Proc. Indian Acad. Sci. (Math. Sci.) Vol. 121, No. 1, February 2011, pp. 1–18.

On counting twists of a character appearing in its associated Weil representation

K VISHNU NAMBOOTHIRI

Department of Mathematics and Statistics, University of Hyderabad,
Hyderabad 500 046, India
Department of Collegiate Education, Government of Kerala, Kerala, India
E-mail: kvnamboothiri@gmail.com

MS received 4 September 2010

Abstract. Consider an irreducible, admissible representation π of GL(2,F) whose restriction to GL(2,F)$^+$ breaks up as a sum of two irreducible representations $\pi_+ + \pi_-$. If $\pi = r_\theta$, the Weil representation of GL(2,F) attached to a character θ of K^* does not factor through the norm map from K to F, then $\chi \in \widehat{K^*}$ with $(\chi \cdot \theta^{-1})|_{F^*} = \omega_{K/F}$ occurs in $r_{\theta +}$ if and only if $\epsilon(\theta\chi^{-1}, \psi_0) = \epsilon(\overline{\theta}\chi^{-1}, \psi_0) = 1$ and in $r_{\theta -}$ if and only if both the epsilon factors are -1. But given a conductor n, can we say precisely how many such χ will appear in π? We calculate the number of such characters at each given conductor n in this work.

Keywords. Nonarchimedian local field; irreducible; admissible representation of $GL(2, F)$; ϵ-factor of a character; Weil representation; number of characters appearing in its restriction.

1. Introduction

Let F be a nonarchimedean local field of characteristic not two and K a separable quadratic extension. Then if $K = F(x_0)$ with x_0 an element of K^* whose trace to F is 0 we have an embedding of K^* into GL(2,F) given by

$$a + bx_0 \mapsto \begin{bmatrix} a & bx_0^2 \\ b & a \end{bmatrix}.$$

Let GL(2,F)$^+$ be the subgroup of index 2 in GL(2,F) consisting of those matrices whose determinant is in $N_{K/F}(K^*)$ where $N_{K/F}$ is the usual norm map from K to F. In [4], Prasad considered irreducible, admissible representations π of GL(2,F) whose restriction to GL(2,F)$^+$ breaks up as a sum of two irreducible representations $\pi_+ + \pi_-$. There he gave a characterization of characters χ of K^* occurring in the restriction of π to K^*. It is immediate that if a character χ occurs in such a restriction then $\chi|_{F^*}$ must be the central character of π. Hence if π is supercuspidal then $\pi = r_\theta$, the Weil representation of GL(2,F) attached to a character θ of K^* which does not factor through the norm map from K to F. He showed that χ occurs in $r_{\theta +}$ if and only if $\epsilon(\theta\chi^{-1}, \psi_0) = \epsilon(\overline{\theta}\chi^{-1}, \psi_0) = 1$

and in $r_{\theta -}$ if and only if both the epsilon factors are -1 ($\overline{\theta}$ is the Galois conjugate of θ). What he proved exactly is the following:

Theorem 1.1. *Let r_θ be an irreducible admissible representation of* $\mathrm{GL}(2, F)$ *associated to a regular character θ of K^*. Fix embeddings of K^* in* $\mathrm{GL}(2, F)^+$ *and in ${D^{*+}}_F$, and choose a nontrivial additive character ψ of F, and an element x_0 of K^* with* $\mathrm{tr}(x_0) = 0$. *Then the representation r_θ of* $\mathrm{GL}(2, F)$ *decomposes as $r_\theta = r_{\theta +} \oplus r_{\theta -}$ when restricted to* $\mathrm{GL}(2, F)^+$ *and the representation ${r_\theta}'$ of ${D^*}_F$ decomposes as ${r_\theta}' = {r_\theta}'_+ \oplus {r_\theta}'_-$ when restricted to ${D^{*+}}_F$, such that for a character χ of K^* with $(\chi \cdot \theta^{-1})|_{F^*} = \omega_{K/F}$, χ appears in $r_{\theta +}$ if and only if $\epsilon(\theta\chi^{-1}, \psi_0) = \epsilon(\overline{\theta}\chi^{-1}, \psi_0) = 1$, χ appears in $r_{\theta -}$ if and only if $\epsilon(\theta\chi^{-1}, \psi_0) = \epsilon(\overline{\theta}\chi^{-1}, \psi_0) = -1$, χ appears in ${r_\theta}'_+$ if and only if $\epsilon(\theta\chi^{-1}, \psi_0) = 1$ and $\epsilon(\overline{\theta}\chi^{-1}, \psi_0) = -1$, and χ appears in ${r_\theta}'_-$ if and only if $\epsilon(\theta\chi^{-1}, \psi_0) = -1$ and $\epsilon(\overline{\theta}\chi^{-1}, \psi_0) = 1$.*

Here D_F^* is the unique quarternion division algebra over F. This result was proved only in the odd residue characteristic case in [4]. Proof in the even residue characteristic case appeared independently in [5] and [3].

We have, by definition of central character, $\theta|_{F^*} = \omega_{r_\theta}\omega$ where ω_{r_θ} is the central character of r_θ and $\omega = \omega_{K/F}$. Since $\theta|_{F^*} \neq \omega_{r_\theta}$ the character θ cannot occur in $r_\theta|_{K^*}$. A necessary condition for a character λ of K^* to occur in $r_\theta|_{K^*}$ is that its restriction to F^* should be equal to the central character ω_{r_θ}. The question we would like to ask at this point is whether θ twisted by some character λ of K^* can occur in r_θ with $\lambda|_{F^*} = \omega$. Note that such a twist satisfies the said necessary condition. Making it more precise, it means, whether there exist some λ such that $\lambda\theta$ occurs in $r_\theta|_{K^*}$. We prove some results which give an affirmative answer to this question. In fact, we try to count at each conductor level precisely how many characters occur in $r_{\theta +}$ and $r_{\theta -}$. It is not really surprising to see that the necessary condition is not sufficient to guarantee the occurrence of a character. Our computations on the local ϵ-factors are sometimes long, but by no means they are complicated. We feel that we have performed all kinds of computations possible using the ϵ-factors of characters. Lending the words of Tunnell [7] the results here in this exposition are presented as an 'entertainment'. The main results in the exposition are in the last two sections.

2. Notations

Our notations are consistent with those used in [3] more or less because we depend heavily on not only the results in [3], but also the computations performed there.

Throughout this paper F will be a nonarchimedian local field of characteristic $\neq 2$ and K a quadratic extension of F. The image of $x \in K$ under the nontrivial element of the Galois group of K over F is denoted by $\overline{x}$. For a local field F, O_F will be the ring of integers in F, $P_F = \pi_F O_F$ the unique prime ideal in O_F and π_F a uniformizer, i.e., an element in P_F whose valuation is one, i.e., $v_F(\pi_F) = 1$. The cardinality of the residue field of F is denoted by q and $U_F = O_F - P_F$ is the group of units in O_F. Let $P_F^i = \{x \in F : v_F(x) \geq i\}$ and for $i \geq 0$ define $U_F^i = 1 + P_F^i$ (with the proviso that $U_F^0 = U_F$).

Conductor of an additive character ψ of F or K is $n(\psi)$ if ψ is trivial on $P^{-n(\psi)}$, but nontrivial on $P^{-n(\psi)-1}$. Fix an additive character ψ of F of conductor zero (with no loss of generality, as in [3]) and let $\psi_K = \psi \circ \mathrm{tr}_{K/F}$ where $\mathrm{tr}_{K/F}$ or simply 'tr' is the trace map from K to F. By $N_{K/F}$ or simply N we mean the norm map from K to F and by $d_{K/F}$ or simply d the differential exponent of K over F which is such that $\mathrm{tr} P_K^{-d} \subseteq O_F$ but $\mathrm{tr} P_K^{-d-1} \nsubseteq O_F$. The conductor of ψ_K is d. For a character χ of F^* or K^* by $a(\chi)$ we mean the conductor of χ, i.e., $a(\chi)$ is the smallest integer $n \geq 0$ such that χ is trivial on U^n. We say that χ is unramified if $a(\chi)$ is zero. Also, if χ_1 and χ_2 are two characters of F then $a(\chi_1\chi_2) \leq \max(a(\chi_1), a(\chi_2))$. Equality holds if $a(\chi_1) \neq a(\chi_2)$. Furthermore, $a(\chi) = a(\chi^{-1})$.

A character θ of K^* is regular if it does not factor through the norm map from K to F. This guarantees that $\theta \neq \overline{\theta}$. The F-valuation of 2, $v_F(2)$, will always be denoted by t. Therefore, $2 = \pi_F^t u$, $u \in U_F$. By x_0 we will always denote a nonzero element of K with trace 0. Define ψ_0 by $\psi_0(x) = \psi(\mathrm{tr}[-x x_0/2])$ for $x \in K$. Then ψ_0 is an additive character of K trivial on F.

If G is a locally compact abelian group by $\hat{G}$ we mean the group of characters of G. Denote by $\omega_{K/F}$, or simply ω the character of F^* associated to K by class field theory, i.e., it is the unique nontrivial character of $F^*/N(K^*)$.

If X is a finite set, by $|X|$ we will mean the number of elements in X.

3. Some useful results

Deligne [1] described how the epsilon factor changes under twisting by a character of small conductor in the theorem:

Theorem 3.1. *Let α, β be two characters of a local field F such that $a(\alpha) \geq 2a(\beta)$. Let y_α be an element of F^* such that $\alpha(1+x) = \psi(y_\alpha x)$ for $v_F(x) \geq \frac{a(\alpha)}{2}$ (if $a(\alpha) = 0$, let $y_\alpha = \pi_F^{-n(\psi)}$). Then $\epsilon(\alpha\beta, \psi) = \beta^{-1}(y_\alpha)\epsilon(\alpha, \psi)$.*

Note that $v_F(y_\alpha) = -a(\alpha) - n(\psi)$.

From [3] we have

Lemma 3.2. *If $a(\chi) \geq 2a(\tilde{\omega})$, $\chi|_{F^*} = \omega$, then*

$$\epsilon(\chi, \psi_0) = \tilde{\omega}(-x_0/2)\tilde{\omega}^{-1}(y_\chi), \tag{1}$$

where $\chi \cdot \tilde{\omega}^{-1}(1+x) = \psi_K(y_{\chi\tilde{\omega}^{-1}} x)$.

Here y_χ is as in Theorem 3.1.

The *main theorem* in [3] states the following:

Theorem 3.3. *Let K be a separable quadratic extension of a local field F of characteristic not two. Let ψ be a nontrivial additive character of F, and $x_0 \in K^*$ such that $\mathrm{tr}(x_0) = 0$. Define an additive character ψ_0 of K by $\psi_0(x) = \psi(\mathrm{tr}[-x x_0/2])$. Then*

$$\epsilon(\omega, \psi) \frac{\omega\left(\frac{x-\overline{x}}{x_0-\overline{x}_0}\right)}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}} = \sum_{\chi \in S} \chi(x), \tag{2}$$

$x \in K^* - F^*$ where as is usual, the summation on the right is by partial sums over all characters of K^* of conductor $\leq n$.

We have the following result obtained by combining Corollary 7.2 and the calculations given at the end of §7 in [3].

Theorem 3.4. *Let $x = 1 + \pi_F^{r-1}\pi_K x'$ where $x' \in U_F$. Then*

$$\sum_{\chi \in S(2r+2m)} \chi(x) = \begin{cases} -q^{r-1}, & \text{if } m = 0 \\ 0, & \text{if } m = 1, 2, \ldots \\ & \text{and } m \neq d-1 \\ \omega(-1)\epsilon(\omega, \psi)\dfrac{\omega\left(\frac{x-\overline{x}}{x_0-\overline{x}_0}\right)}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}}, & \text{if } m = d-1 \end{cases}$$

and

$$\sum_{\chi' \in S'(2r+2m)} \chi'(x) = \begin{cases} -q^{r-1}, & \text{if } m = 0 \\ 0, & \text{if } m = 1, 2, \ldots \\ & \text{and } m \neq d-1 \\ -\omega(-1)\epsilon(\omega, \psi)\dfrac{\omega\left(\frac{x-\overline{x}}{x_0-\overline{x}_0}\right)}{\left|\frac{(x-\overline{x})^2}{x\overline{x}}\right|_{F^*}^{\frac{1}{2}}}, & \text{if } m = d-1. \end{cases}$$

Note that Namboothiri and Tandon [3] defined S to be the set $\{\chi \in K^* : \chi|_{F^*} = \omega, \epsilon(\chi, \psi_0) = 1\}$ and $S(l) = \{\chi \in S : a(\chi) = l\}$, analogously defined as S' and $S'(l)$ with the property that $\epsilon(\chi, \psi_0) = -1$. For computational convenience, we slightly changed our definition of S to denote the set $\{\chi \in \widehat{K^*} : \chi|_{F^*} = \omega, \epsilon(\chi^{-1}, \psi_0) = 1\}$ and $S(l) = \{\chi \in S : a(\chi) = l\}$. Analogously, we redefined S' and $S'(l)$. Because of this change in notations, we have an extra term $\omega(-1)$ in the above version compared to the one appeared in [3]. This is due to the fact that $\chi\overline{\chi} = 1$ since their restriction to F^* is ω and so $\epsilon(\chi^{-1}, \psi_0) = \omega(-1)\epsilon(\chi, \psi_0)$. We define $S_l = S(l) \cup S'(l)$.

When K/F is ramified, the following result can be verified trivially by applying Lemma 5.1 in [3].

Lemma 3.5. *Let $\chi \in \widehat{F^*}$ and ψ a nontrivial character of $(F, +)$.*

(1) *If $n < a(\chi) + n(\psi)$, then*

$$\sum_{u \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(u)\psi\left(\pi_F^{-n}u\right) = 0.$$

(2) *If $n > a(\chi) + n(\psi)$, then*

$$\sum_{u \in \frac{U_F}{U_F^{n}}} \chi^{-1}(u)\psi\left(\pi_F^{-n}u\right) = 0.$$

We also have the following theorem from [3].

Theorem 3.6. $|S(l)| = |S'(l)|$ *for each feasible* l, *that is when* $l = 2d-1$ *or* $l = 2f$ *with* $f \geq d$.

We use Theorem 1.1 to determine whether $\chi \in S$ is such that $\chi\theta$ occurs in $r_{\theta+}$ or $r_{\theta-}$. By this theorem, $\chi\theta$ occurs in $r_{\theta+}$ if and only if $\epsilon(\theta(\chi\theta)^{-1}, \psi_0) = \epsilon(\chi^{-1}, \psi_0) = 1 = \epsilon(\overline{\theta}(\chi\theta)^{-1}, \psi_0) = \epsilon\big(\chi^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\big)$ and $\chi\theta$ occurs in $r_{\theta-}$ if and only if $\epsilon(\chi^{-1}, \psi_0) = -1 = \epsilon\big(\chi^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\big)$. Note that a character $\chi\theta$ can occur in r_θ if and only if it occurs in either $r_{\theta+}$ or in $r_{\theta-}$. Also, if $\chi \in S(l)$ for some l then $\chi\theta$ can occur in r_θ if and only if it occurs in $r_{\theta+}$. Furthermore if $\chi \in S(l)$, then $\chi\theta$ cannot occur in $r_{\theta-}$ since for that χ, $\epsilon(\chi^{-1}, \psi_0) = +1$. Since its multiplicity cannot exceed 1 in r_θ, it is so in $r_{\theta+}$ and $r_{\theta-}$.

Now we are ready to start our counting. We divide the proof mainly into two cases: K/F ramified and K/F unramified.

4. Counting the twists when K/F is ramified

It is known (see, for instance, §3 of [2]) that if d is odd then $d = 2t+1$ and there exists a uniformizer, denoted by π_K such that tr $\pi_K = 0$. Let $x_0 = \pi_K$. In this case π_K^2 is a uniformizer of F which we denote by π_F and $N\pi_K = -\pi_F$. If d is even (which can only happen if the residue characteristic is 2), then $O_K = O_F[\pi_K]$ where π_K is a uniformizer of K which satisfies the Eisenstein polynomial $X^2 - u'\pi_F^s X - \pi_F$ with $s \leq t$. Again $N\pi_K = -\pi_F$. In this case $d = 2s$ and $\pi_K = \frac{\pi_F^s u'}{2}(1+x_0)$ where x_0 is a unit of trace 0. We note that $n(\psi_0)$ is equal to 2 if d is odd and $2(s-t)$ if d is even. So $n(\psi_0)$ is always even. Note also that if $\chi|_{F^*} = \omega$, then $a(\chi)$ is either $2d-1$ or it is even, say $2f$, with $f \geq d$.

We know that (see [3]) if $\chi \in \widehat{F^*}$ and ψ is a nontrivial additive character of F, then

$$\epsilon(\chi, \psi) = \chi(c)q^{-a(\chi)/2} \sum_{y \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(y)\psi(y/c), \tag{3}$$

where $v_F(c) = a(\chi) + n(\psi)$. In particular, since $a(\omega) = d$ and we have chosen ψ such that $n(\psi) = 0$ we have

$$\epsilon(\omega, \psi) = \omega\big(\pi_F^d\big)q^{-d/2} \sum_{y \in \frac{U_F}{U_F^d}} \omega(y)\psi\big(\pi_F^{-d}y\big). \tag{4}$$

This expression is obtained by normalizing the Haar measure given in the expression for ϵ-factor in [6] such that the volume of O_F is 1.

To start off, we have the following simple lemma.

Lemma 4.1. *For a regular character* θ *of* K^*, $a\big(\frac{\theta}{\overline{\theta}}\big)$ *is always even.*

Proof. Suppose $a\big(\frac{\theta}{\overline{\theta}}\big) = 2r+1$, $r \geq 0$. Then it has to be nontrivial on $\frac{U_K^{2r}}{U_K^{2r+1}}$. But $\frac{\theta}{\overline{\theta}}(1 + \pi_F^r a) = \frac{\theta(1+\pi_F^r a)}{\overline{\theta}(1+\pi_F^r a)} = 1$, where $a \in \mathbb{F}_q$ which is a contradiction. $\square$

4.1 *Twist by characters of odd conductor*

We reserve the symbols $\tilde{\omega}$ and $\tilde{\omega}_{K/F}$ to denote elements of S_{2d-1}.

Lemma 4.2. *If $\frac{\theta}{\bar{\theta}} = (-1)^{v_K}$, then no $\tilde{\omega}\theta$ can occur in r_θ.*

Proof. By Theorem 1.1, $\tilde{\omega}\theta$ can occur in $r_{\theta+}$ if and only if $\epsilon\big(\frac{\bar{\theta}}{\theta}\tilde{\omega}^{-1}, \psi_0\big) = \epsilon(\tilde{\omega}^{-1}, \psi_0) = 1$ and in $r_{\theta-}$ if and only if $\epsilon\big(\frac{\bar{\theta}}{\theta}\tilde{\omega}^{-1}, \psi_0\big) = \epsilon(\tilde{\omega}^{-1}, \psi_0) = -1$. Since $\frac{\bar{\theta}}{\theta}$ unramified, we have

$$
\begin{aligned}
\epsilon\left(\frac{\bar{\theta}}{\theta}\tilde{\omega}^{-1}, \psi_0\right) &= \frac{\bar{\theta}}{\theta}(\pi_K)^{a(\tilde{\omega}^{-1})+n(\psi_0)}\epsilon\big(\tilde{\omega}^{-1}, \psi_0\big) \\
&= \frac{\bar{\theta}}{\theta}(\pi_K)^{2d-1}\epsilon\big(\tilde{\omega}^{-1}, \psi_0\big) \text{ (since } n(\psi_0) \text{ even)} \\
&= -\epsilon(\tilde{\omega}^{-1}, \psi_0)
\end{aligned}
$$

which shows that $\epsilon(\tilde{\omega}^{-1}, \psi_0) = -\epsilon\big(\tilde{\omega}^{-1}\frac{\bar{\theta}}{\theta}, \psi_0\big)\ \forall\, \tilde{\omega} \in S(2d-1)$. Similarly $\epsilon(\tilde{\omega}^{-1}, \psi_0) = -\epsilon\big(\tilde{\omega}^{-1}\frac{\bar{\theta}}{\theta}, \psi_0\big)$ for all $\tilde{\omega} \in S'(2d-1)$. So $\tilde{\omega}\theta$ can occur neither in $r_{\theta+}$ nor in $r_{\theta-}$ for any $\tilde{\omega} \in S_{2d-1}$. Therefore it cannot occur in r_θ. □

Theorem 4.3. *Let $0 \neq a\big(\frac{\theta}{\bar{\theta}}\big) < a(\tilde{\omega})$. Then among all $\tilde{\omega} \in S(2d-1)$ half and only half will be such that $\tilde{\omega}\theta$ occur in $r_{\theta+}$ and among all $\tilde{\omega} \in S'(2d-1)$ half and only half will be such that $\tilde{\omega}\theta$ occur in $r_{\theta-}$.*

Remark. When $d = 1$, $a(\tilde{\omega}) = 1$. Therefore, since $a\big(\frac{\theta}{\bar{\theta}}\big) \neq 0$, this theorem is not applicable in the $d = 1$ case.

Proof. We show that $\sum_{\tilde{\omega}\in S(2d-1)} \epsilon\left(\tilde{\omega}^{-1}\frac{\bar{\theta}}{\theta}, \psi_0\right) = 0$ so that half of $\tilde{\omega} \in S(2d-1)$ will be such that $\epsilon\big(\tilde{\omega}^{-1}\frac{\bar{\theta}}{\theta}, \psi_0\big) = +1$ and the other half will be -1. The first half will occur in $r_{\theta+}$. The remaining half will not occur either in $r_{\theta+}$ or in $r_{\theta-}$. The other part of the proof is similar.

Note that $n(\psi_0)$ is always even irrespective of d. Also, $a(\tilde{\omega}^{-1}) = a\big(\tilde{\omega}^{-1}\frac{\bar{\theta}}{\theta}\big)$. Taking $c = \pi_F^{d+\frac{n(\psi_0)}{2}}\pi_K^{-1}$, in eq. (3) we have that if $\tilde{\omega} \in S(2d-1)$, then

$$
\begin{aligned}
\epsilon\left(\tilde{\omega}^{-1}\frac{\bar{\theta}}{\theta}, \psi_0\right) = {} & q^{-\frac{2d-1}{2}}\tilde{\omega}^{-1}\frac{\bar{\theta}}{\theta}\left(\pi_F^{d+\frac{n(\psi_0)}{2}}\pi_K^{-1}\right) \\
& \times \sum_{y\in\frac{U_K}{U_K^{2d-1}}} \tilde{\omega}\frac{\theta}{\bar{\theta}}(y)\psi_0\left(\pi_F^{-\left(d+\frac{n(\psi_0)}{2}\right)}\pi_K y\right).
\end{aligned}
$$

Write $y \in \frac{U_K}{U_K^{2d-1}}$ as $y = y_1(1+\pi_F^{r-1}\pi_K y_2)$, $r \geq 1$, $y_1 \in \frac{U_F}{U_F^d}$, $y_2 = 0$ or $y_2 \in \frac{U_F}{U_F^d}$. Also note that $\frac{\theta}{\bar{\theta}}$ is trivial on F^*. Summing over $S(2d-1)$, we get

$$\begin{aligned}
&\sum_{\tilde{\omega}\in S(2d-1)} \epsilon\left(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right)\\
&\quad = q^{\left(-\frac{2d-1}{2}\right)}\omega\left(\pi_F^{d+\frac{n(\psi_0)}{2}}\right)\frac{\overline{\theta}}{\theta}(\pi_K)\sum_{y_1,y_2,r,\tilde{\omega}}\Bigg[\tilde{\omega}(\pi_K y_1(1+\pi_F^{r-1}\pi_K y_2))\\
&\qquad \times \frac{\theta}{\overline{\theta}}\big(1+\pi_F^{r-1}\pi_K y_2\big)\psi_0\left(\pi_F^{-(d+\frac{n(\psi_0)}{2})}\pi_K y_1(1+\pi_F^{r-1}\pi_K y_2)\right)\Bigg].
\end{aligned}$$

But from the identity in Theorem 3.3 and the fact that we have to only consider characters in S with odd conductor (which is equal to $2d-1$) when $v_K(x) = 1$, it follows that

$$\begin{aligned}
&\sum_{\tilde{\omega}} \tilde{\omega}(\pi_K y_1(1+\pi_F^{r-1}\pi_K y_2))\\
&\quad = \begin{cases} \omega(-1)\epsilon(\omega,\psi)q^t\omega(y_1), & \text{if } d = 2t+1\\ \omega(-1)\epsilon(\omega,\psi)q^{s-\frac{1}{2}}\times\omega(\pi_F^{s-t}uu'y_1(1+\pi_F^{s+r-1}u'y_2)), & \text{if } d = 2s.\end{cases}
\end{aligned}$$

Also,

$$\begin{aligned}
&\psi_0\left(\pi_F^{-\left(d+\frac{n(\psi_0)}{2}\right)}\pi_K y_1(1+\pi_F^{r-1}\pi_K y_2)\right)\\
&\quad = \begin{cases} \psi(-\pi_F^{-d}y_1) \text{ if } d = 2t+1\\ \psi(-\pi_F^{-d}u^{-1}u'x_0^2 y_1(1+\pi_F^{r+s-1}u'y_2)) \text{ if } d = 2s.\end{cases}
\end{aligned}$$

Let $d = 2t+1$. If we keep y_2 fixed,

$$\begin{aligned}
&\sum_{y_1,\tilde{\omega}} \tilde{\omega}(\pi_K y_1(1+\pi_F^{r-1}\pi_K y_2))\psi_0\left(\pi_F^{-\left(d+\frac{n(\psi_0)}{2}\right)}\pi_K(1+\pi_F^{r-1}\pi_K y_2)\right)\\
&\quad = \epsilon(\omega,\psi)q^t\sum_{y_1}\omega(-y_1)\psi\big(-\pi_F^{-d}y_1\big)\\
&\quad = \epsilon(\omega,\psi)q^t\epsilon(\omega,\psi)\omega\big(\pi_F^d\big)q^d
\end{aligned}$$

which is a multiple of $\epsilon(\omega,\psi)$ independent of y_1 and y_2. So

$$\sum_{\tilde{\omega}\in S(2d-1)}\epsilon\left(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta},\psi_0\right) = C\sum_{r,y_2}\frac{\theta}{\overline{\theta}}\Big(\big(1+\pi_F^{r-1}\pi_K y_2\big) = C\times 0 = 0,$$

since $\frac{\theta}{\overline{\theta}}$ is a nontrivial character of $\frac{U_K}{U_F U_K^{a(\frac{\theta}{\overline{\theta}})}}$ and $a\left(\frac{\theta}{\overline{\theta}}\right) \le 2d-2$. Here C is a constant independent of y_1 and y_2. Similarly if $d = 2s$, then if we again keep y_2 fixed,

$$\begin{aligned}\sum_{y_1,\tilde{\omega}} &\tilde{\omega}(\pi_K y_1(1+\pi_F^{r-1}\pi_K y_2))\psi_0\left(\pi_F^{-\left(d+\frac{n(\psi_0)}{2}\right)}\pi_K(1+\pi_F^{r-1}\pi_K y_2)y_1\right)\\ &= \epsilon(\omega,\psi)q^{s-\frac{1}{2}}\omega(-\pi_F^{s-t}uu')\\ &\quad\times\sum_{y_1}\omega((1+\pi_F^{s+r-1}u'y_2)y_1)\psi(-\pi_F^{-d}u^{-1}u'x_0^2y_1(1+\pi_F^{r+s-1}u'y_2))\end{aligned}$$

which is again a constant multiple of $\epsilon(\omega,\psi)$ independent of y_1 and y_2. So

$$\sum_{\tilde{\omega}\in S(2d-1)}\epsilon\left(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta},\psi_0\right) = C'\sum_{r,y_2}\frac{\theta}{\overline{\theta}}\left((1+\pi_F^{r-1}\pi_K y_2\right) = C'\times 0 = 0,$$

where C' is a constant multiple of $\epsilon(\omega,\psi)$. This completes the proof of the theorem. □

COROLLARY 4.4

The number of $\tilde{\omega}\in S_{2d-1}$ *such that* $\tilde{\omega}\theta$ *occurs in* r_θ *is* $|S_{2d-1}|/2 = |S(2d-1)| = |S'(2d-1)|$.

Proof. This is clear since occurring in r_θ means occurring in either $r_{\theta+}$ or in $r_{\theta-}$. Equality follows from Theorem 3.6. □

Lemma 4.5. *If* $a\left(\frac{\theta}{\overline{\theta}}\right) > a(\tilde{\omega})$ *then the number of* $\tilde{\omega}\in S_{2d-1}$ *such that* $\tilde{\omega}\theta$ *occurs in* r_θ *is* $|S_{2d-1}|/2$.

Proof. This is quite easy to verify. In this case, $a\left(\frac{\theta}{\overline{\theta}}\right) = a\left(\tilde{\omega}^{-1}\frac{\theta}{\overline{\theta}}\right)$. Note that $a\left(\frac{\theta}{\overline{\theta}}\right)$ is even. So if

$$\epsilon\left(\tilde{\omega}^{-1},\psi_0\right) \neq \epsilon\left(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta},\psi_0\right), \tag{5}$$

consider the character $\mu = (-1)^{v_K}$ of K^* and take $\tilde{\omega}_2^{-1} = \tilde{\omega}^{-1}\mu$. If we consider the expression for epsilon factors on both sides of eq. (5), since $a\left(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta}\right)$ is even, no π_K is present but only π_F on the RHS of this equation. Therefore the twist by μ will not make any difference on the RHS. But on the LHS, an extra $\mu(\pi_K) = -1$ will appear changing the sign of LHS. Similarly if $\epsilon(\tilde{\omega}^{-1},\psi_0) = \epsilon\left(\tilde{\omega}^{-1}\frac{\overline{\theta}}{\theta},\psi_0\right)$, we can make them unequal by the same sort of twisting. So for half of $\tilde{\omega}\in S_{2d-1}$, the corresponding epsilon factors are equal and for the other half they are unequal. □

4.2 *Twist by characters of even conductor*

Note that if $a(\lambda) = 2f \ge 2d$, then in the expression for $\epsilon(\lambda^{-1},\psi_0)$ there is no π_K, but only π_F.

Theorem 4.6. *Let $\lambda \in S(2f+2d)$, $f \geq 0$, $a\left(\frac{\theta}{\overline{\theta}}\right) \leq a(\lambda) - 2d = 2f$. Then all the elements in $\{\lambda\theta : \lambda \in S(2f+2d)\}$ will occur in $r_{\theta+}$. Similarly if $\lambda' \in S'(2f+2d)$, then all the elements in $\{\lambda'\theta : \lambda' \in S'(2f+2d)\}$ will occur in $r_{\theta-}$. Therefore the number of $\lambda\theta$ where $\lambda \in S_{2f+2d}$ occurring in r_θ is $|S_{2f+2d}|$.*

Proof. Consider the two sums $\sum_{\lambda\in S(2f+2d)} \epsilon(\lambda^{-1}, \psi_0)$ and $\sum_{\lambda\in S(2f+2d)} \epsilon\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right)$. We have

$$\begin{aligned}
&\sum_{\lambda\in S(2f+2d)} \epsilon\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right) \\
&\quad = q^{-f-d}\omega\left(\pi_F^{f+d+\frac{n(\psi_0)}{2}}\right) \sum_{\lambda\in S(2f+2d)} \sum_{y\in\frac{U_K}{U_K^{2f+2d}}} \\
&\qquad \times \lambda(y)\frac{\theta}{\overline{\theta}}(y)\psi_0\left(\pi_F^{-\left(f+d+\frac{n(\psi_0)}{2}\right)}y\right).
\end{aligned}$$

In this summation, by Theorem 3.4, $\sum_{\lambda\in S(2f+2d)} \lambda(y) \neq 0$ only for two types of y's:

(1) when $y = y_1(1+\pi_F^f\pi_K y_2)$, $y_1, y_2 \in U_F$, and
(2) when $y = y_1(1+\pi_F^{f+d-1}\pi_K y_2)$, $y_1, y_2 \in U_F$ or $y_2 = 0$.

But since $a\left(\frac{\theta}{\overline{\theta}}\right) \leq 2f$ and $\frac{\theta}{\overline{\theta}} = 1$ on F^* we have $\frac{\theta}{\overline{\theta}}$ trivial on these y's. So both the sums are independent of $\frac{\theta}{\overline{\theta}}$ and so they are the same. That is,

$$\sum_{\lambda\in S(2f+2d)} \epsilon(\lambda^{-1}, \psi_0) = \sum_{\lambda\in S(2f+2d)} \epsilon\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right)$$

which means

$$\epsilon(\lambda^{-1}, \psi_0) = \epsilon\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right) \quad \forall\lambda \in S(2f+2d)$$

since $\epsilon(\lambda^{-1}, \psi_0) = 1$ for each $\lambda \in S(2f+2d)$. The remaining part follows similarly. □

COROLLARY 4.7

If $\frac{\theta}{\overline{\theta}} = (-1)^{v_K}$, then all $\lambda \in S(2f+2d)$ are such that $\lambda\theta$ occur in $r_{\theta+}$. Similarly all $\lambda' \in S'(2f+2d)$ are such that $\lambda'\theta$ occur in $r_{\theta-}$.

Proof. It follows by taking $a\left(\frac{\theta}{\overline{\theta}}\right) = 0$ in the above theorem. □

Note. The above corollary shows the difference between characters of even conductor and characters of odd conductor. This corollary is extremely opposite to Lemma 4.2.

Let $\lambda \in S(2f+2d),\ 2f < a\left(\frac{\theta}{\overline{\theta}}\right) < a(\lambda)$. Note that if $d=1$, then no such θ exists. So we have $d \geq 2$ and so q is even. By definition, we have

$$\epsilon\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right) = q^{-f-d}\omega\left(\pi_F^{f+d+\frac{n(\psi_0)}{2}}\right)\sum_{\lambda\in S(2f+2d)}\sum_{y\in\frac{U_K}{U_K^{2f+2d}}}\lambda(y)\frac{\theta}{\overline{\theta}}(y)\psi_0 \times\left(\pi_F^{-\left(f+d+\frac{n(\psi_0)}{2}\right)}y\right).$$

Again, by Theorem 3.4, the sum $\sum_{\lambda\in S(2f+2d)}\lambda(y)\neq 0$ only for two types of y's:

(1) $y = y_1\left(1+\pi_F^{f+d-1}\pi_K y_2\right),\ y_1 \in \frac{U_F}{U_F^{f+d}},\ y_2 \in \mathbb{F}_q$;

(2) $y = y_1\left(1+\pi_F^{f}\pi_K y_2\right),\ y_1 \in \frac{U_F}{U_F^{f+d}},\ y_2 \in \frac{U_F}{U_F^{d}}$.

Consider the first type of y's. $\frac{\theta}{\overline{\theta}}$ is trivial on the these y's. Now

$$\sum_{\lambda\in S(2f+2d)}\lambda(y)=\omega(y_1)\sum_{\lambda\in S(2f+2d)}\lambda\left(1+\pi_F^{f+d-1}\pi_K y_2\right)=-q^{f+d-1}\omega(y_1).$$

Also

$$\psi_0\left(\pi_F^{-f-d-\frac{n(\psi_0)}{2}}y\right) = \begin{cases}\psi\left(-\pi_F^{-1}y_1y_2\right), & \text{if } d=2t+1\\ \psi\left(-\pi_F^{-1}u^{-1}u'x_0^2y_1y_2\right), & \text{if } d=2s.\end{cases}$$

Therefore

$$\sum_{\lambda\in S(2f+2d)}\lambda(y)\psi_0\left(\pi_F^{-f-d-\frac{n(\psi_0)}{2}}y\right)$$
$$=\begin{cases}-q^{f+d-1}\sum_{y_2\in\mathbb{F}_q}\sum_{y_1\in\frac{U_F}{U_F^{f+d}}}\omega(y_1)\psi\left(-\pi_F^{-1}y_1y_2\right)\\ \qquad\text{if } d \text{ odd},\\ -q^{f+d-1}\sum_{y_2\in\mathbb{F}_q}\sum_{y_1\in\frac{U_F}{U_F^{f+d}}}\omega(y_1)\psi\left(-\pi_F^{-1}u^{-1}u'x_0^2y_1y_2\right)\\ \qquad\text{if } d \text{ even},\end{cases}$$
$$=\begin{cases}-q^{f+d-1}\sum_{y_2\in\mathbb{F}_q}\omega(-y_2)\sum_{y_1\in\frac{U_F}{U_F^{f+d}}}\omega(y_1)\psi\left(\pi_F^{-1}y_1\right)\\ \qquad\text{if } d \text{ odd},\\ -q^{f+d-1}\sum_{y_2\in\mathbb{F}_q}\omega\left(-y_2x_0^2uu'\right)\sum_{y_1\in\frac{U_F}{U_F^{f+d}}}\omega(y_1)\psi\left(\pi_F^{-1}y_1\right)\\ \qquad\text{if } d \text{ even}.\end{cases}$$

Since $a(\omega) = d \neq 1$, by Lemma 3.5, $\sum_{y_1 \in \frac{U_F}{U_F^{f+d}}} \omega(y_1)\psi(\pi_F^{-1} y_1) = 0$. So

$$\sum_{\lambda \in S(2f+2d)} \lambda(y) \sum_{y_1 \in \frac{U_F}{U_F^{f+d}}} \lambda(y)\psi_0\left(\pi_F^{-f-d-\frac{n(\psi_0)}{2}} y\right) = 0.$$

Consider the second type of y's. On these, we have

$$\begin{aligned}
&\sum_{\lambda \in S(2f+2d)} \lambda\left(y_1\left(1+\pi_F^f \pi_K y_2\right)\right)\\
&\quad = \begin{cases} \omega(-1)\omega(y_1)\omega\big(\pi_F^f y_2\big) q^{f+t+\frac{1}{2}} \epsilon(\omega,\psi), \text{ if } d \text{ odd} \\ \omega(-1)\omega(y_1 y_2 u u')\omega\big(\pi_F^{f+s-t} y_2\big) q^{f+s}\epsilon(\omega,\psi), \text{ if } d \text{ even} \end{cases}\\
&\qquad\qquad\qquad\qquad\qquad \text{by Theorems 3.3 and 3.4}
\end{aligned}$$

and

$$\psi_0\left(\pi_F^{-f-d-\frac{n(\psi_0)}{2}} y_1\big(1+\pi_F^f \pi_K y_2\big)\right) = \begin{cases} \psi(-\pi_F^{-d} y_1 y_2) \text{ if } d \text{ odd} \\ \psi(-\pi_F^{-d} y_1 y_2 u^{-1} u' x_0^2) \text{ if } d \text{ even.} \end{cases}$$

Let $d = 2t+1$. Then

$$\begin{aligned}
&\sum_{\lambda \in S(2f+2d)} \sum_{y_1 \in \frac{U_F}{U_F^{f+d}}} \sum_{y_2 \in \frac{U_F}{U_F^d}} \lambda(y)\frac{\theta}{\overline{\theta}}(y)\psi_0(\pi_F^{-f-d-1} y)\\
&= \omega(-1)\omega\big(\pi_F^f\big) q^{f+t+\frac{1}{2}}\epsilon(\omega,\psi)\\
&\quad \times \sum_{y_1 \in \frac{U_F}{U_F^{f+d}}} \sum_{y_2 \in \frac{U_F}{U_F^d}} \omega(y_1 y_2)\psi\big(-\pi_F^{-d} y_1 y_2\big)\frac{\theta}{\overline{\theta}}\big(1+\pi_F^f \pi_K y_2\big)\\
&= q^f q^{f+t+\frac{1}{2}}\omega\big(\pi_F^f\big)\epsilon(\omega,\psi)\\
&\quad \times \sum_{y_2 \in \frac{U_F}{U_F^d}} \frac{\theta}{\overline{\theta}}\big(1+\pi_F^f \pi_K y_2\big) \sum_{y_1 \in \frac{U_F}{U_F^d}} \omega(y_1 y_2)\psi\big(\pi_F^{-d} y_1 y_2\big)\\
&= q^{2f} q^{\frac{d}{2}}\omega(\pi_F^f)\epsilon(\omega,\psi)\epsilon(\omega,\psi)\omega(\pi_F^d) q^{\frac{d}{2}} \sum_{y_2 \in \frac{U_F}{U_F^d}} \frac{\theta}{\overline{\theta}}(1+\pi_F^f \pi_K y_2)\\
&= q^{2f+d}\omega(\pi_F^{f+d})\epsilon(\omega,\psi)^2 \sum_{y_2 \in \frac{U_F}{U_F^d}} \frac{\theta}{\overline{\theta}}(1+\pi_F^f \pi_K y_2).
\end{aligned}$$

If $d = 2s$ we will get the same sum with an extra $\omega(-1)$ factor. Now if $a\left(\frac{\theta}{\overline{\theta}}\right) \geq 2f + 4$, then

$$\sum_{y_2 \in \frac{U_F}{U_F^d}} \frac{\theta}{\overline{\theta}}(1 + \pi_F^f \pi_K y_2) = 0 \text{ and so } \sum_{\lambda \in S(2f+2d)} \epsilon\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right) = 0.$$

Therefore half of elements in $\{\lambda\theta : \lambda \in S(2f + 2d)\}$ will appear in $r_{\theta+}$. Similarly, half of elements in $\{\lambda'\theta : \lambda' \in S'(2f + 2d)\}$ will appear in $r_{\theta-}$.

Let $a\left(\frac{\theta}{\overline{\theta}}\right) = 2f + 2$. Then

$$\sum_{y_2 \in \frac{U_F}{U_F^d}} \frac{\theta}{\overline{\theta}}(1 + \pi_F^f \pi_K y_2) = q^{d-1} \sum_{a \in \mathbb{F}_q} \frac{\theta}{\overline{\theta}}(1 + \pi_F^f \pi_K a) = -q^{d-1}.$$

Therefore if $d = 2t + 1$, then

$$\sum_{\lambda \in S(2f+2d)} \sum_{y_1 \in \frac{U_F}{U_F^{f+d}}} \sum_{y_2 \in \frac{U_F}{U_F^d}} \lambda(y)\frac{\theta}{\overline{\theta}}(y)\psi_0(\pi_F^{-f-d-1} y)$$

$$= -q^{2f+2d-1}\omega(\pi_F^{f+d})\epsilon(\omega, \psi)^2.$$

So

$$\sum_{\lambda \in S(2f+2d)} \epsilon\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right) = q^{-f-d}\omega\left(\pi_F^{f+d}\right) \times -q^{2f+2d-1}\omega\left(\pi_F^{f+d}\right)\epsilon(\omega, \psi)^2$$

$$= -q^{f+d-1}\epsilon(\omega, \psi)^2.$$

Similarly we will get $\sum_{\lambda \in S(2f+2d)} \epsilon(\lambda^{-1}, \psi_0) = (q - 1)q^{f+d-1}\epsilon(\omega, \psi)^2$. (This is because in place of $\sum_{y_2 \in \frac{U_F}{U_F^d}} \frac{\theta}{\overline{\theta}}(1 + \pi_F^f \pi_K y_2)$, we have $|\frac{U_F}{U_F^d}|$.)

But

$$\sum_{\lambda \in S(2f+2d)} \epsilon(\lambda^{-1}, \psi_0) = |S(2f + 2d)| = (q - 1)q^{f+d-1}.$$

So $\epsilon(\omega, \psi)^2 = 1$. (If $d = 2s$, instead of this, we have $\omega(-1)\epsilon(\omega, \psi)^2 = 1$.) Therefore number of λ such that $\lambda\theta$ appear in $r_{\theta+}$ is $\sum_{\lambda \in S(2f+2d)} \left(\epsilon(\lambda^{-1}, \psi_0) + \epsilon\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right)\right) = \frac{(q-1)-1}{2}q^{f+d-1} = \frac{q-2}{2}q^{f+d-1}$. If $d = 2s$, we can show that the sum is $\frac{q-2}{2}q^{f+d-1}$.

So in $r_{\theta+}$, the number of $\lambda\theta$ occurring where $\lambda \in S(2f + 2d)$ is $\frac{q-2}{2}q^{f+d-1}$. Similarly in $r_{\theta-}$, the number of $\lambda'\theta$ occurring where $\lambda' \in S'(2f + 2d)$ is $\frac{q-2}{2}q^{f+d-1}$. We summarize the above computations in the following two theorems.

Theorem 4.8. *Let $\lambda \in S(2f+2d)$, $2f+2 < a\left(\frac{\theta}{\overline{\theta}}\right) < a(\lambda)$. Then among all $\lambda\theta$ where $\lambda \in S(2f+2d)$ exactly half will occur in $r_{\theta+}$. Similarly, let $\lambda' \in S'(2f+2d)$, $2f+2 < a\left(\frac{\theta}{\overline{\theta}}\right) < a(\lambda')$. Then among all $\lambda'\theta$ where $\lambda' \in S'(2f+2d)$ exactly half will occur in $r_{\theta-}$. Therefore the number of $\lambda\theta$ where $\lambda \in S_{2f+2d}$ occurring in r_θ is $|S_{2f+2d}|/2$.*

Theorem 4.9. *Let $\lambda \in S(2f+2d)$, $a\left(\frac{\theta}{\overline{\theta}}\right) = 2f+2 < a(\lambda)$. Then the number of $\lambda\theta$ appearing in $r_{\theta+}$ where $\lambda \in S(2f+2d)$ is $\frac{q-2}{2}q^{f+d-1}$. Similarly, let $\lambda' \in S'(2f+2d)$, $a\left(\frac{\theta}{\overline{\theta}}\right) = 2f+2 < a(\lambda')$. Then the number of $\lambda'\theta$ appearing in $r_{\theta-}$ where $\lambda' \in S'(2f+2d)$ is $\frac{q-2}{2}q^{f+d-1}$. The number of $\lambda\theta$ where $\lambda \in S_{2f+2d}$ occurring in r_θ is therefore $(q-2)q^{f+d-1}$.*

Note. These two theorems are not valid for $d=1$ since no $\frac{\theta}{\overline{\theta}}$ satisfies the condition in the theorem.

Theorem 4.10. *Let $a(\lambda) = 2f+2d < a\left(\frac{\theta}{\overline{\theta}}\right) = 2m < a(\lambda)+2d$. Then the number of $\lambda\theta$ with $\lambda \in S_{2f+2d}$ appearing in r_θ is $|S(2f+2d)| = |S_{2f+2d}|/2$.*

Proof. Here $a\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}\right) = a\left(\frac{\theta}{\overline{\theta}}\right)$. Using the definition of ϵ-factors, we have

$$\sum_{\lambda\in S(2f+2d)} \epsilon\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right)$$

$$= q^{-m}\omega\left(\pi_F^{m+\frac{n(\psi_0)}{2}}\right)\sum_{\lambda\in S(2f+2d)}\sum_{y\in\frac{U_K}{U_K^{2m}}}\lambda(y)\frac{\theta}{\overline{\theta}}(y)\psi_0\left(\pi_F^{-(m+\frac{n(\psi_0)}{2})}y\right).$$

Recall that, from Theorem 3.4 the sum $\sum_{\lambda\in S(2f+2d)}\lambda(y) \neq 0$ only for three types of y's:

(1) $y = y_1(1+\pi_F^f\pi_K y_2)$, $y_1 \in \frac{U_F}{U_F^m}$, $y_2 \in \frac{U_F}{U_F^{m-f}}$,

(2) $y = y_1(1+\pi_F^{f+d-1}\pi_K y_2)$, $y_1 \in \frac{U_F}{U_F^m}$, $y_2 \in \frac{U_F}{U_F^{m-f-d+1}}$,

(3) $y = y_1(1+\pi_F^{f+d}\pi_K y_2)$, $y_1 \in \frac{U_F}{U_F^m}$, $y_2 \in \frac{U_F}{U_F^{m-f-d}}$ or $y_2 = 0$.

But on the third type of y's, λ is just ω since $a(\lambda) = 2f+2d$. On the second type of y's,

$$\sum_{\lambda\in S(2f+2d)}\lambda(y) = \omega(y_1)\sum_{\lambda\in S(2f+2d)}\lambda(1+\pi_F^{f+d-1}\pi_K y_2)$$

$$= \omega(y_1)(-q^{f+d-1}) \text{ (by Theorem 3.4).}$$

So $\sum_{\lambda\in S(2f+2d)}\lambda(y)$ is independent of λ on these y's. Finally consider the first type of y's. Let $d=2t+1$.

$$\begin{aligned}\sum_{\lambda\in S(2f+2d)}\lambda(y_1(1+\pi_F^f\pi_K y_2)) &= \omega(y_1)\sum_{\lambda\in S(2f+2d)}\lambda(1+\pi_F^f\pi_K y_2)\\ &= \omega(-1)\omega(y_1)\epsilon(\omega,\psi)\omega(\pi_F^f y_2)q^{f+t+\frac{1}{2}},\\ &\quad\text{by Theorems 3.3 and 3.4.}\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{\lambda\in S(2f+2d)}\sum_{y_1\in\frac{U_F}{U_F^m}}\sum_{y_2\in\frac{U_F}{U_F^{m-f}}}\lambda(y_1(1+\pi_F^f\pi_K y_2))\frac{\theta}{\bar{\theta}}(y_1(1+\pi_F^f\pi_K y_2))\\ &\times\psi_0(\pi_F^{-m-1}y_1(1+\pi_F^f\pi_K y_2)) = \epsilon(\omega,\psi)\omega(\pi_F^f)q^{f+t+\frac{1}{2}}\\ &\times\sum_{y_1\in\frac{U_F}{U_F^m}}\sum_{y_2\in\frac{U_F}{U_F^{m-f}}}\omega(-y_1y_2)\psi\left(-\pi_F^{-m+f}y_1y_2\right)\frac{\theta}{\bar{\theta}}\left(1+\pi_F^f\pi_K y_2\right).\end{aligned}$$

In this sum, we have

$$\begin{aligned}&\sum_{y_1\in\frac{U_F}{U_F^m}}\omega(y_1y_2)\psi(-\pi_F^{-m+f}y_1y_2)\\ &=\sum_{y_1\in\frac{U_F}{U_F^{m-f}}}\omega(y_1y_2)\psi(-\pi_F^{-m+f}y_1y_2)\left|\frac{U_F^{m-f}}{U_F^m}\right|.\end{aligned}$$

Since $a(\omega)=d$ and $m-f>d$, by Lemma 3.5, the above sum is zero.

Now if $d=2s$, we have $n(\psi_0)=2(s-t)$. Also, here the trace 0 element x_0 is a unit and $\pi_K=\frac{\pi_F^s u'}{2}(1+x_0)$. Considering y's first type, we have

$$\begin{aligned}\psi_0(\pi_F^{-\left(m+\frac{n(\psi_0)}{2}\right)}y) &= \psi_0(\pi_F^{-m-s+t}y_1(1+\pi_F^f\pi_K y_2))\\ &= \psi\left(-\frac{\pi_F^{-m-s+t}}{2}y_1 tr\, x_0(1+\pi_F^f\pi_K y_2)\right)\\ &= \psi\left(-\frac{\pi_F^{-m-s+t}}{2}y_1x_0^2\cdot 2\pi_F^f y_2\frac{\pi_F^s}{2}u'\right)\\ &= \psi(-\pi_F^{-m+f}u^{-1}u'y_1y_2x_0^2)\end{aligned}$$

and so

$$\sum_{\lambda\in S(2f+2d)}\lambda(y_1(1+\pi_F^f\pi_K y_2))=\omega(-1)\epsilon(\omega,\psi)\omega(\pi_F^{f+s-t})q^{f+s}\omega(y_1y_2uu')$$

by Theorems 3.3 and 3.4.

Therefore

$$\sum_{\lambda\in S(2f+2d)}\sum_{y_1\in\frac{U_F}{U_F^m}}\sum_{y_2\in\frac{U_F}{U_F^{m-f}}}[\lambda(y_1(1+\pi_F^f\pi_K y_2))\frac{\theta}{\overline{\theta}}(y_1(1+\pi_F^f\pi_K y_2))$$

$$\times\,\psi_0(\pi_F^{-m-s+t}y_1(1+\pi_F^f\pi_K y_2))]=\epsilon(\omega,\psi)\omega(\pi_F^{f+s-t})q^{f+s}$$

$$\times\sum_{y_2\in\frac{U_F}{U_F^{m-f}}}\frac{\theta}{\overline{\theta}}(1+\pi_F^f\pi_K y_2)\sum_{y_1\in\frac{U_F}{U_F^m}}\omega(y_1y_2u^{-1}u')\psi(-\pi_F^{-m+f}u^{-1}u'y_1y_2x_0^2).$$

The sum $\sum_{y_1\in\frac{U_F}{U_F^m}}\omega(y_1y_2u^{-1}u')\psi(\pi_F^{-m+f}u^{-1}u'y_1y_2)=0$ as in the $d=2t+1$ case since $m-f>d$. So the sum over the first type of y's become zero. So in both d odd and d even cases, the sum $\sum_{\lambda\in S(2f+2d)}\epsilon(\lambda^{-1}\frac{\overline{\theta}}{\theta},\psi_0)$ depends only on second and third type of y's and is independent of λ. Suppose this sum is n. Using similar arguments, we have $\sum_{\lambda'\in S'(2f+2d)}\epsilon(\lambda'^{-1}\frac{\overline{\theta}}{\theta},\psi_0)=n$. So the number of $+1$'s in $\{\epsilon(\lambda^{-1}\frac{\overline{\theta}}{\theta},\psi_0):\lambda\in S(2f+2d)\}=\frac{|S(2f+2d)|+n}{2}$. Similarly, number of -1's in $\{\epsilon(\lambda'^{-1}\frac{\overline{\theta}}{\theta},\psi_0):\lambda'\in S'(2f+2d)\}=-\frac{-|S(2f+2d)|+n}{2}$. Therefore, the number of $\lambda\theta$ appearing in $r_{\theta+}$ is $\frac{|S(2f+2d)|+n}{2}$, number of $\lambda'\theta$ appearing in $r_{\theta-}$ is $\frac{|S(2f+2d)|-n}{2}$. Total number of $\lambda\theta$ appearing in r_θ is $|S(2f+2d)|$. □

When $a(\lambda)$ is too small compared to $a\left(\frac{\theta}{\overline{\theta}}\right)$ the occurrence of $\lambda\theta$ in $r_{\theta+}$ or $r_{\theta-}$ depends only on θ.

Theorem 4.11. *Suppose $\lambda\in S(2m)$, $m\geq d$ and $a\left(\frac{\theta}{\overline{\theta}}\right)=2n\geq a(\lambda)+2d$. Then either all the elements in $\{\lambda\theta:\lambda\in S(2m)\}$ will occur in $r_{\theta+}$ or all the elements in $\{\lambda'\theta:\lambda'\in S'(2m)\}$ will occur in $r_{\theta-}$ and not in both. Therefore the number of $\lambda\theta$ where $\lambda\in S_{2m}$ occurring in r_θ is $|S_{2m}|/2$.*

Proof. We have that if $\chi\in\widehat{K^*}$ with $\chi|_{K^*}=\omega$ and $a(\chi)\geq 2a(\tilde{\omega})$ then $\epsilon(\chi,\psi_0)=\tilde{\omega}(-x_0/2)\tilde{\omega}^{-1}(y_\chi)$ where

$$y_\chi=\begin{cases}\pi_F^{-f-\frac{d-1}{2}}\pi_K a_0(\chi)(1+a_1(\chi)\pi_K)(1+a_2(\chi)\pi_F)... & \text{if } d \text{ is odd,}\\[2ex] \pi_F^{-f-\frac{d}{2}}x_0a_0(\chi)(1+a_1(\chi)\pi_K)(1+a_2(\chi)\pi_F)... & \text{if } d \text{ is even.}\end{cases}$$

Here $a\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}\right) = a\left(\frac{\overline{\theta}}{\theta}\right) \geq 4d > 2a(\tilde{\omega})$. Therefore

$$\begin{aligned}\epsilon\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right) &= \tilde{\omega}(-x_0/2)\\ &\times \tilde{\omega}^{-1}\left(\pi_F^{-f-\frac{d-1}{2}}\pi_K a_0\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}\right)\left(1+a_1\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}\right)\pi_K\right)\ldots\right.\\ &\times\left.\left(1+a_{2d-2}\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}\right)\pi_F^{d-1}\right)\right),\end{aligned}$$

if d odd.

But note that $\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}\right)|_{U_K^{2n-2d+1}}$ determines $a_i\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}\right)$ for $i = 0, 1, \ldots, 2d-2$ and on $U_K^{2n-2d+1}$, $\lambda^{-1}\frac{\overline{\theta}}{\theta} = \frac{\overline{\theta}}{\theta}$. Therefore $\epsilon\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right)$ is independent of λ or $\epsilon\left(\lambda^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right) = \epsilon\left(\frac{\overline{\theta}}{\theta}, \psi_0\right)$. Now suppose that $\epsilon(\lambda^{-1}, \psi_0) \neq \epsilon\left(\frac{\overline{\theta}}{\theta}, \psi_0\right) = -1$ for one $\lambda \in S(2m)$. Then for all $\lambda' \in S'(2m)$, $\epsilon(\lambda'^{-1}, \psi_0) = -1 = \epsilon\left(\frac{\overline{\theta}}{\theta}, \psi_0\right) = \epsilon\left(\lambda'^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right)$. Therefore $\{\lambda'\theta : \lambda' \in S'(2m)\}$ will occur in $r_{\theta-}$. On the other hand, if $\epsilon(\lambda^{-1}, \psi_0) = \epsilon\left(\frac{\overline{\theta}}{\theta}, \psi_0\right) = 1$, for one λ it is the same for all other $\lambda \in S(2m)$. This proves the theorem. $\square$

COROLLARY 4.12

Suppose $a(\lambda) = 2f + 2d < a\left(\frac{\theta}{\overline{\theta}}\right) = 2m$. *If* n *= number of* $\lambda\theta$, $\lambda \in S(2f+2d)$, *appearing in* $r_{\theta+}$ *then number of* $\lambda'\theta$, $\lambda' \in S'(2f+2d)$, *appearing in* $r_{\theta-}$ *is* $|S(2f+2d)| - n = |S'(2f+2d)| - n$. *Also, if* $2m > a(\lambda) + 2d$, *then either* $n = 0$ *or* $n = |S(2f+2d)|$.

Proof. Follows easily from the above two theorems. $\square$

Only one case is left now for us to handle in this exposition viz. $a\left(\frac{\theta}{\overline{\theta}}\right) = a(\lambda)$. In this case we are not giving an exact count, but still we provide a lower bound in the next theorem. Note that our calculations deal much with $a\left(\frac{\theta}{\overline{\theta}}\lambda\right)$ and it is difficult to find when the two characters have equal conductor.

Theorem 4.13. *If* $a\left(\frac{\theta}{\overline{\theta}}\right) = a(\lambda) = 2f + 2d$, $\lambda|_{F^*} = \omega$ *then the number of* $\lambda\theta$ *appearing in* r_θ *is greater than or equal to* q^{f+d-1}.

Proof. Note that $S(2f+2d) \cup S'(2f+2d) = \left\{\frac{\overline{\theta}}{\theta}\chi : \chi|_{F^*} = \omega, a(\chi) = 2d-1, 2d, 2d+2, \ldots, 2f+2d-2\right\} \cup \left\{\frac{\overline{\theta}}{\theta}\chi : \chi|_{F^*} = \omega, a(\chi) = 2f+2d,\ \chi|_{U_K^{2f+2d-1}} \neq \frac{\theta}{\theta}|_{U_K^{2f+2d-1}}\right\}$. Now a $\frac{\overline{\theta}}{\theta}\chi \cdot \theta = \chi\overline{\theta}$ will appear in r_θ if and only if $\epsilon\left(\chi^{-1}, \psi_0\right) = \epsilon\left(\chi^{-1}\frac{\theta}{\overline{\theta}}, \psi_0\right)$. So the number of $\chi\overline{\theta}$ appearing in r_θ where $a(\chi) = 2d-1, 2d, \ldots, 2f+2d$ is greater than or equal to $|S(2d-1)| + |S(2d)| + S(2d+2)| + \cdots + |S(2f+2d-2)| = q^{d-1} + (q-1)q^{d-1} + (q-1)q^d + \cdots + (q-1)q^{f+d-2} = q^{f+d-1}$ by Corollary 4.4, Lemma 4.5

and Theorems 4.10 and 4.11. Note that we are not considering χ's with conductor equal to $2f + 2d$ and that is why we are unable to claim equality. $\square$

Remark. If $q = 2$, there is no χ such that $\chi|_{U_K^{2f+2d-1}} \neq \frac{\overline{\theta}}{\theta}|_{U_K^{2f+2d-1}}$. So equality holds in the theorem.

5. The unramified case

Suppose K over F is unramified and let $\chi \in \widehat{F^*}$ be such that $\chi|_{K^*} = \omega$. Let $\tilde{\omega}$ be an extension of ω trivial on U_K and -1 on any uniformizer of K. Note that $a\left(\frac{\theta}{\overline{\theta}}\right) \neq 0$. Otherwise, since $\pi_K = \pi_F \in F$ in this case, $\frac{\theta}{\overline{\theta}}(\pi_K) = 1$ so that $\frac{\theta}{\overline{\theta}}$ is trivial. Then $\theta = \overline{\theta}$ contradicting the regularity of θ. So $a\left(\frac{\theta}{\overline{\theta}}\right) \geq 1$.

We divide our counting into mainly three cases:

Case 1. $a\left(\frac{\theta}{\overline{\theta}}\right) < a(\chi)$. We have $\epsilon\left(\chi^{-1}, \psi_0\right) = \tilde{\omega}(-x_0/2)\tilde{\omega}^{-1}(y_{\chi^{-1}})$ by eq. (1). Since $\tilde{\omega}$ is trivial on units in the unramified case, let $y_{\chi^{-1}} = \pi_F^{-a(\chi)}$. So we have $\epsilon\left(\chi^{-1}, \psi_0\right) = (-1)^{a(\chi)+t}$ where $t = v_F(2)$. Since $a\left(\frac{\theta}{\overline{\theta}}\right) < a(\chi)$ we have $\epsilon\left(\chi^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right) = (-1)^{a(\chi)+t} = \epsilon\left(\chi^{-1}, \psi_0\right)$. So all the χ's are such that all $\chi\theta$ will occur in $r_{\theta+}$ or all will occur in $r_{\theta-}$ depending on whether $a(\chi)$ is even or odd.

Case 2. $a(\chi) < a\left(\frac{\theta}{\overline{\theta}}\right)$. In this case, $a\left(\chi^{-1}\frac{\overline{\theta}}{\theta}\right) = a\left(\frac{\theta}{\overline{\theta}}\right)$. So $\epsilon\left(\chi^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right) = (-1)^{a\left(\frac{\theta}{\overline{\theta}}\right)+t}$. Also, $\epsilon\left(\chi^{-1}\frac{\overline{\theta}}{\theta}, \psi_0\right) = (-1)^{a(\chi)+t}$. $\chi\theta$ will occur in r_θ if and only if $a(\chi) = a(\frac{\theta}{\overline{\theta}}) (\bmod 2)$.

Case 3. $a(\chi) = a\left(\frac{\theta}{\overline{\theta}}\right)$. Here we have two possibilities:

(1) $a\left(\chi^{-1}\frac{\overline{\theta}}{\theta}\right) < a(\chi)$ or $a(\chi) < a\left(\chi^{-1}\frac{\theta}{\overline{\theta}}\right)$: In this case, if $a(\chi) = a\left(\frac{\theta}{\overline{\theta}}\right) (\bmod 2)$ then $\chi\theta$ will occur in r_θ.
(2) $a\left(\chi^{-1}\frac{\overline{\theta}}{\theta}\right) = a(\chi)$: In this case $\chi\theta$ will occur in r_θ.

Remark. Since by Theorem 1.1, $\lambda\theta$ appears in $r_{\theta+}$ (respectively $r_{\theta-}$) if and only if $\lambda\theta$ does not appear in $r'_{\theta+}$ (respectively $r'_{\theta-}$). All the theorems proved in this paper have their obvious D_F^* analogues.

Acknowledgements

The author would like to thank R Tandon, University of Hyderabad, India for some tedious discussions and helpful suggestions and D Prasad, Tata Institute of Fundamental Research, India for motivating him towards the problems they discussed in this exposition.

References

[1] Deligne P, Les Constantes locales de l'èquation fonctionelle de la fonction Ld'Artin d'une representation orthogonale, *Invent. Math.* **35** (1976) 299–316
[2] Kameswari P A and Tandon R, A converse theorem for epsilon factors, *J. Number Theory* **89** (2001) 308–323

[3] Namboothiri K Vishnu and Tandon R, Completing an extension of Tunnell's theorem, *J. Number Theory* **128** (2008) 1622–1636

[4] Prasad D, On an extension of a theorem of Tunnell, *Compos. Math.* **94** (1994) 19–28

[5] Prasad D, Relating invariant linear forms and local epsilon factors via global methods, with an appendix by H Saito, *Duke Math. J.* **138(2)** (2007) 233–261

[6] Tate J, Number theoretic background, in automorphic forms, representations and L-function, *AMS Proc. Symp. Pure Math.* **33(2)** (1979) 3–26

[7] Tunnell J, Local epsilon factors and characters of $GL(2)$, *Am. J. Math.* **105** (1983) 1277–1307