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Gromov hyperbolicity in Cartesian product graphs

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Abstract. If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. The space X is δ-*hyperbolic* (in the Gromov sense) if any side of T is contained in a δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. If X is hyperbolic, we denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e. $\delta(X) = \inf\{\delta \geq 0\colon X \text{ is } \delta\text{-hyperbolic}\}$. In this paper we characterize the product graphs $G_1 \times G_2$ which are hyperbolic, in terms of G_1 and G_2: the product graph $G_1 \times G_2$ is hyperbolic if and only if G_1 is hyperbolic and G_2 is bounded or G_2 is hyperbolic and G_1 is bounded. We also prove some sharp relations between the hyperbolicity constant of $G_1 \times G_2$, $\delta(G_1)$, $\delta(G_2)$ and the diameters of G_1 and G_2 (and we find families of graphs for which the inequalities are attained). Furthermore, we obtain the precise value of the hyperbolicity constant for many product graphs.

Keywords. Infinite graphs; Cartesian product graphs; connectivity; geodesics; Gromov hyperbolicity.

1. Introduction

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance [2–4, 7–9, 11, 12, 18–22, 24, 26, 27, 29, 30].

The theory of Gromov's spaces was used initially for the study of finitely generated groups, where it was demonstrated to have an enormous practical importance. This theory was applied principally to the study of automatic groups (see [23]), that play an important role in sciences of the computation. Another important application of these spaces is secure transmission of information by internet (see [18, 19]). In particular, the hyperbolicity also plays an important role in the spread of viruses through the network (see [18, 19]). The hyperbolicity is also useful in the study of DNA data (see [7]).

In recent years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring–Osgood j-metric is Gromov hyperbolic; and the Vuorinen j-metric is not Gromov hyperbolic except in the punctured space (see [13]). The study of Gromov hyperbolicity of the quasi-hyperbolic and the Poincaré metrics is the subject of [1, 5, 14–17, 24, 25, 27–30]. In particular, in [24, 27, 29, 30] it is proved the equivalence of the hyperbolicity of Riemannian manifolds and the hyperbolicity of a simple graph; hence, it is useful to know hyperbolicity criteria for graphs.

In our study on hyperbolic graphs we use the notations of [10]. We now give the basic facts about Gromov's spaces. If γ is a continuous curve in a metric space (X, d), we say that γ is a *geodesic* if it is an isometry, i.e. $d(\gamma(t), \gamma(s)) = s - t$ for every $t < s$. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[xy]$ any such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected. If X is a graph, we use the notation $[u, v]$ for the edge of a graph joining the vertices u and v.

In order to consider a graph G as a geodesic metric space, we must identify any edge $[u, v] \in E(G)$ with the real interval $[0, l]$ (if $l := L([u, v])$); hence, if we consider $[u, v]$ as a graph with just one edge, then it is isometric to $[0, l]$. Therefore, any point in the interior of the edge $[u, v]$ is a point of G. A connected graph G is naturally equipped with a distance or, more precisely, metric defined on its points, induced by taking shortest paths in G. Then, we see G as a metric graph.

Along the paper we just consider (finite or infinite) graphs with edges of length 1, which are connected and locally finite (i.e., every vertex has finite degree). These conditions guarantee that the graph is a geodesic space. We do not allow loops and multiple edges in the graphs.

If X is a geodesic metric space and $J = \{J_1, J_2, \dots, J_n\}$ is a polygon, with sides $J_j \subseteq X$, we say that J is *δ-thin* if for every $x \in J_i$ we have that $d(x, \cup_{j\neq i} J_j) \leq \delta$. We denote by $\delta(J)$ the sharp thin constant of J, i.e. $\delta(J) := \inf\{\delta \geq 0\colon J \text{ is } \delta\text{-thin}\}$. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$. The space X is *δ-hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e. $\delta(X) := \sup\{\delta(T)\colon T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$. If X is hyperbolic, then $\delta(X) = \inf\{\delta \geq 0\colon X \text{ is } \delta\text{-hyperbolic}\}$.

A *bigon* is a geodesic triangle $\{x_1, x_2, x_3\}$ with $x_2 = x_3$. Therefore, every bigon in a δ-hyperbolic geodesic metric space is δ-thin.

Remark 1. There are several definitions of Gromov hyperbolicity (see e.g. [6, 10]). These different definitions are equivalent in the sense that if X is δ_A-hyperbolic with respect to the definition A, then it is δ_B-hyperbolic with respect to the definition B, and there exist universal constants c_1, c_2 such that $c_1\delta_A \leq \delta_B \leq c_2\,\delta_A$. However, for a fixed $\delta \geq 0$, the set of δ-hyperbolic graphs with respect to the Definition A, is different, in general, from the set of δ-hyperbolic graphs with respect to the Definition B. We have chosen this definition since it has a deep geometric meaning (see e.g. [10]).

The following are interesting examples of hyperbolic spaces. The real line $\mathbb{R}$ is 0-hyperbolic: in fact, any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously, and therefore we can conclude that $\mathbb{R}$ is 0-hyperbolic. The Euclidean plane $\mathbb{R}^2$ is not hyperbolic: it is clear that equilateral triangles can be drawn with arbitrarily large diameter, so that $\mathbb{R}^2$ with the Euclidean metric is not hyperbolic. This argument can be generalized in a similar way to higher dimensions: a normed vector space E is hyperbolic if and only if dim $E = 1$. Every arbitrary length metric tree is 0-hyperbolic: in fact, all points of a geodesic triangle in a tree belong simultaneously to two sides of the triangle. Every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic. Every simply connected complete Riemannian manifold with sectional curvature verifying $K \leq -k^2$, for some positive constant k, is hyperbolic. We refer to [6, 10] for more background and further results.

If D is a closed connected subset of X, we always consider in D the *inner metric* obtained by the restriction of the metric in X, that is

$$\begin{aligned} d_D(z, w) &:= \inf\{L_X(\gamma)\colon \gamma \subset D \text{ is a continuous curve joining } z \text{ and } w\} \\ &\geq d_X(z, w). \end{aligned}$$

Consequently, $L_D(\gamma) = L_X(\gamma)$ for every curve $\gamma \subset D$.

We would like to point out that deciding whether or not a space is hyperbolic is usually extraordinarily difficult: Notice that, first of all, we have to consider an arbitrary geodesic triangle T, and calculate the minimum distance from an arbitrary point P of T to the union of the other two sides of the triangle to which P does not belong to. And then we have to take supremum over all the possible choices for P and then over all the possible choices for T. Without disregarding the difficulty of solving this minimax problem, notice that, in general, the main obstacle is that we do not know the location of geodesics in the space. Therefore, it is interesting to obtain inequalities relating the hyperbolicity constant and other parameters of graphs.

In §3 of this paper we find several lower and upper bounds for the hyperbolicity constant of $G_1 \times G_2$, involving $\delta(G_1)$, $\delta(G_2)$ and the diameters of G_1 and G_2; the main results of this kind are Theorems 13 and 18. These results allow us to obtain Theorem 21, the main result of the paper; it characterizes the product graphs $G_1 \times G_2$ which are hyperbolic, in terms of G_1 and G_2: the product graph $G_1 \times G_2$ is hyperbolic if and only if G_1 is hyperbolic and G_2 is bounded or G_2 is hyperbolic and G_1 is bounded. We also find families of graphs for which many of the inequalities in §3 are attained. Furthermore, in §4 we obtain the precise value of the hyperbolicity constant for many product graphs.

2. The distance in product graphs

Before starting the study of the hyperbolicity of product graphs, it will be very useful to study first the distance function in product graphs.

DEFINITION 1

Let G_1, G_2 be two connected locally finite graphs with edges of length 1 without loops nor multiple edges. We define $G_1 \times G_2$ as the graph with vertices $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $[(u_1, u_2), (v_1, v_2)] \in E(G_1 \times G_2)$ if and only if we have either $u_1 = v_1 \in V(G_1)$ and $[u_2, v_2] \in E(G_2)$ or $u_2 = v_2 \in V(G_2)$ and $[u_1, v_1] \in E(G_1)$. We consider that every edge of $G_1 \times G_2$ has length 1.

Remark 2. A point (u, v) belongs to $G_1 \times G_2$ if and only if we have $u \in G_1$ and $v \in V(G_2)$ or $v \in G_2$ and $u \in V(G_1)$.

The following result allows to compute the distance between any two points in $G_1 \times G_2$.

PROPOSITION 3

For every graph G_1, G_2 we have

(a) *If $u_1, v_1 \notin V(G_1), u_1, v_1 \in [a_1, b_1] \in E(G_1)$, and $u_2 \neq v_2$, then,*

$$d_{G_1 \times G_2}((u_1, u_2), (v_1, v_2)) \\ = d_{G_2}(u_2, v_2) + \min\{d_{G_1}(u_1, a_1) + d_{G_1}(v_1, a_1), d_{G_1}(u_1, b_1) + d_{G_1}(v_1, b_1)\}.$$

(b) *If $u_2, v_2 \notin V(G_2), u_2, v_2 \in [a_2, b_2] \in E(G_2)$, and $u_1 \neq v_1$, then,*

$$d_{G_1 \times G_2}((u_1, u_2), (v_1, v_2)) \\ = d_{G_1}(u_1, v_1) + \min\{d_{G_2}(u_2, a_2) + d_{G_2}(v_2, a_2), d_{G_2}(u_2, b_2) + d_{G_2}(v_2, b_2)\}.$$

(c) *Otherwise, we have*

$$d_{G_1 \times G_2}((u_1, u_2), (v_1, v_2)) = d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2).$$

Proof. We will prove each item separately.

- Let us start with the case (a). Since $u_1, v_1 \notin V(G_1), u_1, v_1 \in [a_1, b_1] \in E(G_1)$, and $u_2 \neq v_2$. Then $u_2, v_2 \in V(G_2)$ and the two shortest possible paths to go from (u_1, u_2) to (v_1, v_2) have lengths $d_{G_1}(b_1, u_1) + d_{G_2}(u_2, v_2) + d_{G_1}(b_1, v_1)$, and $d_{G_1}(a_1, u_1) + d_{G_2}(u_2, v_2) + d_{G_1}(a_1, v_1)$; therefore,

$$d_{G_1 \times G_2}((u_1, u_2), (v_1, v_2)) \\ = d_{G_2}(u_2, v_2) + \min\{d_{G_1}(u_1, a_1) + d_{G_1}(v_1, a_1), d_{G_1}(u_1, b_1) + d_{G_1}(v_1, b_1)\}.$$

- By symmetry, we also have (b).
- In order to prove (c), let us distinguish the following cases:

  (i) $u_2 = v_2 \in V(G_2)$,
  (ii) $u_1 \in V(G_1), v_2 \in V(G_2)$,
  (iii) $u_2, v_2 \in V(G_2), u_1, v_1$ do not belong to the same edge,
  (i′) $u_1 = v_1 \in V(G_1)$,
  (ii′) $u_2 \in V(G_2), v_1 \in V(G_1)$,
  (iii′) $u_1, v_1 \in V(G_1), u_2, v_2$ do not belong to the same edge.

  It is clear by Remark 2 that if $(u_1, u_2), (v_1, v_2)$ are in case (c), then they are either in (i), (ii), (iii), (i′), (ii′) or (iii′).
  In (i), we have

$$\begin{aligned} d_{G_1 \times G_2}((u_1, u_2), (v_1, u_2)) &= d_{G_1 \times \{u_2\}}((u_1, u_2), (v_1, u_2)) \\ &= d_{G_1}(u_1, v_1) = d_{G_1}(u_1, v_1) + d_{G_2}(u_2, u_2). \end{aligned}$$

  In (ii),

$$\begin{aligned} &d_{G_1 \times G_2}((u_1, u_2), (v_1, v_2)) \\ &\quad = d_{G_1 \times G_2}((u_1, u_2), (u_1, v_2)) + d_{G_1 \times G_2}((u_1, v_2), (v_1, v_2)) \\ &\quad = d_{\{u_1\} \times G_2}((u_1, u_2), (u_1, v_2)) + d_{G_1 \times \{v_2\}}((u_1, v_2), (v_1, v_2)) \\ &\quad = d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2). \end{aligned}$$

In order to prove (iii), let u be any vertex of a geodesic in G_1 joining u_1 with v_1. Then

$$\begin{aligned} &d_{G_1\times G_2}((u_1,u_2),(v_1,v_2)) \\ &\quad = d_{G_1\times G_2}((u_1,u_2),(u,u_2)) + d_{G_1\times G_2}((u,u_2),(u,v_2)) \\ &\qquad + d_{G_1\times G_2}((u,v_2),(v_1,v_2)) \\ &\quad = d_{G_1}(u_1,u) + d_{G_2}(u_2,v_2) + d_{G_1}(u,v_1) \\ &\quad = d_{G_1}(u_1,v_1) + d_{G_2}(u_2,v_2). \end{aligned}$$

The cases (i′), (ii′), (iii′) are similar to (i), (ii), (iii) by symmetry.

COROLLARY 4

For every graph G_1, G_2 we have

(a) *If $u_1, v_1 \notin V(G_1)$, $u_1, v_1 \in [a_1, b_1] \in E(G_1)$, and $u_2 \neq v_2$, then*

$$d_{G_1\times G_2}((u_1,u_2),(v_1,v_2)) \le d_{G_2}(u_2,v_2) + 1.$$

(b) *If $u_2, v_2 \notin V(G_2)$, $u_2, v_2 \in [a_2, b_2] \in E(G_2)$, and $u_1 \neq v_1$, then*

$$d_{G_1\times G_2}((u_1,u_2),(v_1,v_2)) \le d_{G_1}(u_1,v_1) + 1.$$

Proof. It suffices to prove case (a), since case (b) is similar. Note that, by Proposition 3 we have

$$\begin{aligned} &d_{G_1\times G_2}((u_1,u_2),(v_1,v_2)) \\ &\quad = d_{G_2}(u_2,v_2) + \min\{d_{G_1}(u_1,a_1) + d_{G_1}(v_1,a_1), d_{G_1}(u_1,b_1) + d_{G_1}(v_1,b_1)\}. \end{aligned}$$

It suffices to prove that $\min\{d_{G_1}(u_1,a_1) + d_{G_1}(v_1,a_1), d_{G_1}(u_1,b_1) + d_{G_1}(v_1,b_1)\} \le 1$. In fact, we have that $d_{G_1}(u_1,a_1) + d_{G_1}(v_1,a_1) + d_{G_1}(u_1,b_1) + d_{G_1}(v_1,b_1) = 2$; this implies that $d_{G_1}(u_1,a_1) + d_{G_1}(v_1,a_1) \le 1$ or $d_{G_1}(u_1,b_1) + d_{G_1}(v_1,b_1) \le 1$; therefore, $\min\{d_{G_1}(u_1,a_1) + d_{G_1}(v_1,a_1), d_{G_1}(u_1,b_1) + d_{G_1}(v_1,b_1)\} \le 1$. □

These results allow us to obtain information about the geodesics in $G_1 \times G_2$.

COROLLARY 5

Let us consider the projection $P_j\colon G_1 \times G_2 \longrightarrow G_j$ for $j = 1, 2$.

(a) *If γ is a geodesic joining x and y in $G_1 \times G_2$, then for each $j = 1, 2$ there exists a geodesic γ^* in G_j joining $P_j(x)$ and $P_j(y)$, with $\gamma^* \subseteq P_j(\gamma)$ and $d_{G_j}(p, \gamma^*) \le 1/2$ for every $p \in P_j(\gamma)$.*
(b) *If γ is a geodesic joining two points of $G_1 \times G_2$ in the case* (c) *of Proposition* 3, *then $P_j(\gamma)$ is a geodesic in G_j for $j = 1, 2$.*

Theorem 6. *For every graph G_1, G_2 we have*

(i) $d_{G_1\times G_2}((u_1,u_2),(v_1,v_2)) = d_{G_1}(u_1,v_1) + d_{G_2}(u_2,v_2)$, *for every* $(u_1,u_2), (v_1,v_2) \in V(G_1 \times G_2)$,
(ii) $d_{G_1}(u_1,v_1) + d_{G_2}(u_2,v_2) \le d_{G_1\times G_2}((u_1,u_2),(v_1,v_2)) \le d_{G_1}(u_1,v_1) + d_{G_2}(u_2,v_2) + 1$, *for every* $(u_1,u_2), (v_1,v_2) \in G_1 \times G_2$.

Proof. It is clear that (i) is a direct consequence of case (c) in Proposition 3.

In order to prove (ii), it suffices to check it for case (a) in Proposition 3 (since case (b) is similar). This is equivalent to prove that

$$\begin{aligned} d_{G_1}(u_1, v_1) &\le \min\{d_{G_1}(u_1, a_1) + d_{G_1}(v_1, a_1), d_{G_1}(u_1, b_1) + d_{G_1}(v_1, b_1)\} \\ &\le d_{G_1}(u_1, v_1) + 1. \end{aligned}$$

The second inequality is a direct consequence of Corollary 4. The first inequality is a consequence of the triangle inequality, since

$$\begin{aligned} d_{G_1}(u_1, v_1) &\le d_{G_1}(a_1, u_1) + d_{G_1}(a_1, v_1), \\ d_{G_1}(u_1, v_1) &\le d_{G_1}(b_1, u_1) + d_{G_1}(b_1, v_1). \end{aligned} \qquad \square$$

Although it is simple to check that $\operatorname{diam}_{G_1\times G_2} V(G_1 \times G_2) = \operatorname{diam}_{G_1} V(G_1) + \operatorname{diam}_{G_2} V(G_2)$, it is not so simple to compute $\operatorname{diam}_{G_1\times G_2}(G_1 \times G_2)$.

Theorem 7. *Let G_1, G_2 be any graph. If $\operatorname{diam}'_G G := \sup\{d_G(u, v), u \in G, v \in V(G)\}$, then we have*

$$\begin{aligned} \operatorname{diam}_{G_1\times G_2}(G_1 \times G_2) = \max\{&\operatorname{diam}_{G_1} V(G_1) + \operatorname{diam}_{G_2} G_2, \operatorname{diam}_{G_2} V(G_2) \\ &+ \operatorname{diam}_{G_1} G_1, \operatorname{diam}'_{G_1} G_1 + \operatorname{diam}'_{G_2} G_2\}. \end{aligned}$$

Proof. We can assume that G_j has at least two vertices, since otherwise $\operatorname{diam}_{G_1\times G_2}(G_1 \times G_2) = \operatorname{diam}_{G_j} G_j$, for some $j \in \{1, 2\}$ and the formula holds. Parts (a) and (b) of Proposition 3 give

$$\begin{aligned} &\sup\{d_{G_1\times G_2}((u_1, u_2), (v_1, v_2))\colon (u_1, u_2) \text{ and } (v_1, v_2) \text{ hold either } (a) \text{ or } (b)\} \\ &\quad \le \max\{1 + \operatorname{diam}_{G_1} G_1, 1 + \operatorname{diam}_{G_2} G_2\}. \end{aligned}$$

Since G_j has at least two vertices, $\operatorname{diam}_{G_j} V(G_j) \ge 1$, $j = 1, 2$, which implies that

$$\begin{aligned} &\sup\{d_{G_1\times G_2}((u_1, u_2), (v_1, v_2))\colon (u_1, u_2) \text{ and } (v_1, v_2) \text{ hold either } (a) \text{ or } (b)\} \\ &\quad \le \max\{\operatorname{diam}_{G_1} V(G_1) + \operatorname{diam}_{G_2} G_2, \operatorname{diam}_{G_2} V(G_2) + \operatorname{diam}_{G_1} G_1\}. \end{aligned}$$

In case (c), we have

$$\begin{aligned} &\sup\{d_{G_1\times G_2}((u_1, u_2), (v_1, v_2))\colon (c) \text{ holds}\} \\ &\quad = \sup\{d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2)\colon (c) \text{ holds}\}. \end{aligned}$$

We denote by (i), (ii), (iii), (i′), (ii′), (iii′) the cases in the proof of Proposition 3.

If $u_2 = v_2 \in V(G_2)$, then

$$\begin{aligned} &\sup\{d_{G_1\times G_2}((u_1, u_2), (v_1, u_2))\colon (i) \text{ holds}\} \\ &\quad \le \sup\{d_{G_1}(u_1, v_1)\colon u_1, v_1 \in G_1\} \\ &\quad \le \operatorname{diam}_{G_1} G_1 \le \operatorname{diam}_{G_1} G_1 + \operatorname{diam}_{G_2} V(G_2). \end{aligned}$$

If $u_1 \in V(G_1)$, $v_2 \in V(G_2)$, then

$$\begin{aligned} &\sup\{d_{G_1\times G_2}((u_1, u_2), (v_1, v_2))\colon \text{(ii) holds}\} \\ &\quad = \sup\{d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2)\colon \text{(ii) holds}\} \end{aligned}$$

$$\begin{aligned}&\le \sup\{d_{G_1}(u_1, v_1)\colon u_1 \in V(G_1)\} + \sup\{d_{G_2}(u_2, v_2)\colon v_2 \in V(G_2)\}\\ &\le \operatorname{diam}'_{G_1} G_1 + \operatorname{diam}'_{G_2} G_2.\end{aligned}$$

If $u_2, v_2 \in V(G_2)$ and u_1, v_1 do not belong to the same edge, then

$$\begin{aligned}&\sup\{d_{G_1\times G_2}((u_1, u_2), (v_1, v_2))\colon \text{(iii) holds}\}\\ &\quad = \sup\{d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2)\colon \text{(iii) holds}\}\\ &\quad \le \sup\{d_{G_1}(u_1, v_1)\colon u_1, v_1 \in G_1\} + \sup\{d_{G_2}(u_2, v_2)\colon u_2, v_2 \in V(G_2)\}\\ &\quad = \operatorname{diam}_{G_1} G_1 + \operatorname{diam}_{G_2} V(G_2).\end{aligned}$$

The cases (i′), (ii′), (iii′) are treated in the same way. Therefore,

$$\begin{aligned}&\sup\{d_{G_1\times G_2}((u_1, u_2), (v_1, v_2))\colon \text{(c) holds}\}\\ &\quad \le \max\{\operatorname{diam}_{G_1} V(G_1) + \operatorname{diam}_{G_2} G_2, \operatorname{diam}_{G_2} V(G_2)\\ &\qquad + \operatorname{diam}_{G_1} G_1, \operatorname{diam}'_{G_1} G_1 + \operatorname{diam}'_{G_2} G_2\}.\end{aligned}$$

Combining (a), (b), (c), we deduce that

$$\begin{aligned}&\operatorname{diam}_{G_1\times G_2}(G_1 \times G_2)\\ &\quad \le \max\{\operatorname{diam}_{G_1} V(G_1) + \operatorname{diam}_{G_2} G_2, \operatorname{diam}_{G_2} V(G_2)\\ &\qquad + \operatorname{diam}_{G_1} G_1, \operatorname{diam}'_{G_1} G_1 + \operatorname{diam}'_{G_2} G_2\}.\end{aligned}$$

Let $u_1, v_1 \in G_1, u_2, v_2 \in G_2$ be such that $d_{G_1}(u_1, v_1) = \operatorname{diam}'_{G_1} G_1$ and $d_{G_2}(u_2, v_2) = \operatorname{diam}'_{G_2} G_2$ with $u_1 \in V(G_1), v_2 \in V(G_2)$. Then

$$\begin{aligned}d_{G_1\times G_2}((u_1, u_2), (v_1, v_2)) &= d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2)\\ &= \operatorname{diam}'_{G_1} G_1 + \operatorname{diam}'_{G_2} G_2.\end{aligned}$$

Now let $u_1, v_1 \in V(G_1)$ be such that $d_{G_1}(u_1, v_1) = \operatorname{diam}_{G_1} V(G_1)$, and let us choose $u_2, v_2 \in G_2$ such that $d_{G_2}(u_2, v_2) = \operatorname{diam}_{G_2} G_2$ (u_2, v_2 are not in the interior of the same edge). Then

$$\begin{aligned}d_{G_1\times G_2}((u_1, u_2), (v_1, v_2)) &= d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2)\\ &= \operatorname{diam}_{G_1} V(G_1) + \operatorname{diam}_{G_2} G_2.\end{aligned}$$

Changing the role of u_1, v_1 and u_2, v_2 we also obtain

$$\begin{aligned}d_{G_1\times G_2}((u_1, u_2), (v_1, v_2)) &= d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2)\\ &= \operatorname{diam}_{G_1} G_1 + \operatorname{diam}_{G_2} V(G_2).\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{diam}_{G_1\times G_2}(G_1 \times G_2) \ge \max\{&\operatorname{diam}_{G_1} V(G_1) + \operatorname{diam}_{G_2} G_2, \operatorname{diam}_{G_2} V(G_2)\\ &+ \operatorname{diam}_{G_1} G_1, \operatorname{diam}'_{G_1} G_1 + \operatorname{diam}'_{G_2} G_2\}.\end{aligned}$$

This inequality completes the proof. □

We can deduce several results from Theorem 7. The first one says that $\operatorname{diam}_{G_1} G_1 + \operatorname{diam}_{G_2} G_2$ is a good approximation for the diameter of $G_1 \times G_2$.

COROLLARY 8

For every graph G_1, G_2 we have

$$\begin{aligned}\operatorname{diam}_{G_1} G_1 + \operatorname{diam}_{G_2} G_2 - 1 &\le \operatorname{diam}_{G_1 \times G_2}(G_1 \times G_2)\\ &\le \operatorname{diam}_{G_1} G_1 + \operatorname{diam}_{G_2} G_2.\end{aligned}$$

Proof. We always have

$$\begin{aligned}&\max\{\operatorname{diam}_{G_1} V(G_1) + \operatorname{diam}_{G_2} G_2, \operatorname{diam}_{G_2} V(G_2)\\ &\quad + \operatorname{diam}_{G_1} G_1, \operatorname{diam}'_{G_1} G_1 + \operatorname{diam}'_{G_2} G_2\}\\ &\le \operatorname{diam}_{G_1} G_1 + \operatorname{diam}_{G_2} G_2.\end{aligned}$$

On the other hand, every graph G with edges of length 1 satisfies

$$\operatorname{diam}_G G \le \operatorname{diam}'_G G + 1/2, \quad \operatorname{diam}_G G \le \operatorname{diam}_G V(G) + 1.$$

Therefore,

$$\begin{aligned}&\operatorname{diam}_{G_1} G_1 + \operatorname{diam}_{G_2} G_2 - 1\\ &\le \max\{\operatorname{diam}_{G_1} V(G_1) + \operatorname{diam}_{G_2} G_2, \operatorname{diam}_{G_2} V(G_2)\\ &\quad + \operatorname{diam}_{G_1} G_1, \operatorname{diam}'_{G_1} G_1 + \operatorname{diam}'_{G_2} G_2\},\end{aligned}$$

and Theorem 7 gives the result. □

Furthermore, we can characterize the graphs for which the diameter of $G_1 \times G_2$ is equal to $\operatorname{diam}_{G_1} G_1 + \operatorname{diam}_{G_2} G_2$.

COROLLARY 9

The equality $\operatorname{diam}_{G_1 \times G_2}(G_1 \times G_2) = \operatorname{diam}_{G_1} G_1 + \operatorname{diam}_{G_2} G_2$ *holds if and only if we have* $\operatorname{diam}_{G_1} G_1 = \operatorname{diam}_{G_1} V(G_1)$, *or* $\operatorname{diam}_{G_2} G_2 = \operatorname{diam}_{G_2} V(G_2)$, *or* $\operatorname{diam}_{G_j} G_j = \operatorname{diam}'_{G_j} G_j$ *for* $j = 1, 2$.

COROLLARY 10

If T is any tree and G is any graph, then

$$\operatorname{diam}_{T \times G}(T \times G) = \operatorname{diam}_T T + \operatorname{diam}_G G.$$

3. Bounds for the hyperbolicity constant

The following result will be useful.

Theorem 11 (Theorem 8 in [26]). *In any graph G the inequality $\delta(G) \le \frac{1}{2} \operatorname{diam} G$ holds and it is sharp.*

Corollary 8 and Theorem 11 give the following result.

COROLLARY 12

For every graph G_1, G_2, *we have* $\delta(G_1 \times G_2) \leq \frac{1}{2}(\mathrm{diam}_{G_1} G_1 + \mathrm{diam}_{G_2} G_2)$, *and the inequality is sharp.*

Theorem 26 provides a family of examples for which the equality in Corollary 12 is attained: $P_m \times C_n$ with $m - 1 \leq [n/2]$.

We have the following upper bound for $\delta(G_1 \times G_2)$.

Theorem 13. *For every graph* G_1, G_2 *we have*

$$\delta(G_1 \times G_2) \leq \min\{\max\{1/2 + \mathrm{diam}_{G_2} V(G_2), \delta(G_1) + \mathrm{diam}'_{G_2} G_2\},\\ \max\{1/2 + \mathrm{diam}_{G_1} V(G_1), \delta(G_2) + \mathrm{diam}'_{G_1} G_1\}\},$$

and the inequality is sharp.

Proof. By symmetry, it suffices to show that $\delta(G_1 \times G_2) \leq \max\{1/2 + \mathrm{diam}_{G_2} V(G_2)$, $\delta(G_1) + \mathrm{diam}'_{G_2} G_2\}$. We can assume that $\mathrm{diam}_{G_2} V(G_2) \geq 1$, since if G_2 has a single vertex, then $G_1 \times G_2$ is isometric to G_1 and the inequality is direct. If $\delta(G_1) + \mathrm{diam}'_{G_2} G_2 = \infty$, then the inequality holds. Hence, without loss of generality we can assume that G_1 is hyperbolic and G_2 is bounded. Let $T_0 = \{\gamma_1, \gamma_2, \gamma_3\}$ be any geodesic triangle in $G_1 \times G_2$.

Let P_1 be the projection $P_1 \colon G_1 \times G_2 \longrightarrow G_1$ and $\gamma'_j := P_1(\gamma_j)$. By Corollary 5 there exist geodesics $\gamma^*_j \subseteq \gamma'_j$ $(j = 1, 2, 3)$ joining the images by P_1 of the vertices of T_0, such that γ'_j is contained in a 1/2-neighborhood of γ^*_j, for $j = 1, 2, 3$.

Assume first that $\gamma'_1 = \gamma^*_1$, i.e. that γ'_1 is a geodesic in G_1. Consider the geodesic triangle $T^* = \{\gamma'_1, \gamma^*_2, \gamma^*_3\}$. Since G_1 is hyperbolic, then $d_{G_1}(a, \gamma^*_2 \cup \gamma^*_3) \leq \delta(G_1)$, for every $a \in \gamma'_1$. Let now $(u, v) \in G_1 \times G_2$ be any point in γ_1. Let us consider $p \in \gamma^*_2 \cup \gamma^*_3 \subseteq \gamma'_2 \cup \gamma'_3$ with $d_{G_1}(u, \gamma^*_2 \cup \gamma^*_3) = d_{G_1}(u, p) \leq \delta(G_1)$ and $q \in G_2$ with $(p, q) \in \gamma_2 \cup \gamma_3$.

If u, p belong to the interior of the same edge $[a_1, b_1] \in E(G_1)$, then $v, q \in V(G_2)$. If $v = q$, then

$$d_{G_1 \times G_2}((u, v), \gamma_2 \cup \gamma_3) \leq d_{G_1 \times G_2}((u, v), (p, v))\\ = d_{G_1}(u, p) < 1 \leq \mathrm{diam}_{G_2} V(G_2) + 1/2.$$

If $v \neq q$, then $[a_1, b_1] \times \{q\} \subseteq \gamma_2 \cup \gamma_3$, since otherwise there exists a point of γ_1 in the interior of the edge $[a_1, b_1] \times \{q\}$ and hence, γ'_1 is not a geodesic in G_1. Consequently,

$$d_{G_1 \times G_2}((u, v), \gamma_2 \cup \gamma_3) \leq d_{G_1 \times G_2}((u, v), [a_1, b_1] \times \{q\})\\ \leq \mathrm{diam}_{G_2} V(G_2) + 1/2.$$

If v, q belong to the interior of the same edge $[a_2, b_2] \in E(G_2)$, then Corollary 4 gives $d_{G_1 \times G_2}((u, v), (p, q)) \leq d_{G_1}(u, p) + 1 \leq \delta(G_1) + \mathrm{diam}_{G_2} V(G_2)$.

If u, p do not belong to the interior of the same edge in G_1 and v, q do not belong to the interior of the same edge in G_2, then Proposition 3 gives

$$d_{G_1 \times G_2}((u, v), (p, q)) = d_{G_1}(u, p) + d_{G_2}(v, q).$$

Assume that $d_{G_2}(v, q) \leq \mathrm{diam}'_{G_2} G_2$; then $d_{G_1 \times G_2}((u, v), (p, q)) \leq \delta(G_1) + \mathrm{diam}'_{G_2} G_2$. Assume now that $d_{G_2}(v, q) > \mathrm{diam}'_{G_2} G_2$; then v, q are not vertices of G_2, and q belongs to the interior of the same edge $[a_3, b_3] \in E(G_2)$. If (p, a_3) or (p, b_3) belongs to $\gamma_2 \cup \gamma_3$, without loss of generality we can assume that (p, a_3) belongs to $\gamma_2 \cup \gamma_3$, and we have

$$\begin{aligned} d_{G_1\times G_2}((u, v), \gamma_2 \cup \gamma_3) &\le d_{G_1\times G_2}((u, v), (p, a_3)) \\ &= d_{G_1}(u, p) + d_{G_2}(v, a_3) \le \delta(G_1) + \operatorname{diam}'_{G_2} G_2. \end{aligned}$$

If $(p, a_3), (p, b_3) \notin \gamma_2 \cup \gamma_3$, then $\gamma_2 \cup \gamma_3 \subset \{p\} \times [a_3, b_3]$ and hence, $\gamma_1 \subset \{p\} \times [a_3, b_3]$. Therefore, we have $d_{G_1\times G_2}((u, v), \gamma_2 \cup \gamma_3) = 0$.

Assume now that γ_1' is not a geodesic in G_1. By Proposition 3, γ_1 joins two points (u_1, u_2) and (v_1, v_2) with u_1, v_1 in the interior of some edge $[\alpha_1, \beta_1]$ and $u_2, v_2 \in V(G_2)$; furthermore, $L(\gamma_1) \le 1 + d_{G_2}(u_2, v_2) \le 1 + \operatorname{diam}_{G_2} V(G_2)$.

If $(u, v) \in \gamma_1$, then $d_{G_1\times G_2}((u, v), \gamma_2 \cup \gamma_3) \le L(\gamma_1)/2 \le (1 + \operatorname{diam}_{G_2} V(G_2))/2$.

Therefore, $\delta(T_0) \le \max\{1/2 + \operatorname{diam}_{G_2} V(G_2), \delta(G_1) + \operatorname{diam}'_{G_2} G_2\}$, for every geodesic triangle T_0 in $G_1 \times G_2$, and consequently $\delta(G_1 \times G_2) \le \max\{1/2 + \operatorname{diam}_{G_2} V(G_2), \delta(G_1) + \operatorname{diam}'_{G_2} G_2\}$.

In order to check that the inequality is sharp, it suffices to note that the inequality in Theorem 15 (which is a particular case of this one) is sharp. □

We have the following consequences of Theorem 13.

Theorem 14. *For every graph G_1, G_2 we have*

$$\begin{aligned} \delta(G_1 \times G_2) &\le \min\{\max\{1/2, \delta(G_1)\} \\ &\qquad + \operatorname{diam}'_{G_2} G_2, \max\{1/2, \delta(G_2)\} + \operatorname{diam}_{G_1} G_1\} \\ &\le 1/2 + \min\{\delta(G_1) + \operatorname{diam}'_{G_2} G_2, \delta(G_2) + \operatorname{diam}'_{G_1} G_1\}. \end{aligned}$$

The following bound for the hyperbolicity constant will be very useful. It is a consequence of Theorem 13 and the inequality $\operatorname{diam}'_G G \le \operatorname{diam}_G V(G) + 1/2$.

Theorem 15. *If T is any tree and G is any graph, then*

$$\delta(T \times G) \le \operatorname{diam}_G V(G) + 1/2$$

and the inequality is sharp.

Theorem 24 gives that the equality in Theorem 15 is attained for every tree and every graph with $\operatorname{diam}_T T > \operatorname{diam}_G V(G)$.

Theorem 16. *For every graph G_1, G_2 which are not trees we have*

$$\delta(G_1 \times G_2) \le \min\{\delta(G_1) + \operatorname{diam}'_{G_2} G_2, \delta(G_2) + \operatorname{diam}'_{G_1} G_1\}.$$

Proof. Theorem 11 in [22] gives that if G is not a tree, then $\delta(G) \ge 3/4$. This fact and Theorem 13 give the result. □

We say that a subgraph Γ of G is *isometric* if $d_\Gamma(x, y) = d_G(x, y)$ for every $x, y \in \Gamma$. The following result will be useful.

Lemma 17 (*Lemma* 5 *in* [26]). *If Γ is an isometric subgraph of G, then $\delta(\Gamma) \le \delta(G)$.*

We also have the following lower bounds for $\delta(G_1 \times G_2)$.

Theorem 18. *For every graph G_1, G_2 we have*

(a) $\delta(G_1 \times G_2) \geq \max\{\delta(G_1), \delta(G_2)\}$,
(b) $\delta(G_1 \times G_2) \geq \min\{\operatorname{diam}_{G_1} V(G_1), \operatorname{diam}_{G_2} V(G_2)\}$,
(c) $\delta(G_1 \times G_2) \geq \min\{\operatorname{diam}_{G_1} V(G_1), \operatorname{diam}_{G_2} V(G_2)\} + 1/2$, *if* $\operatorname{diam}_{G_1} V(G_1) \neq \operatorname{diam}_{G_2} V(G_2)$,
(d) $\delta(G_1 \times G_2) \geq \frac{1}{2} \min\{\delta(G_1) + \operatorname{diam}_{G_2} V(G_2), \delta(G_2) + \operatorname{diam}_{G_1} V(G_1)\}$.

Furthermore, inequalities in (b) *and* (c) *are sharp, as the first and second item in Theorem* 23 *show.*

Proof. Part (a) is immediate: $G_1 \times \{v\}$ and $\{u\} \times G_2$ are isometric supbgraphs of $G_1 \times G_2$ for every $(u, v) \in V(G_1 \times G_2)$; then Lemma 17 gives that $\delta(G_1 \times G_2) \geq \delta(G_1 \times \{v\}) = \delta(G_1)$ and $\delta(G_1 \times G_2) \geq \delta(\{u\} \times G_2) = \delta(G_2)$. Hence, we obtain $\delta(G_1 \times G_2) \geq \max\{\delta(G_1), \delta(G_2)\}$.

In order to prove (b), let $D := \min\{\operatorname{diam}_{G_1} V(G_1), \operatorname{diam}_{G_2} V(G_2)\}$. If $D < \infty$, let us consider a geodesic square $K := \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ in $G_1 \times G_2$ with sides of length D; then $T := \{\gamma_1, \gamma_2, \gamma\}$ is a geodesic triangle in $G_1 \times G_2$, where $\gamma := \gamma_3 \cup \gamma_4$. It is clear that the midpoint $p = \gamma_3 \cap \gamma_4$ of γ satisfies $d_{G_1 \times G_2}(p, \gamma_1 \cup \gamma_2) = D$; therefore $\delta(T) \geq D$, and consequently $\delta(G_1 \times G_2) \geq D$. If $D = \infty$, we can repeat the same argument for any integer N instead of D, and we obtain $\delta(G_1 \times G_2) \geq N$, for every N: hence, $\delta(G_1 \times G_2) = \infty = D$.

In order to prove (c), we can assume that $D < \infty$, since if $D = \infty$ then part (b) gives the result. Without loss of generality we can assume that $\operatorname{diam}_{G_1} V(G_1) < \operatorname{diam}_{G_2} V(G_2)$. Let us consider a geodesic rectangle $R := \sigma_1 \cup \sigma_2 \cup \sigma_3 \cup \sigma_4$ in $G_1 \times G_2$ with $L(\sigma_1) = L(\sigma_3) = \operatorname{diam}_{G_1} V(G_1)$ and $L(\sigma_2) = L(\sigma_4) = \operatorname{diam}_{G_2} V(G_2)$. Then $B := \{\sigma, \gamma\}$ is a geodesic bigon in $G_1 \times G_2$, where $\sigma := \sigma_1 \cup \sigma_2$, $\gamma := \sigma_3 \cup \sigma_4$. Let p be the point in σ_2 with $d_{G_1 \times G_2}(p, \sigma_1 \cap \sigma_2) = 1/2$; then

$$\begin{aligned} d_{G_1 \times G_2}(p, \gamma) &= 1/2 + \operatorname{diam}_{G_1} V(G_1) \\ &= 1/2 + \min\{\operatorname{diam}_{G_1} V(G_1), \operatorname{diam}_{G_2} V(G_2)\}. \end{aligned}$$

Consequently, $\delta(G_1 \times G_2) \geq \delta(B) \geq 1/2 + \min\{\operatorname{diam}_{G_1} V(G_1), \operatorname{diam}_{G_2} V(G_2)\}$.

In order to prove (d), let $E := \max\{\delta(G_1), \delta(G_2)\}$. Then from parts (a) and (b), we have

$$\begin{aligned} \delta(G_1 \times G_2) &\geq \max\{D, E\} \geq \frac{1}{2}(D + E) \\ &= \frac{1}{2} \min\{\operatorname{diam}_{G_1} V(G_1) + E, \operatorname{diam}_{G_2} V(G_2) + E\}, \\ &\geq \frac{1}{2} \min\{\operatorname{diam}_{G_1} V(G_1) + \delta(G_2), \delta(G_1) + \operatorname{diam}_{G_2} V(G_2)\}. \quad \square \end{aligned}$$

Note that the items (a) and (d) will play an important qualitative role in the rest of the paper.

Corollary 12 and Theorem 18 allow us to give lower and upper bounds for $\delta(G_1 \times G_2)$ just in terms of distances in G_1 and G_2.

COROLLARY 19

For every graph G_1, G_2 we have

$$\min\{\operatorname{diam}_{G_1} V(G_1), \operatorname{diam}_{G_2} V(G_2)\} \le \delta(G_1 \times G_2) \le \frac{1}{2}(\operatorname{diam}_{G_1} G_1 + \operatorname{diam}_{G_2} G_2).$$

Theorems 14 and 18 give that $\delta(G_1 \times G_2)$ is equivalent, in a precise way, to $\min\{\delta(G_1) + \operatorname{diam} G_2, \delta(G_2) + \operatorname{diam} G_1\}$.

COROLLARY 20

For every graph G_1, G_2 we have

$$\frac{1}{2}\min\{\delta(G_1) + \operatorname{diam}_{G_2} V(G_2), \delta(G_2) + \operatorname{diam}_{G_1} V(G_1)\}$$

$$\le \delta(G_1 \times G_2) \le \frac{1}{2} + \min\{\delta(G_1) + \operatorname{diam}'_{G_2} G_2, \delta(G_2) + \operatorname{diam}'_{G_1} G_1\}.$$

Corollary 20 allows to obtain the main result on this topic: the characterization of the hyperbolic graphs $G_1 \times G_2$.

Theorem 21. *For every graph G_1, G_2 we have that $G_1 \times G_2$ is hyperbolic if and only if G_1 is hyperbolic and G_2 is bounded or G_2 is hyperbolic and G_1 is bounded.*

Many parameters γ of graphs satisfy the inequality $\gamma(G_1 \times G_2) \ge \gamma(G_1) + \gamma(G_2)$. Therefore, one could think that the inequality $\delta(G_1 \times G_2) \ge \delta(G_1) + \delta(G_2)$ holds for every graph G_1, G_2. However, this is false, as the following example shows:

Example 22. $\delta(P \times C_3) < \delta(P) + \delta(C_3)$, where P is the Petersen graph.

Proof. We have that

$$\operatorname{diam}_P V(P) = 2,\ \operatorname{diam}'_P P = 5/2,\ \operatorname{diam}_P P = 3,$$
$$\operatorname{diam}_{C_3} V(C_3) = 1,\ \operatorname{diam}'_{C_3} C_3 = \operatorname{diam}_{C_3} C_3 = 3/2.$$

Theorem 11 in [26] gives that $\delta(P) = 3/2$ and $\delta(C_3) = 3/4$. Theorem 7 gives $\operatorname{diam}_{P \times C_3}(P \times C_3) = 4$ and by Theorem 11, we obtain $\delta(P \times C_3) \le 2 < 3/2 + 3/4 = \delta(P) + \delta(C_3)$. □

4. Computation of the hyperbolicity constant for some product graphs

We obtain in this section the value of the hyperbolicity constant for many product graphs.

Theorem 23. *The following graphs have these precise values of δ:*

- $\delta(P_n \times P_n) = n - 1$, *for every* $n \ge 2$.
- $\delta(P_m \times P_n) = \min\{m, n\} - 1/2$, *for every* $m, n \ge 2$ *with* $m \ne n$.
- $\delta(Q_n) = n/2$, *for every* $n \ge 2$.

- $\delta(C_m \times C_n) = (m+n)/4$, *for every* $m, n \geq 3$.
- $\delta(T_1 \times T_2) = \delta(P_{1+\operatorname{diam}T_1} \times P_{1+\operatorname{diam}T_2})$, *for every trees* T_1, T_2, *i.e.*,

$$\delta(T_1 \times T_2) = \begin{cases} \operatorname{diam}_{T_1} T_1, & \text{if } \operatorname{diam}_{T_1} T_1 = \operatorname{diam}_{T_2} T_2, \\ \min\{\operatorname{diam}_{T_1} T_1, \operatorname{diam}_{T_2} T_2\} + 1/2, & \text{if } \operatorname{diam}_{T_1} T_1 \neq \operatorname{diam}_{T_2} T_2. \end{cases}$$

Proof. We can see $P_m \times P_n$ as the subset of points (a, b) in the Cayley graph of $\mathbb{Z} \times \mathbb{Z} \subset \mathbb{R}^2$ with $0 \leq a \leq m-1$, $0 \leq b \leq n-1$. Given any geodesic triangle T in $P_m \times P_n$, the bigon B with vertices $x = (0, 0)$, $y = (m-1, n-1)$ and geodesics $\gamma_1 = [(0, 0)(0, n-1)] \cup [(0, n-1)(m-1, n-1)]$, $\gamma_2 = [(0, 0)(m-1, 0)] \cup [(m-1, 0)(n-1, n-1)]$ verifies $\delta(T) \leq \delta(B)$.

If $m = n$, then $\delta(B) = d_{P_n \times P_n}((0, n-1), \gamma_2) = n-1$. Therefore $\delta(P_n \times P_n) = n-1$.

If $m \neq n$, then without loss of generality we can assume that $m < n$. We have $\delta(B) = d_{P_m \times P_n}((0, m-1/2), \gamma_2) = m-1/2$; therefore, $\delta(P_m \times P_n) = m-1/2 = \min\{m, n\}-1/2$.

First of all let us prove by induction that $\operatorname{diam}_{Q_n} Q_n = n$. For $n = 1$, we have $\operatorname{diam}_{Q_1} Q_1 = \operatorname{diam}_{P_2} P_2 = 1$.

Now suppose that $\operatorname{diam}_{Q_n} Q_n = n$. Since $Q_{n+1} = Q_n \times P_2$, by Theorem 7 we deduce that $\operatorname{diam}_{Q_{n+1}} Q_{n+1} = n+1$. Therefore, we have proved $\operatorname{diam}_{Q_n} Q_n = n$. Consequently, Theorem 11 gives that $\delta(Q_n) \leq n/2$. In order to prove the reverse inequality, we consider Q_n contained in $[0, 1]^n \subset \mathbb{R}^n$. Let us define two n-dimensional vectors x_i, y_i for each $0 \leq i \leq n$ as follows:

$x_i = (0, \ldots, 0, \overbrace{1, \ldots, 1}^{i})$ has i-times '1' in the i last components and '0' in the rest of the components,

$y_i = (\overbrace{1, \ldots, 1}^{i}, 0, \ldots, 0)$ has i-times '1' in the i first components and '0' in the rest of the components.

Now we consider the paths $\gamma_1 := x_0, x_1, \ldots, x_n$, $\gamma_2 := y_0, y_1, \ldots, y_n$. If $n = 2p$, let $z := x_p$; then $d_{Q_n}(x_p, y_p) = n$ and $d_{Q_n}(x_p, y_j) = n - |p-j| \geq n/2$ for $0 \leq j \leq n$. If $n = 2p+1$, let $z := \frac{x_p + x_{p+1}}{2}$; then

$$\begin{aligned} d_{Q_n}(z, y_j) &= 1/2 + \min\{d_{Q_n}(x_p, y_j), d_{Q_n}(x_{p+1}, y_j)\} \\ &= \min\{n - |p+1-j|, n - |p-j|\} + 1/2 \\ &= n - \min\{|p+1-j|, |p-j|\} + 1/2 \geq n/2 \end{aligned}$$

for $0 \leq j \leq n$. Let us consider the bigon $B = \gamma_1 \cup \gamma_2$. In any case, we have that the midpoint z of γ_1 is at distance $n/2$ from γ_2; therefore, $\delta(Q_n) \geq n/2$ and, consequently, $\delta(Q_n) = n/2$.

Consider the graph $C_m \times C_n$. Corollary 9 gives $\operatorname{diam}_{C_m \times C_n}(C_m \times C_n) = (m+n)/2$. Thus Theorem 11 gives $\delta(C_m \times C_n) \leq (m+n)/4$.

Let us denote by $u_1, u_2, \ldots, u_m$ (respectively, $v_1, v_2, \ldots, v_n$) the (consecutive) vertices in C_m (respectively, in C_n) with $d_{C_m}(u_i, u_{i+1}) = 1$ for every $1 \leq i \leq m-1$ (respectively, $d_{C_n}(v_j, v_{j+1}) = 1$ for every $1 \leq j \leq n-1$).

First, let us assume that m or n is even; by symmetry, without loss of generality we can assume that m is even.

If n is even, let $z = v_{n/2+1}$; then we define

$$\gamma_1 := [u_1 u_{m/2+1}] \times \{v_1\} \cup \{u_{m/2+1}\} \\ \times [v_1 z] \text{ (with } u_2 \in [u_1 u_{m/2+1}] \text{ and } v_2 \in [v_1 z])$$

and

$$\gamma_2 := [u_{m/2+1} u_1] \times \{z\} \cup \{u_1\} \\ \times [z v_1] \text{ (with } u_m \in [u_{m/2+1} u_1] \text{ and } v_n \in [z v_1]).$$

If n is odd, let z be the midpoint of $[v_{(n+1)/2}, v_{(n+3)/2}]$; then

$$\gamma_1 := [u_1 u_{m/2+1}] \times \{v_1\} \cup \{u_{m/2+1}\} \\ \times [v_1 z] \text{ (with } u_2 \in [u_1 u_{m/2+1}] \text{ and } v_2 \in [v_1 z])$$

and

$$\gamma_2 := \{u_{m/2+1}\} \times [z v_{(n+3)/2}] \cup [u_{m/2+1} u_1] \\ \times \{v_{(n+3)/2}\} \cup \{u_1\} \times [v_{(n+3)/2} v_1] \text{ (with } u_m \in [u_{m/2+1} u_1]).$$

We have that $B = \{\gamma_1, \gamma_2\}$ is a bigon and $L(\gamma_1) = L(\gamma_2) = (m+n)/2$. If p is the midpoint of γ_1, then $d_{C_m \times C_n}(p, \gamma_2) = (m+n)/4$; therefore, $\delta(C_m \times C_n) \geq \delta(B) \geq (m+n)/4$.

Now, let us assume that m, n are odd. Let w be the midpoint of $[u_{(m+1)/2}, u_{(m+3)/2}]$ and z the midpoint of $[v_{(n+1)/2}, v_{(n+3)/2}]$; then we define

$$\gamma_1 := [w u_1] \times \{v_1\} \cup \{u_1\} \times [v_1 z] \text{ (with } u_2 \in [w u_1] \text{ and } v_2 \in [v_1 z])$$

and

$$\gamma_2 := \{u_1\} \times [z v_{(n+3)/2}] \cup [u_1 u_{(m+3)/2}] \\ \times \{v_{(n+3)/2}\} \cup \{u_{(m+3)/2}\} \times [v_{(n+3)/2} v_1] \cup [u_{(m+3)/2} w] \times \{v_1\}.$$

We have that $L(\gamma_1) = L(\gamma_2) = (m+n)/2$. If p is the midpoint of γ_1, then $d_{C_m \times C_n}(p, \gamma_2) = (m+n)/4$; therefore, $\delta(C_m \times C_n) \geq \delta(B) \geq (m+n)/4$.

Then we conclude in any case that $\delta(C_m \times C_n) = (m+n)/4$.

Now let us consider the graph $T_1 \times T_2$. We have that there exists an isometric subgraph Γ_j of T_j with Γ_j isometric to $P_{1+\mathrm{diam} T_j}$, for $j = 1, 2$; then $\Gamma_1 \times \Gamma_2$ is an isometric subgraph of $T_1 \times T_2$ and Lemma 17 gives $\delta(P_{1+\mathrm{diam} T_1} \times P_{1+\mathrm{diam} T_2}) \leq \delta(T_1 \times T_2)$. In order to prove the reverse inequality let us consider two cases.

If $\mathrm{diam}_{T_1} T_1 = \mathrm{diam}_{T_2} T_2$, then $\mathrm{diam}_{T_1 \times T_2}(T_1 \times T_2) = 2\mathrm{diam}_{T_1} T_1$ by Corollary 9. Now, Theorem 11 and the first item of Theorem 23 give

$$\delta(T_1 \times T_2) \leq \frac{1}{2} \mathrm{diam}_{T_1 \times T_2}(T_1 \times T_2) \\ = \mathrm{diam}_{T_1} T_1 = \delta(P_{1+\mathrm{diam} T_1} \times P_{1+\mathrm{diam} T_2}).$$

If $\mathrm{diam}_{T_1} T_1 \neq \mathrm{diam}_{T_2} T_2$, by symmetry we can assume that $\mathrm{diam}_{T_1} T_1 > \mathrm{diam}_{T_2} T_2$. Then Theorem 15 and the second item of Theorem 23 give

$$\delta(T_1 \times T_2) \leq \mathrm{diam}_{T_2} T_2 + 1/2 \\ = 1 + \mathrm{diam}_{T_2} T_2 - 1/2 = \delta(P_{1+\mathrm{diam} T_1} \times P_{1+\mathrm{diam} T_2}). \qquad \square$$

Theorem 24. *Let T be any tree and G be any graph. Then*

$$\delta(T \times G) = \begin{cases} (\operatorname{diam}_T T + \operatorname{diam}_G G)/2, & \text{if } \operatorname{diam}_T T = \operatorname{diam}_G V(G) > \operatorname{diam}_G G - 1, \\ \operatorname{diam}_G V(G) + 1/2, & \text{if } \operatorname{diam}_T T > \operatorname{diam}_G V(G). \end{cases}$$

Proof. If $\operatorname{diam}_T T > \operatorname{diam}_G V(G)$, then the formula is a direct consequence of Theorem 15 and the third item of Theorem 18.

Assume now that $\operatorname{diam}_T T = \operatorname{diam}_G V(G) > \operatorname{diam}_G G - 1$. First of all, note that Corollary 12 gives $\delta(T \times G) \le (\operatorname{diam}_T T + \operatorname{diam}_G G)/2$.

In order to prove the reverse inequality, we consider two cases: $\operatorname{diam}_G G = \operatorname{diam}_G V(G)$ and $\operatorname{diam}_G G = \operatorname{diam}_G V(G) + 1/2$.

If $\operatorname{diam}_G G = \operatorname{diam}_G V(G)$, then $\operatorname{diam}_G G = \operatorname{diam}_T T$ and the second item of Theorem 18 gives

$$\delta(T \times G) \ge \operatorname{diam}_T T = \frac{1}{2}(\operatorname{diam}_T T + \operatorname{diam}_G G).$$

If $\operatorname{diam}_G G = \operatorname{diam}_G V(G) + 1/2$, then there exist $v \in V(G)$ and $w \in [a, b] \in E(G)$ with $d_G(v, w) = \operatorname{diam}_G G$ (note that w is the midpoint of $[a, b]$). Let γ_a, γ_b be two geodesics joining v and w, with $a \in \gamma_a$ and $b \in \gamma_b$.

Let $u_1, u_2 \in V(T)$ such that $d_T(u_1, u_2) = \operatorname{diam}_T T$. We define $\sigma_1 := [u_2 u_1] \times \{v\} \cup \{u_1\} \times \gamma_a$, $\sigma_2 := \{u_1\} \times [wb] \cup [u_1 u_2] \times \{b\} \cup \{u_2\} \times [bv]$; σ_1, σ_2 are geodesics joining (u_2, v) and (u_1, w) with length $\operatorname{diam}_T T + \operatorname{diam}_G G$. If p is the midpoint of σ_1, it is easy to check that $d_{T \times G}(p, V(T \times G)) = 1/4$ and $\delta(T \times G) \ge d_{T \times G}(p, \sigma_2) = (\operatorname{diam}_T T + \operatorname{diam}_G G)/2$. $\square$

Note that $\delta(T \times G) = (\operatorname{diam}_T T + \operatorname{diam}_G G)/2$ does not hold for every tree T and every graph G with $\operatorname{diam}_T T = \operatorname{diam}_G V(G) = \operatorname{diam}_G G - 1$, as shown in the following example.

Let G be a graph obtained from two graphs G_1, G_2 isomorphic to C_3 by identifying a vertex of G_1 with a vertex of G_2. It is clear that $\operatorname{diam}_G V(G) = 2$ and $\operatorname{diam}_G G = 3$. One can check that $\delta(P_3 \times G) = 9/4 < 5/2 = (\operatorname{diam}_{P_3} P_3 + \operatorname{diam}_G G)/2$.

Theorem 25. *Let us consider any tree T with at least three vertices and any complete graph K_n with $n \ge 4$. Then $\delta(T \times K_n) = 3/2$.*

Proof. The result is a consequence of Theorem 24, since $\operatorname{diam}_T T \ge 2 > 1 = \operatorname{diam}_{K_n} V(K_n)$. $\square$

Theorem 26.

$$\delta(P_m \times C_n) = \begin{cases} (m - 1 + n/2)/2, & \text{if } m - 1 \le [n/2], \\ [n/2] + 1/2, & \text{if } m - 1 > [n/2], \end{cases}$$

for every $m \ge 2, n \ge 3$, where $[x] := \max\{k \in \mathbb{Z}\colon k \le x\}$.

Proof. First of all, by Theorem 24 if $m - 1 > [n/2]$, then $\delta(P_m \times C_n) = [n/2] + 1/2$.

Assume now that $m - 1 \le [n/2]$ and define $\{x\} := \min\{|x - y|\colon y \in \mathbb{Z}\}$. Theorem 7 gives that $\operatorname{diam}_{P_m \times C_n}(P_m \times C_n) = m - 1 + n/2$; hence, Theorem 11 gives $\delta(P_m \times C_n) \le (m - 1 + n/2)/2$.

In order to prove the reverse inequality, let us denote by $u_1, u_2, \dots, u_m$ (respectively, $v_1, v_2, \dots, v_n$) the vertices of P_m (respectively, the vertices of C_n) with $d_{P_m}(u_i, u_{i+1}) = 1$

for every $1 \leq i \leq m-1$ (respectively, $d_{C_n}(v_j, v_{j+1}) = 1$ for every $1 \leq j \leq n-1$). Let $x := v_1$ and y be the point in C_n such that $d_{C_n}(x, y) = n/2$. Let us define the geodesics $\sigma_1 := [(u_m, x)(u_1, x)] \cup [(u_1, x)(u_1, y)]$, where $[(u_1, x)(u_1, y)]$ is the geodesic containing (u_1, v_2), $\sigma_2 := [(u_1, y)(u_m, y)] \cup [(u_m, y)(u_m, x)]$ if n is even (with $(u_m, v_{n-1}) \in [(u_m, y)(u_m, x)]$), and $\sigma_2 := [(u_1, y)(u_1, v_{(n+3)/2})] \cup [(u_1, v_{(n+3)/2})(u_m, v_{(n+3)/2})] \cup [(u_m, v_{(n+3)/2})(u_m, x)]$ if n is odd; note that σ_1, σ_2 are geodesics of length $m-1+n/2$.

Now let us consider the bigon $B = \{\sigma_1, \sigma_2\}$. Then the midpoint p of σ_1 satisfies

$$d_{P_m \times C_n}(p, \sigma_2) = \min\{(m-1+n/2)/2, [n/2] + \{(m-1+n/2)/2\}\};$$

therefore,

$$\delta(P_m \times C_n) \geq \min\{(m-1+n/2)/2, [n/2] + \{(m-1+n/2)/2\}\}.$$

We will now show that $(m-1+n/2)/2 \leq [n/2] + \{(m-1+n/2)/2\}$. If n is even, then $[n/2] = n/2$ and $m-1 \leq n/2$; hence, $m-1+n/2 \leq n$ and

$$(m-1+n/2)/2 \leq n/2 = [n/2] \leq [n/2] + \{(m-1+n/2)/2\}.$$

If n is odd, then $[n/2] = (n-1)/2$ and $(m-1+n/2)/2 \in (\mathbb{N}+1/4) \cup (\mathbb{N}+3/4)$; hence, $\{(m-1+n/2)/2\} = 1/4$, $m-1+n/2 \leq (n-1)/2 + n/2 = n-1+1/2$ and

$$(m-1+n/2)/2 \leq (n-1)/2 + 1/4 = [n/2] + \{(m-1+n/2)/2\}.$$

Therefore, $\delta(P_m \times C_n) \geq (m-1+n/2)/2$ and we conclude that $\delta(P_m \times C_n) = (m-1+n/2)/2$. □

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