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Group graded associated ideals with flat base change of rings and short exact sequences

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Abstract. This paper deals with the study of behaviour of G-associated ideals and strong Krull G-associated ideals with flat base change of rings and behaviour of G-associated ideals with short exact sequences over rings graded by finitely generated abelian group G.

Keywords. Graded ring; graded module; Noetherian ring; G-associated ideals.

1. Introduction

The study of graded rings arises naturally out of the study of affine schemes and allows us to formalize (and unify) arguments by induction [15]. However, this is not just an algebraic trick. The concepts of grading in algebra, in particular graded modules are essential in the study of homological algebraic aspect of rings. In recent years, rings with a group-graded structure have become increasingly important and, consequently, the graded analogues of different concepts are widely studied [2, 3, 5, 7–9, 11–14]. As a result, graded analogue of different concepts are being developed in recent research. The graded primary decomposition of graded modules has been covered in [1] for the case of gradings by torsion-free abelian groups. Let A be a commutative Noetherian ring which is graded by a finitely generated abelian group G. In [1], it is shown that for G torsion free, the associated primes of G-graded modules are G-graded as well. Moreover, for any two G-graded, finitely generated A-modules M and N, where N is a submodule of M, there exists a primary decomposition $N = \bigcap_{i \in I} Q_i$ in M such that Q_i are G-graded. If G has torsion, such a statement is wrong in general. For example, see the introduction of [12]. However, for homogeneous coordinate rings, one also has to consider gradings by groups with torsion. A generalization of the treatment of [1] to the case of gradings by general finitely generated abelian groups has been given in [12]. The graded version of the local criterion of flatness was discussed in Appendix 3 of [4].

This paper has two objectives. First, in §3 we discuss the application of the theory developed in [12] for G-associated ideals, that is, the behaviour of G-associated ideals (Ass^G) with short exact sequences. Second, in §4 we consider the generalized notions of G-associated prime ideals for not necessarily Noetherian rings and introduce strong Krull G-associated ideals (Ass^{SG}) with flat base change of rings, over rings graded by finitely generated abelian groups which do not seem to have appeared in literature. The reason to discuss the strong Krull G-associated prime ideals is that for non-Noetherian rings the G-graded primary decomposition may not exist. We study the properties of non-Noetherian rings more closely by using strong Krull G-associated prime ideals in §4. These results have applications in algebraic geometry, for instance for the study of toric varieties which are not necessarily Noetherian and which arise in the study of representations of Kac-Moody groups. Using the theory developed in this section we prove Theorem 4.10 as an application on polynomial rings.

In §2 we discuss some preliminary concepts on group graded rings. In §3 we discuss about the behavior of G-associated ideals with short exact sequences. In §4 we introduce the strong Krull G-associated ideals and establish a relationship between strong Krull G-associated ideals and G-associated ideals with the corresponding associated ideals in polynomial rings by using techniques developed in [5].

2. Preliminaries

Let G be a finitely generated abelian group with identity element e and A be a G-graded commutative ring. If not stated otherwise, all A-modules are assumed to be finitely generated. Before we state some results, we introduce some notations and terminologies.

A ring A is called a G-graded ring if there exists a family $\{A_g : g \in G\}$ of additive subgroups of A such that $A = \bigoplus_{g\in G} A_g$ and $A_g A_h \subseteq A_{gh}$ for each $g, h \in G$. An element of a graded ring A is called *homogeneous* if it belongs to $\bigcup_{g\in G} A_g$ and this set of homogeneous elements is denoted by $h(A)$. Also if $x \in A_g$ for some $g \in G$, then we say that x is of *degree* g. Let A be a G-graded ring and M an A-module. We say that M is a G-graded A-module if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \bigoplus_{g\in G} M_g$ (as abelian groups) and $A_g M_h \subseteq M_{gh}$ for $g, h \in G$. Here $A_g M_h$ denotes the additive subgroup of M consisting of all finite sums of elements $a_g s_h$ with $a_g \in A_g$ and $s_h \in M_h$. Also we denote $h(M) = \bigcup_{g\in G} M_g$. A graded ideal I of a graded ring A is an ideal and $I = \bigoplus_{g\in G}(I \cap A_g) = \bigoplus_{g\in G} I_g$. A graded ideal $I \subset A$ is *G-prime* if and only if for every two G-graded ideals J, K, $JK \subset I$ implies $J \subset I$ or $K \subset I$. Let I be a G-graded ideal of A. Then $\mathrm{Gr}(I)$, the graded radical of I, is the set of all $x \in A$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$, where the notation x_g corresponds to the decomposition $x = \sum_{g\in G} x_g$.

As a general fact for G-prime ideals, maximal G-prime ideals always exist as a consequence of Zorn's lemma. We can define nilpotent homogeneous element for the G-graded ring A in a similar way as in the case of non-graded ring and we can observe that the set of all nilpotent homogeneous elements form an ideal of the G-graded ring A. The ideal of nilpotent homogeneous elements of A is called the G-graded nilradical of A and is denoted by $N^G(A)$.

DEFINITION 2.1

Let $A = \bigoplus_{g\in G} A_g$ and $B = \bigoplus_{g\in G} B_g$ be two graded rings. A mapping $f : A \longrightarrow B$ with $f(1_A) = 1_B$ is called a graded homomorphism if $f(A_g) \subseteq B_g$ for all $g \in G$.

Let S be a multiplicatively closed subset of $h(A)$. Then the ring of fractions $S^{-1}A$ is a G-graded ring which is called the G-ring of fractions. Indeed, $S^{-1}A = \bigoplus_{g\in G}(S^{-1}A)_g$ where

$$(S^{-1}A)_g = \{\tfrac{a}{s} : a \in h(A), \quad s \in S \quad \text{and} \quad g = (\deg s)^{-1}(\deg a)\}.$$

Let P be any G-prime ideal of graded ring A and consider the multiplicatively closed subset $S = h(A) - P$. We denote the graded ring of fraction $S^{-1}A$ of A by A_P^G or (A, P) and we call it the G-localization of A. This ring is a G-*local* ring with unique G-maximal ideal $S^{-1}P$. For more definitions and properties of graded ring of fractions of graded rings one can see [9].

DEFINITION 2.2

A G-graded commutative ring A having unique G-maximal ideal is called a G-local ring and if A has finitely many G-maximal ideals, then it is called quasi G-local ring.

DEFINITION 2.3

A G-graded submodule N of M is called G-graded primary or G-primary if $N \neq M$ and for each $a \in h(A)$, the homothety $\lambda_a : M/N \longrightarrow M/N$ is either injective or nilpotent. An ideal I of A is called G-graded primary ideal if it is a G-graded primary submodule of A.

Now we prove the following result:

PROPOSITION 2.4

Let N be a G-primary submodule of M. Then the ideal $P = \langle\{a \in h(A) \mid \lambda_a : M/N \longrightarrow M/N$ is not injective$\}\rangle$ is a G-prime ideal.

Proof. Suppose I and J are any two G-graded ideals of A such that $I \not\subseteq P$ and $J \not\subseteq P$. Then there exist two elements $a, b \in h(A)$ but not in P and $a \in I$, $b \in J$ such that λ_a and λ_b are injective. Then $\lambda_{ab} = \lambda_a\lambda_b$ is injective. Therefore $IJ \not\subseteq P$, since IJ contains the elements of the form $\sum ab$. Hence P is a G-prime ideal. □

DEFINITION 2.5

Let N be a G-primary submodule of M. Then the ideal generated by the set $\{a \in h(A) \mid \lambda_a : M/N \longrightarrow M/N$ is not injective$\}$ is a G-prime ideal and is denoted by $\mathrm{Gr}_M(N)$.

Remark. It is clear from Definition 2.3 that N is a G-graded primary submodule of M if and only if $ax \in N, a \in h(A), x \in M$ implies either $x \in N$ or $a^n M \subset N$ for some $n \geq 1$ i.e., $a \in \mathrm{Gr}(\mathrm{Ann}(M/N))$.

In §3 of [11], Perling discussed some standard facts on graded commutative rings and modules such as graded structure on the tensor product of modules. We follow here the same notation and terminology in the rest of the paper. In fact Definition 2.1 is a special case of [11] by taking the identity map in place of χ and $G = G'$. The graded ring homomorphism $f : A \longrightarrow B$, which preserves gradings between the two G-graded rings A and B, makes B an A-algebra. Tensoring with B, we perform a base change. In §2 of [12] it is proved that for any G-graded module M and if $m \in M$ is homogeneous, then $\mathrm{Ann}(m)$ and $\mathrm{Ann}\, M$ are also homogeneous ideals.

DEFINITION 2.6

Let M be a finitely generated G-graded A-module. An ideal $I \subset A$ is *G-associated* if and only if I is G-prime and $I = \mathrm{Ann}(x)$ for some element $0 \neq x \in M$. We denote the set of all G-associated ideals of M by $\mathrm{Ass}^G\, M$.

3. Behavior of G-associated ideals with short exact sequences

Let A be a Noetherian ring which is graded by a finitely generated Abelian group G. In [12], Perling and Kumar discussed an analogue of primary decomposition which works when G has torsion. More precisely, they have discussed whether for some G-graded modules M and N, where N is a submodule of M, there exists a decomposition $N = \bigcap_{i \in I} Q_i$, where the Q_i are G-graded and $\mathrm{Ann}(M/Q_i)$ are irreducible in a suitable sense. Now we consider some applications of their work on primary decomposition over rings graded by a finitely generated abelian group. For more details the reader may refer to ref. [12].

PROPOSITION 3.1

Let A be a G-graded Noetherian ring and $0 \neq M$ be a finitely generated G-graded A-module. Then $\bigcap_{P \in \mathrm{Ass}^G(M)} P = \mathrm{Gr}(\mathrm{Ann}_A M)$.

Proof. Let $0 = \cap_{i=1}^r N_i$ be a reduced G-primary decomposition of 0 in M, N_i being G-graded P_i primary. Then $\mathrm{Ass}^G(M) = \{P_1, P_2, \ldots, P_r\}$. If $a \in \mathrm{Gr}(\mathrm{Ann}_A M)$, then $a^n M = 0$, for some $n \geq 1$, as M is finitely generated. $\lambda_a : M/N_i \longrightarrow M/N_i$ is nilpotent for each i, $(1 \leq i \leq r)$ i.e. $a \in \cap_{i=1}^r P_i$. Conversely, if $a \in \cap_{i=1}^r P_i$, $\lambda_a : M/N_i \longrightarrow M/N_i$ is nilpotent for each i, so that $a^{n_i} M \subset N_i$, $n_i \geq 1$. If $n = \max_i\{n_i\}$, then $a^n M = 0$ and $a \in \mathrm{Gr}(\mathrm{Ann}_A M)$. □

COROLLARY 3.2

If A is a G-graded Noetherian ring and $N^G(A)$ is the ideal of nilpotent homogeneous elements of A, then $N^G(A) = \bigcap_{P \in \mathrm{Ass}^G(A)} P$.

Proof. By the definition of G-graded nil radical of A (discussed in the preliminaries) and by taking $M = A$ in Proposition 3.1, the result follows. □

DEFINITION 3.3

For G-graded A-module M, the support of M, denoted by $\mathrm{Supp}^G(M)$, is defined as $\mathrm{Supp}^G(M) = \{P \in \mathrm{Spec}^G(A) | M_P^G \neq 0\}$.

Clearly $M = 0$ if and only if $\mathrm{Supp}^G(M) = \emptyset$.

PROPOSITION 3.4

Let A be a Noetherian ring and M a finitely generated G-graded A-module. Then for any G-prime ideal P of A, the following conditions are equivalent:

(1) $P \in \mathrm{Supp}^G(M)$.
(2) $P \supset P_i$ *for some* $P_i \in \mathrm{Ass}^G(M)$.
(3) $P \supset \mathrm{Gr}(\mathrm{Ann}_A M)$.

Proof. Let $\mathrm{Ass}^G(M) = \{P_1, P_2, \dots, P_n\}$. Then $\cap_1^n P_i = \mathrm{Gr}(\mathrm{Ann}_A M)$. Now we prove the following:

(1) $\Rightarrow$ (2). Let $P \in \mathrm{Supp}^G(M)$ and suppose contrary to that $P \not\supset P_i$ for any i. Choose $a_i \in P_i$ such that $a_i \notin P$. Then $a = \Pi_i a_i \in \cap P_i$. We claim that $a \notin P$. Suppose $a \in P$ i.e., $\Pi a_i \in P$. Then $a_i \in P$ for some i, for P is a G-prime ideal which contradicts the assumption. This proves the claim. Then $P \not\supset \cap_i P_i$, i.e. $P \not\supset \mathrm{Gr}(\mathrm{Ann}_A M)$. Hence $M_P^G = 0$, a contradiction.

(2) $\Rightarrow$ (3). If $P \supset P_i$ for some $P_i \in \mathrm{Ass}^G(M)$, then we have $P \supset \mathrm{Gr}(\mathrm{Ann}_A M)$, for $P_i \supset \mathrm{Gr}(\mathrm{Ann}_A M)$.

(3) $\Rightarrow$ (1). Let $P \supset \mathrm{Gr}(\mathrm{Ann}_A M)$ and suppose $M_P^G = 0$. Since M is finitely generated, there exists some $s \in A - P$ with $sM = 0$. This is a contradiction as $P \supset \mathrm{Gr}(\mathrm{Ann}_A M)$. □

COROLLARY 3.5

$\mathrm{Ass}^G(M) \subset \mathrm{Supp}^G(M)$ *and the minimal elements of* $\mathrm{Ass}^G(M)$ *and* $\mathrm{Supp}^G(M)$ *are the same.*

Theorem 3.6. *Let* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *be an exact sequence of G-graded A-modules. Then*

(1) $\mathrm{Supp}^G(M) = \mathrm{Supp}^G(M') \cup \mathrm{Supp}^G(M'')$.
(2) *For any finitely generated A-module L and K*, $\mathrm{Supp}^G(L \otimes_A K) = \mathrm{Supp}^G(L) \cap \mathrm{Supp}^G(K)$.

Proof. Analogous to Bourbaki, § II.4.4 Propositions 16 and 18. □

4. Flat base change of rings

Let A and B be any G-graded ring, which need not be Noetherian. Let $f : A \longrightarrow B$ be a ring homomorphism such that f preserves the gradings and ${}^a f : \mathrm{Spec}^G(B) \longrightarrow \mathrm{Spec}^G(A)$ denote the induced map, where $\mathrm{Spec}^G(A)$ and $\mathrm{Spec}^G(B)$ denote the set of all G-prime ideals of A and B respectively. Let M be a G-graded A-module. Then $M \otimes_A B$ is a G-graded B-module and when we consider it as an A-module, we denote it by ${}_f(M \otimes_A B)$. If M is a G-graded B-module, then G-graded A modules obtained from M by restriction of scalars will be denoted by ${}_f M$. With these notations we have the following.

DEFINITION 4.1

A G-prime ideal P is called strong Krull G-associated to M if for every finitely generated ideal $I \subseteq P$ there exist a homogeneous element $0 \neq m \in M$ such that $I \subseteq \mathrm{Ann}(m) \subseteq P$. We denote the set of all strong Krull G-associated ideals of M by $\mathrm{Ass}^{SG}(M)$.

From the definitions of strong Krull G-associated ideal and G-associated ideal one can observe easily that $\mathrm{Ass}^{SG}(M) \supseteq \mathrm{Ass}^G(M)$. In this section we study the behaviour of Ass^{SG} and Ass^G with flat base change of rings. First we prove local global principle used in our propositions.

Lemma 4.2 (*Graded local global principle*). *Let M be a G-graded A-module. Then the following are equivalent*:

(1) $M = 0$.
(2) $M_P^G = 0$, *for all* $P \in \mathrm{Spec}^G(A)$.
(3) $M_{\mathfrak{m}}^G = 0$, *for all maximal homogeneous ideals* $\mathfrak{m}$ *of* A.

Proof. It is enough to prove (3) $\Rightarrow$ (1). Suppose there exists a homogeneous element $x \in M, x \neq 0$. Let $I = \mathrm{Ann}(x)$. Then $I \neq A$, as $1 \in A$. Choose a maximal homogeneous ideal $\mathfrak{m}$ with $\mathfrak{m} \supset I$. Now $x \in M$ implies $\frac{x}{1} \in M_{\mathfrak{m}}^G = 0$. Then there exists $a = \sum a_g \in h(A) - \mathfrak{m}$ such that $ax = 0$. As x is homogeneous, the degree of each term is distinct, so $a_g x = 0$ for all $g \in G$. This implies that $a \in \mathrm{Ann}(x) \subseteq \mathfrak{m}$, a contradiction. □

COROLLARY 4.3

Let M and N be G-graded A-modules and $f : M \to N$ be a G-graded R-module homomorphism. Then the following are equivalent:

(1) f *is injective* (*surjective*).
(2) $f_P : M_P^G \to N_P^G$ *is injective* (*surjective*), *for all* $P \in \mathrm{Spec}^G(A)$.
(3) $f_{\mathfrak{m}} : M_{\mathfrak{m}}^G \to N_{\mathfrak{m}}^G$ *is injective* (*surjective*), *for all maximal homogeneous ideals* $\mathfrak{m}$ *of* A.

Lemma 4.4 *Let P be a strong Krull G-associated to the G-graded A-module M. Then P behaves perfectly under localization i.e., $S^{-1}P$ remains strong Krull G-associated to G-graded A_P^G-module M_P^G, where $S = h(A) - P$.*

Proof. Since $P \in \mathrm{Ass}^{SG}(M)$, by definition, for every finitely generated ideal I of A, there exists $0 \neq m \in M$ such that $I \subseteq \mathrm{Ann}(m) \subseteq P$. We need to show that $S^{-1}I \subseteq \mathrm{Ann}(\frac{m}{1}) \subseteq S^{-1}P$. Let $\frac{a}{s} \in S^{-1}I$ for some $a \in I$ and $s \in S$. Then $a \in \mathrm{Ann}(m)$ gives us $am = 0$ for some $0 \neq m \in M$. Therefore $s'am = 0$ implies $(\frac{a}{s})(\frac{m}{1}) = 0$ which shows that $\frac{a}{s} \in \mathrm{Ann}(\frac{m}{1})$ and hence $S^{-1}I \subseteq \mathrm{Ann}(\frac{m}{1})$. On the other hand, let $\frac{a}{s} \in \mathrm{Ann}(\frac{m}{1})$ for some $\frac{a}{s} \in A_P^G,\ a \in A,\ s \in S$. Then $(\frac{a}{s})(\frac{m}{1}) = 0$ implies $s'am = 0$ for some $s' \in S$. Therefore $s'a \in \mathrm{Ann}(m) \subseteq P$ and P being G-prime, we have $a \in P$ as $s' \in S = h(A) - P$ which shows that $\frac{a}{s} \in S^{-1}P$. Hence we conclude that $S^{-1}P$ is a strong Krull G-associated to M_P^G from the relation $S^{-1}I \subseteq \mathrm{Ann}(\frac{m}{1}) \subseteq S^{-1}P$. □

By a similar argument we can also prove that the G-associated ideals behave perfectly under localization.

PROPOSITION 4.5

Let $f : A \longrightarrow B$ be a flat ring homomorphism and M be a G-graded B-module. Then ${}^af(\mathrm{Ass}^{SG}(M)) = \mathrm{Ass}^{SG}({}_f(M))$.

Proof. Let $Q \in \mathrm{Ass}^{SG}(M)$ and $P = {}^af(Q)$. Then we need to prove that $P \in \mathrm{Ass}^{SG}({}_f(M))$. By Definition 4.1 we need to show that for any finitely generated ideal I of A contained in P there exists $x \in M$ such that $I \subseteq \mathrm{Ann}_A(x) \subseteq P$. Let I be any finitely generated ideal of A contained in P. Then $IB \subseteq PB \subseteq Q$. Since $Q \in \mathrm{Ass}^{SG}(M)$, again by Definition 4.1, there exists a homogeneous element $x \in M$ such that $IB \subseteq \mathrm{Ann}_B(x) \subseteq Q$. Therefore $I \subseteq f^{-1}(\mathrm{Ann}_B(x)) = \mathrm{Ann}_A(x) \subseteq P$ and $P \in \mathrm{Ass}^{SG}({}_f(M))$. Thus ${}^af(\mathrm{Ass}^{SG}(M)) \subseteq \mathrm{Ass}^{SG}({}_f(M))$. Conversely, let $P \in \mathrm{Ass}^{SG}({}_f(M))$. Since strong Krull G-associated ideals behave perfectly under localization (see Lemma 4.4), we can assume that A is a G-local with G-graded maximal ideal P. Then $PB \neq B$, for otherwise we would have $1 = \sum_{i=1}^n a_i b_i$, with $a_i \in P, b_i \in B$. But then $P \in \mathrm{Ass}^{SG}({}_f(M))$ and we get a non-zero homogeneous element $x \in M$ such that $\langle\{a_1, \ldots, a_n\}\rangle \subseteq \mathrm{Ann}_A(x) \subseteq P$ and therefore $x = 1 \cdot x = \sum_{i=1}^n a_i b_i x = 0$, which is not possible. Now let $Q \in \mathrm{Spec}^G(B)$ be minimal over PB. By localizing at Q we may assume that (B, Q) is a G-local and $Q = \mathrm{Gr}(\mathrm{Ann}(PB))$. We claim that $Q \in \mathrm{Ass}^{SG}(M)$. Let $I \neq 0$ be a finitely generated ideal of B contained in Q. Since $Q = \mathrm{Gr}(\mathrm{Ann}(PB))$ and by remark after Proposition 2.4 there exists an integer $n \geq 1$ and a finitely generated graded ideal J of A contained in P with $I^n \subseteq JB \subseteq PB$. Since $P \in \mathrm{Ass}^{SG}({}_f(M))$, there exists $x \in M$ such that $J \subseteq \mathrm{Ann}_A(x) \subset P$. Then $I^n \subseteq JB \subseteq \mathrm{Ann}_A(x)B \subseteq \mathrm{Ann}_B(x) \subseteq Q$. Choose n such that $I^n x = 0$ but $I^{n-1}x \neq 0$, where $I^0 = A$. If $y \in I^{n-1}$ with $xy \neq 0$, then $I \subseteq \mathrm{Ann}_B(xy) \subseteq Q$ and hence $Q \in \mathrm{Ass}^{SG}(M)$. Thus we have $\mathrm{Ass}^{SG}({}_f(M)) \subseteq {}^af(\mathrm{Ass}^{SG}(M))$. □

Remark 4.1. The proof of Proposition 4.5 shows that any G-prime ideal, which is minimal over PB, in fact belongs to $\mathrm{Ass}^{SG}M$.

PROPOSITION 4.6

Let $f : A \longrightarrow B$ be a flat ring homomorphism and M be a G-graded A-module. Then ${}^af(\mathrm{Ass}^{SG}(M \otimes_A B)) \subseteq \mathrm{Ass}^{SG}(M)$ and equality holds if f is faithfully flat.

Proof. Let $Q \in \mathrm{Ass}^{SG}(M \otimes_A B)$ and $P = {}^a\!f(Q)$. We claim that $P \in \mathrm{Ass}^{SG}(M)$. Since the strong Krull G-associated ideals behave perfectly under localization (see Lemma 4.4), we may pass to $A_P^G \longrightarrow B_Q^G$ and thus to prove the first statement we may reduce to the case that (A, P) and (B, Q) are G-local rings and f is faithfully flat (see page 28, Proposition 1(d) of [1]) and $M \otimes_A B$ is a finitely generated flat module. Suppose $I \subseteq P$ is a finitely generated homogeneous ideal of A. Since $Q \in \mathrm{Ass}^{SG}(M \otimes_A B)$, we have

$$0 \neq \mathrm{Hom}_B(B/IB, M \otimes_A B) \cong \mathrm{Hom}_A(A/I, M) \otimes_A B;$$

the above isomorphism follows from an analogue of the result (page 23, Proposition 11 of [1]). Since B is flat, $\mathrm{Hom}_A(A/I, M) \neq 0$. Therefore we have ${}^a\!f(\mathrm{Ass}^{SG}(M \otimes_A B)) \subseteq \mathrm{Ass}^{SG}(M)$. For the second part, consider the graded analogue of the result (page 32, Proposition 8 of [1]), since f is faithfully flat the map $M \longrightarrow {}_f(M \otimes_A B)$ is injective, and we have $\mathrm{Ass}^{SG}(M) \subseteq \mathrm{Ass}^{SG}({}_f(M \otimes_A B))$. Also, by Proposition 4.5, $\mathrm{Ass}^{SG}({}_f(M \otimes_A B)) = {}^a\!f(\mathrm{Ass}^{SG}(M \otimes_A B))$. Thus

$$\mathrm{Ass}^{SG}(M) = {}^a\!f(\mathrm{Ass}^{SG}(M \otimes_A B)).$$

□

A simple modification of the first part of the proof of Proposition 4.5 studies the behaviour of Ass^G under the base change of rings.

PROPOSITION 4.7

Let $f : A \longrightarrow B$ be a ring homomorphism and M be a G-graded B-module. Then ${}^a\!f(\mathrm{Ass}^G(M)) = \mathrm{Ass}^G({}_f(M))$.

PROPOSITION 4.8

Let $f : A \longrightarrow B$ be a flat A-algebra and M be a G-graded A-module. Then ${}^a\!f(\mathrm{Ass}^G(M \otimes_A B)) \subseteq \mathrm{Ass}^G(M)$.

Proof. Let $Q \in \mathrm{Ass}^G(M \otimes_A B)$ and $P = {}^a\!f(Q)$. Since G-associated ideals behave perfectly under localization, we may pass to $A_P^G \longrightarrow B_Q^G$ and we reduce to the case that (A, P) and (B, Q) are G-local and f is faithfully flat (see page 28, Proposition 1(d) of [1]) and $M \otimes_A B$ is a finitely generated flat module. In this case we must show that $h(P) \subseteq Z_A(M)$, where $Z_A(M) = \{r \in h(A) | r \cdot m = 0 \text{ for some } m \in M, m \neq 0\}$. But if $0 \neq x \in P$ is a homogeneous element, then the exact sequence

$$0 \longrightarrow \mathrm{Ann}_M(x) \longrightarrow M \xrightarrow{x} M$$

gives

$$0 \longrightarrow \mathrm{Ann}_M(x) \otimes_A B \longrightarrow M \otimes_A B \xrightarrow{x \otimes 1} M \otimes_A B.$$

Thus we get $\mathrm{Ann}_M(x) \otimes_A B = \mathrm{Ann}_{M \otimes B}(x \otimes 1) = \mathrm{Ann}_{M \otimes B}(f(x)) \neq 0$, for f is faithfully flat (see page 47, Theorems 7.2 and 7.3 of [8]) and hence $\mathrm{Ann}_M(x) \neq 0$. □

In fact from Propositions 4.5, 4.6, 4.7 and 4.8 one can observe that if M is an A-module, then the strong Krull G-associated ideals of $M \otimes_A B$ contract to strong Krull G-associated ideals of M. We conclude this section with a result which gives a relationship between strong Krull G-associated ideals and G-associated ideals with the corresponding associated ideals in polynomial rings. We need the following result in the proof of Theorem 4.10. This result is the graded version of Theorem 1 of [10]. We state the same without proof.

Theorem 4.9 [10]. *Let $g(x_1, \ldots, x_n)$ be a polynomial with coefficients in the G-graded A-module M and $f(x_1, \ldots, x_n)$ a polynomial with coefficients in the G-graded ring A. Then*

$$C(g)\, C^{k+1}(f) \;=\; C(gf)\, C^k(f)$$

for all large values of k, where $C(f)$ denotes the A-submodule generated by the coefficients of f.

Theorem 4.10. *Let M be a G-graded A-module and T an indeterminate. Then*

(1) $\mathrm{Ass}^G(M \otimes_A A[T]) \subseteq \mathrm{Ass}^{SG}(M \otimes_A A[T])$.
(2) $\{PA[T] : P \in \mathrm{Ass}^{SG}(M)\} \subseteq \mathrm{Ass}^{SG}(M \otimes_A A[T])$.
(3) $\mathrm{Ass}^G(M \otimes_A A[T]) \subseteq \{PA[T] : P \in \mathrm{Ass}^{SG}(M)\}$.

Proof.

(1) Follows from the definition.
(2) Let $f_1, \ldots, f_n \in PA[T]$, $P \in \mathrm{Ass}^{SG}(M)$. Let I be an ideal generated by the coefficients of f_i. Clearly I is finitely generated and $I \subseteq P$. Since $P \in \mathrm{Ass}^{SG}(M)$, by definition of strong Krull G-associated ideal there exists a homogeneous element $y \in M$ such that $I \subseteq \mathrm{Ann}_A(y) = (0 :_A y) \subseteq P$ i.e. y annihilates the coefficients of f_i and $(0 :_A y) \subseteq P$. Let $\varphi : M \longrightarrow M \otimes_A A[T]$ be the map $x \longrightarrow x \otimes 1$. Then $\langle\{f_1, \ldots, f_n\}\rangle \subseteq (0 :_{A[T]} \varphi(y)) \subseteq PA[T]$. Therefore $PA[T] \in \mathrm{Ass}^{SG}(M \otimes_A A[T])$.
(3) We first show that any G-associated ideal Q of $(M \otimes_A A[T])$ will be of the form $Q = PA[T]$, where $P = Q \cap A$ i.e. Q is an extended G-prime. For this we may assume that A is G-local with maximal ideal P. Suppose $Q \neq PA[T]$. Then there exist a monic and irreducible polynomial f modulo $PA[T]$ such that $Q = (P, f)A[T]$. Note that $f \in Q$ and $Q \in \mathrm{Ass}^G(M \otimes_A A[T])$, $M \otimes_A A[T] \cong M[T]$ and by the definition G-associated ideal f is a zero divisor of $M[T]$. Since f is monic, f cannot be a zero divisor. Therefore Q is an extended G-prime. Next we claim that if $PA[T] \in \mathrm{Ass}^G(M \otimes_A A[T])$. Then $P \in \mathrm{Ass}^{SG}(M)$. For this we may again assume that A is G-local. Let $a_0, a_1, \ldots, a_n \in P$. To show $P \in \mathrm{Ass}^{SG}(M)$, it is enough to prove that there exist a homogeneous element $m \in M$ such that $\langle\{a_0, a_1, \ldots, a_n\}\rangle \subseteq (0 :_A m) \subseteq P$. Consider the polynomial $f = a_0 + a_1T + \cdots + a_nT^n$. Clearly $f \in PA[T]$, where $PA[T]$ is a G-associated ideal of $M \otimes_A A[T]$. So by the definition of G-associated ideal, there exists $g = m_0 + m_1T + \cdots + m_kT^k \in M[T]$ such that $f \in (0 :_{A[T]} g) \subseteq PA[T]$. Hence by Theorem 4.9, there exists an integer j such that $C(f)^{j+1}C(g) = C(f)^jC(fg)$, where $C(f)$ denotes the

A-submodule generated by the coefficients of f etc. Since $f \in (0:_{A[T]} g)$, $C(fg) = 0$ and so $C(f)^{j+1}C(g) = C(f)^j C(fg) = 0$. If j is a chosen minimal such that $C(f)^{j+1}C(g) = 0$, then for any non zero $m \in C(f)^j C(g)$ we have

$$(a_0, a_1, \ldots, a_n) = C(f) \subseteq (0 :_A m) \subseteq P.$$

Hence $P \in \mathrm{Ass}^{SG}(M)$.

$\square$

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