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Reduced multiplication modules

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Abstract. An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. As defined for a commutative ring R, an R-module M is said to be reduced if the intersection of prime submodules of M is zero. The prime spectrum and minimal prime submodules of the reduced module M are studied. Essential submodules of M are characterized via a topological property. It is shown that the Goldie dimension of M is equal to the Souslin number of $\mathrm{Spec}(M)$. Also a finitely generated module M is a Baer module if and only if $\mathrm{Spec}(M)$ is an extremally disconnected space; if and only if it is a CS-module. It is proved that a prime submodule N is minimal in M if and only if for each $x \in N$, $\mathrm{Ann}(x) \not\subseteq (N : M)$. When M is finitely generated; it is shown that every prime submodule of M is maximal if and only if M is a von Neumann regular module (VNM); i.e., every principal submodule of M is a summand submodule. Also if M is an injective R-module, then M is a VNM.

Keywords. Multiplication module; reduced module; minimal prime submodule; Zariski topology; extremally disconnected.

1. Introduction

In this paper all rings are commutative with identity and all modules are unitary. An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. Multiplication modules and ideals have been investigated in [2,3,9,12,15,18] and others. A proper submodule P of M is called prime if $rx \in P$, for $r \in R$ and $x \in M$ implying $r \in (P : M)$ or $x \in P$. In this case, $\mathfrak{p} = (P : M)$ is a prime ideal and we say P is a $\mathfrak{p}$-prime submodule of M. We use $\mathrm{Spec}(M)$ for the spectrum of prime submodules of M. For any submodule N of an R-module M, we define $\mathrm{V}(N)$ to be the set of all prime submodules of M containing N. We define the M-radical of a submodule N of an R-module M (denoted $\mathrm{rad}\, N$) to be the intersection of all prime submodules of M containing N, in fact $\mathrm{rad}\, N = \cap \mathrm{V}(N)$. Of course, $\mathrm{V}(M)$ is just the empty set and $\mathrm{V}(0)$ is $\mathrm{Spec}(M)$. Note that for any family of submodules N_λ $(\lambda \in \Lambda)$ of M,

$$\bigcap_{\lambda\in\Lambda} \mathrm{V}(N_\lambda) = \mathrm{V}\left(\sum_{\lambda\in\Lambda} N_\lambda\right).$$

Thus if $\zeta(M)$ denotes the collection of all subsets $\mathrm{V}(N)$ of $\mathrm{Spec}(M)$, then $\zeta(M)$ contains the empty set and $\mathrm{Spec}(M)$, and $\zeta(M)$ is closed under arbitrary intersection. We shall say that M is a module with a Zariski topology, or a top module for short, if $\zeta(M)$ is closed under finite union, i.e., for any submodules N and N' of M, there exists a submodule N''

of M such that $\mathrm{V}(N) \cup \mathrm{V}(N') = \mathrm{V}(N'')$; in this case $\zeta(M)$ satisfies the axioms for the closed subsets of a topological space. It is well-known that every multiplication module is a top module; and the converse holds, if the module is finitely generated (see [12]). The operators 'cl' and 'int' denote the closure and the interior.

An R-module M is said to be reduced if the intersection of all prime submodules of M is equal to zero. If M is a multiplication module, then by Lemma 2.1 and Theorem 2.12 in [3] we have

$$\operatorname{rad} M = \cap \operatorname{Spec}(M) = \sqrt{\operatorname{Ann}(M)}M.$$

Hence M is reduced if and only if $\operatorname{Ann}(M)$ is semiprime; and if and only if $\bar{R} = R/\operatorname{Ann}(M)$ is a reduced ring. For example, every faithful multiplication module over a reduced ring is a reduced module. In particular, every reduced ring R can be considered as a reduced R-module.

Throughout this paper M is a non-zero multiplication R-module and $\bar{R} = R/\operatorname{Ann}(M)$, unless stated otherwise.

In §2, we study the relation between the topological properties of $\operatorname{Spec}(M)$ and the algebraic properties of M.

A set $\{N_\lambda\}_{\lambda\in\Lambda}$ of non-zero submodules in M is said to be independent if $N_\lambda \cap (\sum_{\lambda\neq\lambda'\in\Lambda} N_{\lambda'}) = 0$, i.e., $\sum_{\lambda\in\Lambda} N_\lambda = \bigoplus_{\lambda\in\Lambda} N_\lambda$. The Goldie dimension of M, denoted by $\dim M$ is the smallest cardinal number c such that every independent set of non-zero submodules in M has cardinality less than or equal to c. Also the smallest cardinal number c such that every family of pairwise disjoint non-empty open subsets of a topological space X has cardinality less than or equal to c, is called the cellularity of the space X and denoted by $\mathrm{c}(X)$, see [4]. We show for an R-module M, the cellularity number of $\operatorname{Spec}(M)$ is equal to the Goldie dimension of M.

A submodule N of M is called a closed submodule if it is not essential in a larger submodule, and a module M is said to be a CS-module if every closed submodule is a summand submodule, see [16]. An element $e \in R$ is called a M-idempotent in R if $e^2 \equiv e \pmod{\operatorname{Ann}(M)}$. An R-module M is said to be Baer if for any subset X in M, $\operatorname{Ann}(X)M = eM$, for some M-idempotent $e \in R$. Clearly, every Baer ring R can be considered as a Baer module over R. A topological space is said to be extremally disconnected if every closed set has a closed interior or equivalently, every open set has an open closure, see [6]. We prove that M is a Baer module if and only if $\operatorname{Spec}(M)$ is an extremally disconnected space; if and only if it is a CS-module.

Minimal prime ideals of a reduced commutative ring have been investigated in [7,11] and others. By Corollary 3.3, P is a minimal prime submodule of M if and only if $\bar{\mathfrak{p}}$ is a minimal prime ideal of $\bar{R}$, where $\mathfrak{p} = (P : M)$; in particular, if M is a faithful R-module then P is a minimal prime submodule of M if and only if $\mathfrak{p}$ is a minimal prime ideal of R, see Proposition 3.1 in [17]. In §3, we study minimal prime submodules of M. It is well-known that in a reduced ring R, the prime ideal $\mathfrak{p}$ is minimal if and only if for each $x \in \mathfrak{p}$, $\operatorname{Ann}(x) \nsubseteq \mathfrak{p}$. We generalize this theorem for reduced multiplication modules: A prime submodule N is minimal in M if and only if for each $x \in N$, $\operatorname{Ann}(x) \nsubseteq (N : M)$. We prove that a finitely generated submodule N of M is contained in a minimal prime submodule of M if and only if $\operatorname{Ann}(M) \subsetneq \operatorname{Ann}(N)$.

The notion of a von-Neumann regular ring, VNR, plays a key role in [11]. We generalize this concept for multiplication modules. The definition we use is that every cyclic submodule of M is a summand submodule. We show that every injective R-module is a

VNM. Also if M is a finitely generated R-module, then M is a VNM if and only if M is reduced and every prime submodule of M is maximal.

2. The prime spectrum of reduced multiplication modules

In this section we study essential submodules of M, the topological properties of $\mathrm{Spec}(M)$, the annihilator of subsets of M, and the Goldie dimension of M.

We first give the following lemmas.

Lemma 2.1. *Let P be a proper submodule of M. The following statements are equivalent:*

(1) *P is prime.*
(2) *$(P : M)$ is a prime ideal of R.*
(3) *$P = \mathfrak{p}M$ for some prime ideal $\mathfrak{p}$ of R with* $\mathrm{Ann}(M) \subseteq \mathfrak{p}$.

Proof. See Corollary 2.11 of [3]. □

Lemma 2.2. *Let I be an ideal of R and let N be a submodule of M. Then*

$$\mathrm{V}(N) \cup \mathrm{V}(IM) = \mathrm{V}(IN) = \mathrm{V}(N \cap IM).$$

Proof. See Lemma 3.1 of [12]. □

Lemma 2.3. *Let M be reduced, N a submodule of M and I =* $\mathrm{Ann}(N)$.

(1) $N \cap IM = 0$.
(2) $\mathrm{Ann}(N + IM) = \mathrm{Ann}(M)$.

Proof.

(1) By Lemma 2.2, $\mathrm{V}(N \cap IM) = \mathrm{V}(IN) = \mathrm{V}(0) = \mathrm{Spec}(M)$. Therefore $N \cap IM = 0$.
(2) Suppose that $r \in \mathrm{Ann}(N+IM)$. Since $rN = 0$, $r \in I$. Therefore $r^2 \in rI \subseteq \mathrm{Ann}(M)$, and this implies that $r \in \mathrm{Ann}(M)$, for M is reduced. □

For any subset X of M, we define $\mathrm{D}(X) = \mathrm{Spec}(M)\backslash\mathrm{V}(X)$.

Lemma 2.4. *Let M be reduced, and let X be a subset of M.*

(1) $\sqrt{\mathrm{Ann}(X)} = \mathrm{Ann}(X)$.
(2) $\mathrm{Ann}(X)M = \cap\mathrm{D}(X)$.
(3) $\mathrm{int}\,\mathrm{V}(X) = \mathrm{D}(\mathrm{Ann}(X)M)$.

Proof.

(1) Suppose that $r \in \sqrt{\mathrm{Ann}(X)}$. Then there exists $n > 0$ such that $\mathrm{V}(rX) = \mathrm{V}(r^nX) = \mathrm{Spec}(M)$. Thus $r \in \mathrm{Ann}(X)$.

(2) Suppose that $P \in \mathrm{D}(X)$. Then $\mathrm{Ann}(X) \subseteq (P : M)$. This implies that $\mathrm{Ann}(X)M \subseteq P$, i.e., $\mathrm{Ann}(X)M \subseteq \cap\mathrm{D}(X)$. Conversely, if $y \in \cap\mathrm{D}(X)$, then $Ry = IM$, for some ideal I of R, and Lemma 2.2 implies that

$$\mathrm{Spec}(M) = \mathrm{V}(Ry) \cup \mathrm{V}(X) = \mathrm{V}(IM) \cup \mathrm{V}(\langle X\rangle) = \mathrm{V}(I\langle X\rangle).$$

Hence $I \subseteq \mathrm{Ann}(X)$, i.e., $y \in \mathrm{Ann}(X)M$.

(3) This follows from (2):

$$\mathrm{int}\,\mathrm{V}(X) = \mathrm{Spec}(M)\backslash\mathrm{cl}\,\mathrm{D}(X) = \mathrm{D}(\cap\mathrm{D}(X)) = \mathrm{D}(\mathrm{Ann}(X)M).$$

□

The following theorem is a generalization of Lemma 2.1 in [14].

Theorem 2.5. *Let M be reduced, and let $N \subseteq L$ be two submodules of M. The following statements are equivalent*:

(1) *N is essential in L.*
(2) $\mathrm{Ann}(N) = \mathrm{Ann}(L)$.
(3) $\mathrm{int}\,\mathrm{V}(N) = \mathrm{int}\,\mathrm{V}(L)$.

Proof.

(1) $\Longrightarrow$ (2). Since $N \subseteq L$, clearly $\mathrm{Ann}(N) \supseteq \mathrm{Ann}(L)$. Let $r \in \mathrm{Ann}(N)$ and $r \notin \mathrm{Ann}(L)$. Then $rL \neq 0$ implies that there exists an element $x \in L$ such that $0 \neq rx \in N \cap rL$. Therefore $r^2x = 0$, and this implies that $\mathrm{V}(rx) = \mathrm{V}(r^2x) = \mathrm{Spec}(M)$, i.e., $rx = 0$, which is impossible.
(2) $\Longrightarrow$ (3). This follows from Lemma 2.4(3).
(3) $\Longrightarrow$ (1). Let L' be a submodule contained in L in which $L' \cap N = 0$. We must show that $L' = 0$. Now by Lemma 2.2, we have

$$\mathrm{Spec}(M) = \mathrm{V}(L' \cap N) = \mathrm{V}(J'N),$$

where $J' = (L' : M)$. This implies that $J' \subseteq \mathrm{Ann}(N)$, so by Lemma 2.4(3),

$$\mathrm{D}(J'M) \subseteq \mathrm{int}\,\mathrm{V}(N) = \mathrm{int}\,\mathrm{V}(L).$$

Thus $\mathrm{V}(J'L) = \mathrm{V}(J'M) \cup \mathrm{V}(L) = \mathrm{Spec}(M)$, and hence $J' \subseteq \mathrm{Ann}(L) \subseteq \mathrm{Ann}(L')$. Consequently, $J'L' = J'^2M = 0$, i.e., $J'^2 \subseteq \mathrm{Ann}(M)$. But Lemma 2.4(1) implies that $J' \subseteq \mathrm{Ann}(M)$, and therefore $L' = J'M = 0$. □

The following characterizes the essential submodules of M via a topological property.

COROLLARY 2.6

Let M be reduced, and let N be a non-zero submodule of M. Then N is an essential submodule in M if and only if $\mathrm{int}\,\mathrm{V}(N) = \emptyset$.

The following theorem characterizes the Goldie dimension of reduced modules via a topological property.

Theorem 2.7. *In a reduced module M*, $\dim M = \mathrm{c}(\mathrm{Spec}(M))$.

Proof. Let $\dim M = c$ and $\bigoplus_{\lambda\in\Lambda} N_\lambda$ be a direct sum of submodules in M, where $|\Lambda|$, the cardinality of Λ, is less than or equal to c. Now for each $\lambda \neq \lambda' \in \Lambda$, $\mathrm{D}(N_\lambda) \cap \mathrm{D}(N_{\lambda'}) = \emptyset$, and this implies that $\mathcal{G} = \{\mathrm{D}(N_\lambda) : \lambda \in \Lambda\}$ is a collection of disjoint open sets in $\mathrm{Spec}(M)$, i.e., $\mathrm{c}(\mathrm{Spec}(M)) \geq c$. Now let $\{G_\lambda : \lambda \in \Lambda\}$ be any collection of disjoint open sets in $\mathrm{Spec}(M)$. Then for any $\lambda \in \Lambda$, there exists submodule N_λ such that $G_\lambda = \mathrm{D}(N_\lambda)$. We claim that $\{N_\lambda\}_{\lambda\in\Lambda}$ is an independent set of non-zero submodules in M. Let $x \in N_\lambda \cap (\sum_{\lambda\neq\lambda'\in\Lambda} N_{\lambda'})$. Then

$$x = x_{\lambda_1} + \cdots + x_{\lambda_n},$$

where $x_{\lambda_k} \in N_{\lambda_k}$, $\lambda \neq \lambda_k$ for all $k = 1, 2, ..., n$. Now we have

$$\begin{aligned}\mathrm{D}(x) \subseteq \mathrm{D}(N_\lambda) \cap \mathrm{D}(N_{\lambda_1} + \cdots + N_{\lambda_n}) &= \mathrm{D}(N_\lambda) \cap (\mathrm{D}(N_{\lambda_1}) \cup \cdots \cup \mathrm{D}(N_{\lambda_n}))\\ &= (G_\lambda \cap G_{\lambda_1}) \cup \cdots \cup (G_\lambda \cap G_{\lambda_n}) = \emptyset.\end{aligned}$$

Therefore $x = 0$. This means that $\dim M = c \geq |\Lambda|$, i.e., $c \geq \mathrm{c}(\mathrm{Spec}(M))$. □

Lemma 2.8. *Let M be a finitely generated reduced R-module, and let N be a summand submodule of M. Then there exists a M-idempotent $e \in R$ such that $N = eM$.*

Proof. Suppose that $M = N \oplus N'$. So there are ideals I and I' such that $N = IM$ and $N' = I'M$. Hence $M = (I + I')M$ implies that $(e + e' - 1)M = 0$ for some $e \in I$ and $e' \in I'$. Then $(e^2 - e)M = ee'M \in N \cap N' = 0$, i.e., $e^2 \equiv e$ (mod $\mathrm{Ann}(M)$). Now for any $x \in N$ we have

$$x - ex = e'x \in N \cap N' = 0.$$

This implies that $N = eM$. □

Lemma 2.9. *Let M be a finitely generated reduced R-module. Then $\mathcal{A}$ is a clopen* (*closed and open*) *subset of* $\mathrm{Spec}(M)$ *if and only if there exists a M-idempotent $e \in R$ such that* $\mathcal{A} = \mathrm{V}(eM)$.

Proof. Suppose $\mathcal{A}$ is a clopen subset of $\mathrm{Spec}(M)$ and $N = \cap\mathcal{A}$ and $N' = \cap\mathcal{A}^c$, then $\mathcal{A} = \mathrm{cl}\mathcal{A} = \mathrm{V}(\cap\mathcal{A}) = \mathrm{V}(N)$ and $\mathcal{A}^c = \mathrm{V}(N')$ and $\mathrm{V}(N) \cap \mathrm{V}(N') = \emptyset$. Hence $M = N \oplus N'$, and by Lemma 2.8, there exists a M-idempotent $e \in R$ such that $N = eM$. The converse is trivial. □

It is proved that a reduced ring R is a Baer ring if and only if $\mathrm{Spec}(R)$ is an extremally disconnected space, see [5]. Now, we generalize this fact to finitely generated multiplication modules.

Theorem 2.10. *Let M be a finitely generated reduced R-module. The following statements are equivalent*:

(1) $\mathrm{Spec}(M)$ *is an extremally disconnected space.*
(2) *M is a Baer module.*
(3) *Every non-zero submodule in M is essential in a summand submodule.*
(4) *M is a CS-module.*

Proof.

(1) $\Longrightarrow$ (2). Let X be any subset in M. By (1) and Lemma 2.4(3), $\mathrm{int}\,\mathrm{V}(X) = \mathrm{D}(\mathrm{Ann}(X)M)$ is a clopen subset of $\mathrm{Spec}(M)$. Therefore by Lemma 2.9, there exists a M-idempotent $e \in R$ such that $\mathrm{D}(\mathrm{Ann}(X)M) = \mathrm{V}(eM)$. Now Theorem 4 in [13] and Lemma 2.4(1), (2) implies that

$$\begin{aligned}\mathrm{Ann}(X)M &= \sqrt{\mathrm{Ann}(X)}M = \mathrm{rad}\,\mathrm{Ann}(X)M = \cap\mathrm{V}(\mathrm{Ann}(X)M)\\ &= \cap\mathrm{D}(eM) = \mathrm{Ann}(eM)M = (1-e)M.\end{aligned}$$

(2) $\Longrightarrow$ (3). Let N be a submodule of M. Then by (2), we have

$$\mathrm{Ann}(N)M = eM = \mathrm{Ann}((1-e)M)M$$

for some M-idempotent $e \in R$. So $\mathrm{Ann}(N) = \mathrm{Ann}((1-e)M)$, and hence by Theorem 2.5, N is essential in $(1-e)M$.
(3) $\Longrightarrow$ (4). Let N be a closed submodule. Then by (3), there exists a M-idempotent $e \in R$ in which N is essential in eM. But since N is closed we must have $N = eM$.
(4) $\Longrightarrow$ (1). We note that (4) immediately implies (2), for if X is a subset of M, then the submodule $N = \mathrm{Ann}(X)M$ is a closed submodule in M. To see this, we let N be essential in a larger submodule N', and let $N'' = (N' : M)\langle X\rangle$. Now $(N' : M) \nsubseteq \mathrm{Ann}(X)$ implies that $0 \neq N'' \subseteq N'$, and so $N'' \cap N \neq 0$. But by Lemma 2.3, $N'' \cap N \subseteq \langle X\rangle \cap N = 0$, which is impossible. This shows that $N = \mathrm{Ann}(X)M$ is a closed submodule and by (4), there exists a M-idempotent $e \in R$ such that $N = eM$.

Now we assume (2) and we show that for any closed set $\mathcal{F}$, the interior of $\mathcal{F}$ is closed. Since $\mathcal{F}$ is closed in $\mathrm{Spec}(M)$, then $\mathcal{F} = \mathrm{V}(N)$, for some submodule N of M. Hence by (2), there exists $e \in R$ in which

$$\mathrm{Ann}(N)M = (1-e)M,\ e^2 - e \in \mathrm{Ann}(M).$$

Therefore by Lemma 2.4(3), we have

$$\mathrm{int}\,\mathcal{F} = \mathrm{int}\,\mathrm{V}(N) = \mathrm{D}(\mathrm{Ann}(N)M) = \mathrm{D}((1-e)M) = \mathrm{V}(eM),$$

i.e., the interior of $\mathcal{F}$ is closed. $\square$

Let $\mathrm{B}(\bar{R})$ be the set of idempotents in $\bar{R}$. It is well-known that $B(\bar{R})$ can be made a Boolean algebra. Also it should be recalled that $B(\bar{R})$ is complete if either every subset has an infimum or every subset has a supremum. By Theorem 9 in [1] and Theorm 2.10, we have as follows.

COROLLARY 2.11

Let M be a finitely generated reduced R-module. Then M is a Baer module if and only if for each $x \in M$, $\mathrm{Ann}(x)M$ *is a summand submodule and* $\mathrm{B}(\bar{R})$ *is a complete Boolean algebra.*

COROLLARY 2.12

Let M be a finitely generated reduced R-module. Then M is a Baer module if and only if for each $x \in M$, $\mathrm{Ann}(x)M$ *is a summand submodule and the union of any collection of clopen subsets of* $\mathrm{Spec}(M)$ *is clopen.*

Proof. We note that $\mathrm{Spec}(M)$ is homeomorphic to $\mathrm{Spec}(\bar{R})$; therefore by Corollary 2.11, it is sufficient to show that the union of any collection of clopen subsets of $\mathrm{Spec}(\bar{R})$ is clopen if and only if $\mathrm{B}(\bar{R})$ is a complete Boolean algebra.

Suppose the union of any collection of clopen subsets of $\mathrm{Spec}(M)$ is clopen. Let $B = \{\bar{e}_\lambda : \lambda \in \Lambda\}$ be any subset of $B(\bar{R})$. Since $\mathrm{V}(\bar{e}_\lambda)$ is a clopen subset of $\mathrm{Spec}(\bar{R})$, $\forall \lambda \in \Lambda$, $\mathcal{A} = \bigcup_{\lambda \in \lambda} \mathrm{V}(\bar{e}_\lambda)$ is clopen in $\mathrm{Spec}(\bar{R})$. So there exists $\bar{e} \in \bar{R}$ such that $\mathcal{A} = \mathrm{V}(\bar{e})$. Obviously $\bar{e}$ is the infimum of $\mathrm{B}(\bar{R})$. Conversely, let $\{\mathcal{A}_\lambda : \lambda \in \Lambda\}$ be any collection of clopen sets in $\mathrm{Spec}(\bar{R})$. Then by Lemma 2.9, there are M-idempotent elements $\bar{e}_\lambda \in \bar{R}$ such that $\mathcal{A}_\lambda = \mathrm{V}(\bar{e}_\lambda)$. Let $\bar{e} = \inf\{\bar{e}_\lambda : \lambda \in \Lambda\}$. We have $\mathrm{V}(\bar{e}) = \bigcup_{\lambda \in \Lambda} \mathcal{A}_\lambda$. Therefore $\bigcup_{\lambda \in \lambda} \mathcal{A}_\lambda$ is clopen. □

3. Minimal prime submodules

In this section we generalize some results in [11]. We use $\mathrm{Min}(M)$ for the spectrum of minimal prime submodules of M which is a subspace of $\mathrm{Spec}(M)$. For each $P \in \mathrm{Spec}(M)$, we define $\mathrm{nil}\, P = \cap P'$, where P' ranges over all prime submodules of M contained in P.

Let $\mathfrak{p}$ be a prime ideal of R. Then the set

$$O_{\mathfrak{p}} = \{a \in R : \mathrm{Ann}(a) \nsubseteq \mathfrak{p}\}$$

is an ideal of R contained in $\mathfrak{p}$. It is easy to see that $\sqrt{O_{\mathfrak{p}}} = \mathrm{nil}\, \mathfrak{p}$, so if $\mathfrak{p}$ is a minimal prime ideal of R, $\mathfrak{p} = \sqrt{O_{\mathfrak{p}}}$; in particular, $\mathfrak{p} = O_{\mathfrak{p}}$, when R is reduced, see Proposition 1.1 in [11].

Now, we generalize the above concept to multiplication modules.

DEFINITION 3.1

Let P be a $\mathfrak{p}$-prime submodule of M. We define

$$O_P = \{x \in M : \mathrm{Ann}(x) \nsubseteq \mathfrak{p}\}.$$

It is easy to see that $O_P \subseteq P$.

We consider that $\bar{\mathfrak{p}}$ is the image $\mathfrak{p}$ in $\bar{R}$.

Theorem 3.2. *Let P be a $\mathfrak{p}$-prime submodule of M.*

$$\text{rad } O_P = \text{ nil } P = \sqrt{O_{\bar{\mathfrak{p}}}}M.$$

In particular, if M is reduced, then rad $O_P = O_P$.

Proof. Suppose that P' is a prime submodule of M contained in P. For any $x \in O_P$, we have $\text{Ann}(x) \nsubseteq (P' : M)$. Therefore $x \in P'$, and this implies that rad $O_P \subseteq$ nil P. Also by Lemma 2.1 and Corollary 1.7 in [3] we have

$$\text{nil } P = (\text{ nil } \bar{\mathfrak{p}})M = \sqrt{O_{\bar{\mathfrak{p}}}}M.$$

Now let $\bar{a} \in \sqrt{O_{\bar{\mathfrak{p}}}}$ and $I = (O_P : M)$. Then $\text{Ann}(a^n M) \nsubseteq \mathfrak{p}$. This implies that $a \in \sqrt{I}$. Thus by Lemma 1 in [13] we have

$$\sqrt{O_{\bar{\mathfrak{p}}}}M \subseteq \sqrt{I}M \subseteq \text{rad } IM = \text{rad } O_P,$$

and this completes the proof of the first part.

For the second part, suppose that M is a reduced module. Then $\bar{R} = R/\text{Ann}(M)$ is a reduced ring, and this implies that $\sqrt{O_{\bar{\mathfrak{p}}}} = O_{\bar{\mathfrak{p}}}$. Now we have

$$\text{rad } O_P = \sqrt{O_{\bar{\mathfrak{p}}}}M = O_{\bar{\mathfrak{p}}}M \subseteq O_P.$$

□

COROLLARY 3.3

Let P be a submodule of M. Then P is a minimal prime submodule of M if and only if $P =$ rad O_P. *In particular, if M is reduced, then P is a minimal prime submodule of M if and only if $P = O_P$.*

DEFINITION 3.4

An element $x \in M$ is called quasi-regular if $\text{Ann}(x) = \text{Ann}(M)$; and it is called non-quasi-regular if $\text{Ann}(x) \neq \text{Ann}(M)$.

We denote $\mathrm{E}_{\bar{R}}(M)$ for the injective envelope of M as $\bar{R}$-module.

Theorem 3.5. *Let M be reduced and let $\{P_\lambda\}_{\lambda \in \Lambda}$ be the set of $\mathfrak{p}_\Lambda$-minimal prime submodules of M.*

(1) *$M_{\mathfrak{p}_\lambda}$ is a simple R-module, and injective $\bar{R}$-module.*
(2) *$\mathrm{E}_{\bar{R}}(M)$ is a direct summand of $\Pi_{\lambda \in \Lambda} M_{\mathfrak{p}_\lambda}$.*
(3) *$\bigcup_{\lambda \in \Lambda} P_\lambda$ is the set of all non-quasi-regular elements of M.*

Proof.

(1) By the previous corollary, $P_{\mathfrak{p}_\lambda} = 0$, for any $\lambda \in \Lambda$. On the other hand, by Proposition 1 in [10], $P_{\mathfrak{p}_\lambda}$ is the only prime submodule of $M_{\mathfrak{p}_\lambda}$. So $M_{\mathfrak{p}_\lambda}$ is a simple R-module. By Proposition 1.1 in [11], $\bar{R}_{\mathfrak{p}_\lambda}$ is a field. Hence $M_{\mathfrak{p}_\lambda}$ is an injective $\bar{R}$-module.
(2) It follows from (1) that $\Pi_{\lambda\in\Lambda} M_{\mathfrak{p}_\lambda}$ is an injective $\bar{R}$-module; and that the canonical map $M \longrightarrow \Pi_{\lambda\in\Lambda} M_{\mathfrak{p}_\lambda}$ has kernel equal to $\bigcap_{\lambda\in\Lambda} P_\lambda = 0$, and hence is a monomorphism. Thus the canonical map extends to a monomorphism

$$\mathrm{E}_{\bar{R}}(M) \longrightarrow \Pi_{\lambda\in\Lambda} M_{\mathfrak{p}_\lambda}.$$

(3) It follows from Corollary 3.3 that every element of $\bigcup_{\lambda\in\Lambda} P_\lambda$ is a non-quasi-regular in M. Conversely, let x be a non-quasi-regular in M. Since $\bigcap_{\lambda\in\Lambda} \mathfrak{p}_\lambda = \mathrm{Ann}(M)$, there exists $\mathfrak{p}_{\lambda'}$ such that $\mathrm{Ann}(x) \nsubseteq \mathfrak{p}_{\lambda'}$ and hence $x \in P_{\lambda'}$. □

Theorem 3.6. *Let M be reduced, and let N be a finitely generated submodule of M. Then N is contained in a minimal prime submodule of M if and only if* $\mathrm{Ann}(M) \subsetneq \mathrm{Ann}(N)$.

Proof. Let $N = Rx_1 + \cdots + Rx_n$. Suppose that $N \subseteq P$, a $\mathfrak{p}$-minimal prime submodule of M. So by Corollary 3.3, there exist elements $r_i \in R\setminus\mathfrak{p}$ such that $r_i x_i = 0$ for all $1 \le i \le n$. Let $r = r_1 r_2 \ldots r_n$. Then we have

$$r \in \mathrm{Ann}(N)\setminus\mathfrak{p} \subseteq \mathrm{Ann}(N)\setminus\mathrm{Ann}(M).$$

Conversely, suppose that $\mathrm{Ann}(M) \subsetneq \mathrm{Ann}(N)$. Since M is reduced, there exists a minimal prime ideal $\mathfrak{p} \in \mathrm{V}(\mathrm{Ann}(M))$ such that $\mathrm{Ann}(N) \nsubseteq \mathfrak{p}$. Therefore $\mathfrak{p}M$ is a $\mathfrak{p}$-minimal prime submodule of M and we have $N \subseteq \mathfrak{p}M$. □

DEFINITION 3.7

An R-module M is said to be a von-Neumann regular module (VNM) if every cyclic submodule of M is a summand submodule of M, see page 105 of [8].

Theorem 3.8. *Let M be finitely generated. Then M is a VNM if and only if M is reduced and every prime submodule of M is minimal. In this case every submodule of M is an intersection of prime submodule of M.*

Proof. Assume that M is a VNM. Let N be a submodule of M and $x \in \mathrm{rad}\,N$. By Lemma 2.8 there exists a M-idempotent $e \in R$ such that $Rx = eM$. Hence by Theorem 4 in [13], $eM \subseteq \mathrm{rad}\,N = \sqrt{(N:M)}M$. Therefore Result 2 in [13] implies that $e \in \sqrt{(N:M)}$. Consequently, $e^n \in (N:M)$, for some $n > 0$. Therefore $x \in eM = e^n M \subseteq N$, showing that $N = \mathrm{rad}\,N$. In particular, taking $N = 0$, we see that M is reduced. Now let N be a prime submodule of M. Then $1 - e \in \mathrm{Ann}(x)\setminus(N:M)$. Hence by Corollary 3.3, N is a minimal prime submodule of M.

Conversely, suppose that M is reduced and every prime submodule of M is minimal. Let $0 \neq x \in M$ and $I = \mathrm{Ann}(x)$. Now by Lemma 2.2(1), $Rx \cap IM = 0$ and $\mathrm{Ann}(Rx + IM) = \mathrm{Ann}(M)$. Also by Theorem 3.6, $Rx + IM$ is not contained in any minimal prime submodule of M. Therefore $Rx + IM = M$. Hence Rx is a direct summand of M, and hence M is a VNM. □

Lemma 3.9 (*Chinese remainder theorem*). *Let M be finitely generated, and let $N_1, ..., N_n$ be a finite set of* (*non-trivial*) *submodules of M. If $N_i + N_j = M$ whenever $i \neq j$, then*

$$M/\cap N_i \simeq M/N_1 \oplus \cdots \oplus M/N_n.$$

Proof. For a start, we define a mapping $f : M \longrightarrow M/N_1 \oplus \cdots \oplus M/N_n$ by $f(x) = (x + N_1, ..., x + N_n)$. The reader can painlessly supply a proof that f is a R-homomorphism with $\ker f = \cap N_i$. Our problem is to show that f is onto. Put $I_i = (N_i : M)$ for $i = 1, ..., n$. Fix the index j for the moment. Using the fact that $M = N_i + N_j = (I_i + I_j)M$ when $i \neq j$, there exist elements $a_i \in I_i$ and $a_j \in I_j$ with $a_i + a_j \equiv 1 \pmod{\mathrm{Ann}(M)}$. This ensures that the product

$$r_j = a_1 a_2 ... a_{j-1} a_{j+1} ... a_n \in \bigcap\nolimits_{i \neq j} I_i.$$

Now, pick arbitrary elements $x_i \in M$ $(i = 1, ..., n)$; our contention is that

$$f(x) = (x_1 + N_1, ..., x_n + N_n),$$

where $x = \sum r_i x_i$. To see this, observe that we may write $x + N_j$ as

$$x + N_j = \sum_{i \neq j} (r_i x_i + N_j) + (r_j x_j + N_j) = r_j x_j + N_j.$$

But $r_j x_j + N_j = x_j + N_j$, and the proof is complete. □

DEFINITION 3.10

Let M be an R-module. An element $r \in R$ is called a zero divisor on M, if $rm = 0$, for some $0 \neq m \in M$. We denote the set of zero divisors on M by $\mathrm{Z}(M)$; in fact

$$\mathrm{Z}(M) = \{r \in R : \ rm = 0, \ \text{for some } 0 \neq m \in M\}.$$

Remark. It is well-known that in a reduced ring R, $\mathrm{Z}(R) = \cup \mathfrak{p}$, where $\mathfrak{p}$ varies over $\mathrm{Min}(R)$, see Corollary 2.4 in [7]. This fact is generalizable for multiplication modules. Let M be a finitely generated reduced multiplication R-module. Then

$$\mathrm{Z}(M) = \cup\{\mathfrak{p} \in \min(R) : \ \mathrm{Ann}(M) \subseteq \mathfrak{p}\}.$$

To see this, let $r \in \mathrm{Z}(M)$. Then there exists $P \in \min(M)$ and $m \in M \setminus P$ such that $rm = 0$. Hence $r \in \mathfrak{p} = (P : M) \in \min(R)$. Conversely, suppose that $\mathfrak{p} \in \min(R)$, $\mathrm{Ann}(M) \subseteq \mathfrak{p}$ and $r \in \mathfrak{p}$. By Theorem 3.6, there exists $r' \in \mathrm{Ann}(rM) \setminus \mathrm{Ann}(M)$. Thus $r'm \neq 0$, for some $m \in M$. This implies that $r(r'm) = 0$, i.e., $r \in \mathrm{Z}(M)$. We now have

$$\mathrm{Z}(M) = \cup\{\mathfrak{p} \in \min(R) : \ \bar{\mathfrak{p}} \in \min(\bar{R})\} = \{r \in R : \ \bar{r} \in \mathrm{Z}(\bar{R})\}.$$

Let S be the set of non-zero divisors on M, i.e., $S = R\backslash Z(M)$, Obviously, S is a multiplicative closed subset of R. We call $S^{-1}M$ the total quotient module of M, and we denote it by $\mathrm{T}(M)$. This is a generalization of total quotient rings, see [7].

Theorem 3.11. *Let M be finitely generated, and let $\{P_1, P_2, ..., P_n\}$ be a finite set of distinct minimal prime submodules of M. Let $S = R \setminus \cup_{i=1}^{n} \mathfrak{p}_i$, where $\mathfrak{p}_i = (P_i : M)$; then*

$$S^{-1}M \simeq M_{\mathfrak{p}_1} \oplus \cdots \oplus M_{\mathfrak{p}_n}.$$

Proof. $\{S^{-1}P_1, ..., S^{-1}P_n\}$ is the set of all prime submodules of $S^{-1}M$, and each of them is both maximal and minimal in $S^{-1}M$. Moreover $(S^{-1}M)_{\mathfrak{p}_i} \simeq M_{\mathfrak{p}_i}$, for each $1 \le i \le n$. Thus without loss of generality, we can assume that $\{P_1, P_2, ..., P_n\}$ is the set of all prime submodules of M, and each of them is both maximal and minimal in M. Clearly, for any $1 \le i \le n$, $M/O_{P_i} = M_{\mathfrak{p}_i}$, and by Theorem 3.2, P_i is the only prime submodule of M containing O_{P_i}. Therefore $O_{P_i} + O_{P_j} = M$, $i \neq j$. The annihilator of an element of $\cap_{i=1}^{n} O_{P_i}$ is not contained in any maximal submodule of M and thus $\cap_{i=1}^{n} O_{P_i} = 0$. Hence by Lemma 3.9,

$$M = M/O_{P_1} \oplus \cdots \oplus M/O_{P_n} \simeq M_{\mathfrak{p}_1} \oplus \cdots \oplus M_{\mathfrak{p}_n}.$$

□

Theorem 3.12. *Let M be a finitely generated reduced R-module with a finite number of minimal prime submodules of $\{P_1, P_2, ..., P_n\}$. Let $S = R \setminus \cup_{i=1}^{n} \mathfrak{p}_i$, where $\mathfrak{p}_i = (P_i : M)$; then $\mathrm{T}(M) = M_{\mathfrak{p}_1} \oplus \cdots \oplus M_{\mathfrak{p}_n} \simeq \mathrm{E}_{\bar{R}}(M)$. Hence $\mathrm{T}(M)$ is an injective VNM (in fact a semi-simple module), and $\mathrm{E}_{\bar{R}}(M)$ is a flat R-module.*

Proof. By Theorem 3.11, $\mathrm{T}(M) = M_{\mathfrak{p}_1} \oplus \cdots \oplus M_{\mathfrak{p}_n}$. Thus by Theorem 3.5, $M_{\mathfrak{p}_i}$ is a simple module, and $\mathrm{T}(M)$ is an injective $\bar{R}$-module. Since $\mathrm{T}(M)$ is an essential extension of M, we have $\mathrm{T}(M) = \mathrm{E}_{\bar{R}}(M)$. □

Theorem 3.13. *Let M be a finitely generated reduced R-module. If M is an injective R-module, then M is a VNM. In this case if N is any submodule of M, $I = \mathrm{Ann}(N)$ and $N' = IM$, then $M = \mathrm{E}(N) \oplus N'$.*

Proof. By Lemma 2.2(1), $N \cap N' = 0$; in addition we claim that $N \oplus N' \subseteq M$ is an essential extension. Because, if N'' is a non-zero submodule of M, then $N'' = JM$ for some ideal J of R. So $J(N \oplus N') \subseteq JM = N''$. Now by Lemma 2.2(2), $N'' \cap (N \oplus N') \neq 0$. Thus $M = \mathrm{E}(M) = \mathrm{E}(N \oplus N') = \mathrm{E}(N) \oplus \mathrm{E}(N')$. Now $(\mathrm{E}(N') : M)N \subseteq \mathrm{E}(N) \cap \mathrm{E}(N') = 0$, and thus $\mathrm{E}(N') \subseteq N'$. Hence $N' = \mathrm{E}(N')$, and $M = \mathrm{E}(N) \oplus N'$.

Now suppose that $N = Rx$, $x \in M$. Then by the preceding paragraph and Lemma 2.8, we have $N' = eM$, for some M-idempotent $e \in R$; and hence $Rx \subseteq \mathrm{E}(Rx) = (1-e)M$. Now for any $y \in M$ we have

$$I(1-e)y \in IM \cap (1-e)M = eM \cap (1-e)M = 0.$$

This implies that $I = \mathrm{Ann}(x) \subseteq \mathrm{Ann}((1-e)y)$. Hence there is an R-homomorphism $f : Rx \to R(1-e)y$ with $f(rx) = r(1-e)y$, for all $r \in R$. Since M is injective, f extends to an R-homomorphism from M to M. On the other hand, by hypothesis $N = JM$ for some ideal J of R. Therefore $x = \sum_{i=1}^{n} a_i x_i$ for some $a_i \in J$ and $x_i \in M$. Thus we have

$$(1-e)y = f(x) = \sum_{i=1}^{n} a_i f(x_i) \in JM = N.$$

This implies that $(1-e)M \subseteq Rx$, and so $Rx = (1-e)M$. Thus Rx is a direct summand of M and hence M is a VNM. $\square$

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