle S_n^{-1} P_n f - S^{-1} f, \Theta_i f\rangle|\right)^2 \\
&\leq \sum_{i=1}^{n} \|\Lambda_i (S_n^{-1} P_n f - S^{-1} f)\|^2 . \sum_{i=1}^{n} \|\Lambda_i f\|^2 \\
&\leq B^2 \|S_n^{-1} P_n f - S^{-1} f\|^2 . \|f\|^2.
\end{aligned}
$$

Therefore if (a) is satisfied, then the strong projection method works. $\square$

Theorem 2.6. *Let $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ be a conditional g-Riesz frame. Then $S_n^{-1} P_n f \to S^{-1} f$ as $n \to \infty$ for all $f \in \mathcal{H}$. Consequently, the (strong) projection method works for any conditional g-Riesz frame.*

Proof. Let A, B be the g-Riesz frame bounds for $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$. By assumption, we have $\|S_n^{-1}\| \le \frac{1}{A}$. Let $f \in \mathcal{H}$. We have

$$\begin{aligned}
&\|S_n^{-1} P_n f - S^{-1} f\| \\
&\quad \le \|P_n S^{-1} f - S^{-1} f\| + \|S_n^{-1} P_n f - P_n S^{-1} f\| \\
&\quad = \|P_n S^{-1} f - S^{-1} f\| + \left\| \sum_{i=1}^{n} S_n^{-1} \Lambda_i^* \Lambda_i S^{-1} f - S_n^{-1} P_n f \right\| \\
&\quad \le \|P_n S^{-1} f - S^{-1} f\| + \|S_n^{-1}\| \left\| \sum_{i=1}^{n} \Lambda_i^* \Lambda_i S^{-1} f - P_n f \right\| \\
&\quad \le \|P_n S^{-1} f - S^{-1} f\| + \frac{1}{A} \left\| \sum_{i=1}^{n} \Lambda_i^* \Lambda_i S^{-1} f - P_n f \right\|.
\end{aligned}$$

Since $P_n f \to f$ and $\sum_{i=1}^{n} \Lambda_i^* \Lambda_i S^{-1} f - P_n f \to 0$ as $n \to \infty$, we get $S_n^{-1} P_n f \to S^{-1} f$ as $n \to \infty$. □

3. General method for approximation of the inverse g-frame operator

We recall that in this section, $\{\mathcal{H}_i\}_{i=1}^{\infty}$ is a sequence of finite dimensional Hilbert spaces. In this section, we derive a method for approximation of the inverse g-frame operator which is efficient for all g-frames. We start with the following proposition.

PROPOSITION 3.1

Let $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ be a g-frame for $\mathcal{H}$ with bounds A, B and $\mathcal{K}_n = \operatorname{span}\{\Lambda_i^*(\mathcal{H}_i)\}_{i=1}^{n}$. For any $n \in \mathbb{N}$ and $0 < \alpha < 1$, there exists a number m_n such that

(i) $\alpha A \|f\|^2 \le \sum_{i=1}^{n+m_n} \|\Lambda_i f\|^2$ *for all* $f \in \mathcal{K}_n$;

(ii) *If $P_n : \mathcal{H} \to \mathcal{K}_n$ is the orthogonal projection, then $\{\Lambda_i P_n\}_{i=1}^{n+m_n}$ is a g-frame for $\mathcal{K}_n$ with bounds αA and B. Moreover, $P_n S_{n+m_n} : \mathcal{K}_n \to \mathcal{K}_n$ is the g-frame operator for $\{\Lambda_i P_n\}_{i=1}^{n+m_n}$ with $\|P_n S_{n+m_n}\| \le B$ and $\|(P_n S_{n+m_n})^{-1}\| \le \frac{1}{\alpha A}$.*

Proof.

(i) Let $n \in \mathbb{N}$ and $0 < \alpha < \beta < 1$. Choose $\varepsilon > 0$ such that $\sqrt{\beta A} - \sqrt{B}\,\varepsilon \ge \sqrt{\alpha A}$. Since $\{f \in \mathcal{K}_n : \|f\| = 1\}$ is compact, there is a finite set $\{g_j\}_{j=1}^{k} \subseteq \mathcal{K}_n$ with $\|g_j\| = 1$ such that

$$\{f \in \mathcal{K}_n : \|f\| = 1\} \subseteq \cup_{j=1}^{k} \{f \in \mathcal{K}_n : \|f - g_j\| < \varepsilon\}.$$

Since $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i)\}_{i=1}^{\infty}$ is a g-frame for $\mathcal{H}$, we have

$$A \leq \sum_{i=1}^{\infty} \|\Lambda_i g_j\|^2, \quad j = 1, 2, \ldots, k.$$

Therefore we can choose m_n such that

$$\beta A \leq \sum_{i=1}^{n+m_n} \|\Lambda_i g_j\|^2, \quad j = 1, 2, \ldots, k.$$

Now, let $f \in \mathcal{K}_n$ with $\|f\| = 1$. Then $\|f - g_j\| < \varepsilon$ for some $j \in \{1, 2, \ldots, k\}$. Hence

$$\begin{aligned}\left(\sum_{i=1}^{n+m_n} \|\Lambda_i f\|^2\right)^{\frac{1}{2}} &\geq \left(\sum_{i=1}^{n+m_n} \|\Lambda_i g_j\|^2\right)^{\frac{1}{2}} - \left(\sum_{i=1}^{n+m_n} \|\Lambda_i (f - g_j)\|^2\right)^{\frac{1}{2}} \\ &\geq \sqrt{\beta A} - \sqrt{B}\|f - g_j\| \geq \sqrt{\beta A} - \sqrt{B}\epsilon \geq \sqrt{\alpha A}.\end{aligned}$$

(ii) Since $P_n f = f$ for all $f \in \mathcal{K}_n$, it follows from (i) that

$$\alpha A\|f\|^2 \leq \sum_{i=1}^{n+m_n} \|\Lambda_i f\|^2 = \sum_{i=1}^{n+m_n} \|\Lambda_i P_n f\|^2 \leq \sum_{i=1}^{\infty} \|\Lambda_i P_n f\|^2 \leq B\|f\|^2$$

for all $f \in \mathcal{K}_n$. So $\{\Lambda_i P_n\}_{i=1}^{n+m_n}$ is a g-frame for $\mathcal{K}_n$ with the claimed bounds. Moreover,

$$P_n S_{n+m_n} f = P_n \sum_{i=1}^{n+m_n} \Lambda_i^* \Lambda_i f = \sum_{i=1}^{n+m_n} P_n \Lambda_i^* \Lambda_i P_n f = \sum_{i=1}^{n+m_n} (\Lambda_i P_n)^* (\Lambda_i P_n) f$$

for all $f \in \mathcal{K}_n$. Therefore $P_n S_{n+m_n}$ is the g-frame operator for $\{\Lambda_i P_n\}_{i=1}^{n+m_n}$. The norm estimates follow from Proposition 3.4 in [4]. □

Theorem 3.2. *Let $\{\Lambda_i\}_{i=1}^{\infty}$ be a g-frame with g-frame bounds A, B and g-frame operator S. For $n \in \mathbb{N}$, choose m_n such that*

$$\frac{A}{2}\|f\|^2 \leq \sum_{i=1}^{n+m_n} \|\Lambda_i f\|^2$$

for all $f \in \mathcal{K}_n$. Then

$$(P_n S_{n+m_n})^{-1} P_n f \to S^{-1} f \quad \textit{as } n \to \infty$$

for all $f \in \mathcal{H}$.

Proof. Let $f \in \mathcal{H}$. Since

$$\begin{aligned}S^{-1} f - (P_n S_{n+m_n})^{-1} P_n f &= P_n S^{-1} f - (P_n S_{n+m_n})^{-1} P_n f \\ &\quad + (I - P_n) S^{-1} f\end{aligned}$$

and $(I - P_n)S^{-1}f \to 0$ as $n \to \infty$, it is enough to prove that $\Psi_n = P_n S^{-1} f - (P_n S_{n+m_n})^{-1} P_n f \to 0$ as $n \to \infty$. By Proposition 3.1, we have

$$\begin{aligned}
\|\Psi_n\| &\le \|(P_n S_{n+m_n})^{-1}\| \|S_{n+m_n} P_n S^{-1} f - f\| \\
&\le \frac{2}{A}\Big\{\|S_{n+m_n} P_n S^{-1} f - S_{n+m_n} P_{n+m_n} S^{-1} f\| \\
&\quad + \|S_{n+m_n} P_{n+m_n} S^{-1} f - f\|\Big\} \\
&\le \frac{2}{A}\Big\{B\|P_n S^{-1} f - P_{n+m_n} S^{-1} f\| + \Big\|\sum_{n+m_n+1}^{\infty} \Lambda_i^* \Lambda_i S^{-1} f\Big\|\Big\} \to 0,
\end{aligned}$$

as $n \to \infty$. □

Theorem 3.3. *Let $\{\Lambda_i\}_{i=1}^{\infty}$ be a g-frame for $\mathcal{H}$ with the g-frame operator S. Then the following are equivalent*:

(i) $\sum_{i=1}^{n} S_n^{-1} \Lambda_i^* g_i \to \sum_{i=1}^{\infty} S^{-1} \Lambda_i^* g_i$ *as* $n \to \infty$, *for all* $\{g_i\}_{i=1}^{\infty} \in (\sum_{i=1}^{\infty} \oplus \mathcal{H}_i)_{\ell_2}$.

(ii) *If* $\{g_i\}_{i=1}^{\infty} \in (\sum_{i=1}^{\infty} \oplus \mathcal{H}_i)_{\ell_2}$ *with* $\sum_{i=1}^{\infty} \Lambda_i^* g_i = 0$, *then* $S_n^{-1} \sum_{i=1}^{n} \Lambda_i^* g_i \to 0$ *as* $n \to \infty$.

(iii) $\{\Lambda_i\}_{i=1}^{\infty}$ *is a conditional g-Riesz frame.*

Proof. Since range of T^* is closed, we have $(\sum_{i=1}^{\infty} \oplus \mathcal{H}_i)_{\ell_2} = \mathcal{R}(T^*) \oplus \operatorname{Ker} T$. So for $\{g_i\}_{i=1}^{\infty} \in (\sum_{i=1}^{\infty} \oplus \mathcal{H}_i)_{\ell_2}$, there exist $g \in \mathcal{H}$ and $\{f_i\}_{i=1}^{\infty} \in \operatorname{Ker} T$ such that

$$\{g_i\}_{i=1}^{\infty} = \{\Lambda_i g\}_{i=1}^{\infty} + \{f_i\}_{i=1}^{\infty}.$$

Then

$$\sum_{i=1}^{n} S_n^{-1} \Lambda_i^* g_i = \sum_{i=1}^{n} S_n^{-1} \Lambda_i^* \Lambda_i g + \sum_{i=1}^{n} S_n^{-1} \Lambda_i^* f_i = P_n g + S_n^{-1} \sum_{i=1}^{n} \Lambda_i^* f_i.$$

On the other hand,

$$\sum_{i=1}^{\infty} S^{-1} \Lambda_i^* g_i = \sum_{i=1}^{\infty} S^{-1} \Lambda_i^* \Lambda_i g + S^{-1} \sum_{i=1}^{\infty} \Lambda_i^* f_i = g + S^{-1} \sum_{i=1}^{\infty} \Lambda_i^* f_i.$$

So (i) and (ii) are equivalent.

(i)$\Rightarrow$ (iii). Let us define the operators

$$Q : \Big(\sum_{i=1}^{\infty} \oplus \mathcal{H}_i\Big)_{\ell_2} \to \mathcal{H}, \quad Q\{g_i\} = \sum_{i=1}^{\infty} S^{-1} \Lambda_i^* g_i,$$

$$Q_n : \Big(\sum_{i=1}^{\infty} \oplus \mathcal{H}_i\Big)_{\ell_2} \to \mathcal{H}, \quad Q_n\{g_i\} = \sum_{i=1}^{n} S_n^{-1} \Lambda_i^* g_i.$$

It follows from (i) that the sequence $\{Q_n\}_{n=1}^{\infty}$ converges pointwise to Q. Hence by Banach–Steinhaus theorem, $\sup_n \|Q_n\| < \infty$. For each $f \in \mathcal{H}$ and $\{g_i\} \in (\sum_{i=1}^{\infty} \oplus \mathcal{H}_i)_{\ell_2}$, we have

$$\begin{aligned}\langle f, Q_n\{g_i\}_i\rangle &= \sum_{i=1}^{n} \langle f, S_n^{-1}\Lambda_i^* g_i\rangle \\ &= \sum_{i=1}^{n} \langle f, P_n S_n^{-1}\Lambda_i^* g_i\rangle = \sum_{i=1}^{n} \langle \Lambda_i S_n^{-1} P_n f, g_i\rangle.\end{aligned}$$

Therefore $Q_n^* f = \{h_i\}_{i=1}^{\infty}$ where $h_i = \Lambda_i S_n^{-1} P_n f$ for all $1 \le i \le n$ and $h_i = 0$ for all $i > n$. Hence $Q_n Q_n^* = \sum_{i=1}^{n} S_n^{-1}\Lambda_i^*\Lambda_i S_n^{-1} P_n$ and

$$\|S_n^{-1}\| = \|S_n^{-1} P_n\| = \|Q_n Q_n^*\| \le \|Q_n\|^2.$$

Therefore $\sup_n \|S_n^{-1}\| < \infty$.

(iii)$\to$ (ii). Let $\{\Lambda_i\}_{i=1}^{\infty}$ be a conditional g-Riesz frame and $\{g_i\}_{i=1}^{\infty} \in (\sum_{i=1}^{\infty} \oplus \mathcal{H}_i)_{\ell_2}$ with $\sum_{i=1}^{\infty} \Lambda_i^* g_i = 0$. Since the sequence $\{\|S_n^{-1}\|\}_n$ is bounded and

$$\left\| S_n^{-1} \sum_{i=1}^{n} \Lambda_i^* g_i \right\| \le \|S_n^{-1}\| \left\| \sum_{i=1}^{n} \Lambda_i^* g_i \right\|,$$

we get $S_n^{-1} \sum_{i=1}^{n} \Lambda_i^* g_i \to 0$ as $n \to \infty$. □

Acknowledgments

The authors would like to thank Professor F Bagarello for reading the earlier versions of this manuscript. The authors would also like to thank the referee for his suggestions.

References

[1] Casazza P G and Kutyniok G, Frames of subspaces, *Contemp. Math.* **345** (2004) 87–113

[2] Casazza P G and Christensen O, Approximation of the frame coefficients using finite-demensional methods, *J. Electronic Imaging* **4** (1997) 479–483

[3] Christensen O, An Introduction to Frames and Riesz bases (Birkhauser) (2002)

[4] Christensen O, Frames and pseudo-inverses, *J. Math. Anal. Appl.* **195** (1995) 401–414

[5] Christensen O, Frames and the projection method, *Appl. Comput. Harmon. Anal.* **1** (1993) 50–53

[6] Christensen O, Frames containing a Riesz basis and approximation of the frame coefficients using finite-dimensional methods, *J. Math. Anal. Appl.* **199** (1996) 256–270

[7] Christensen O and Eldar Y C, Oblique dual frames and shift invariant-spaces, *Appl. Comput. Harmon. Anal.* **17** (2004) 48–68

[8] Duffin R J and Schaeffer A C, A class of nonharmonic Fourier series, *Trans. Am. Math. Soc.* **72** (1952) 341–366

[9] Fornasier M, Decompositions of Hibert space: local construction of global frames, Proc. Int. Conf. on Constructive Function Theory, Varna (ed.) B Bojanov (2002) (DARBA, Sofia) (2003) pp. 275–281

[10] Li S and Ogawa H, Pseudoframes for subspaces with applications, *J. Fourier Anal. Appl.* **10** (2004) 409–431

[11] Najati A, Faroughi M H and Rahimi A, G-frames and stability of g-frames in Hilbert spaces, *Meth. Func. Anal. Top.* **4** (2008) 271–286

[12] Sun W, G-frames and g-Riesz bases, *J. Math. Anal. Appl.* **322** (2006) 437–452