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Lifshitz tails for the interband light absorption coefficient

W KIRSCH and M KRISHNA*

Fakultät für Mathematik und Informatik, FernUniversität in Hagen, 58084 Hagen, Germany
*Institute of Mathematical Sciences, Taramani, Chennai 600 113, India
E-mail: krishna@imsc.res.in

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Abstract. In this paper we consider the interband light absorption coefficient (ILAC) for various models. We show that at the lower and upper edges of the spectrum the Lifshitz tails behaviour of the density of states implies similar behaviour for the ILAC at appropriate energies. The Lifshitz tails property is also exhibited at some points corresponding to the internal band edges of the spectrum.

Keywords. Interband light absorption coefficient; random operators; density of states.

1. Introduction

In this work we look for Lifshitz tails behaviour of the interband light absorption coefficient (ILAC) defined in eq. (9). The standard definition of the ILAC involves considering a pair of operators of the form $H_\omega^\pm = \Delta \pm V^\omega$, with Δ the Laplacian on either $\ell^2(\mathbb{Z}^d)$, in the discrete case or on $L^2(\mathbb{R}^d)$ in the continuous case, and taking a random potential V^ω. Restricting these operators $H_\omega^\pm$ to boxes Λ gives operators with discrete spectra so that in any finite region of energy these operators have only finitely many eigenvalues. Using this fact one can define the quantity

$$\frac{1}{\mathrm{Vol}(\Lambda)} \sum_{\lambda_\omega^- + \lambda_\omega^+ \leq E} |\langle \phi_{\omega,\lambda_\omega^-}, \psi_{\omega,\lambda_\omega^+}\rangle|^2,$$

where $\phi_{\omega,\lambda_\omega^-}, \psi_{\omega,\lambda_\omega^+}$ are the eigenfunctions of the operators $H_\omega^\mp$ restricted to the box Λ, corresponding to the eigenvalues $\lambda_\omega^-, \lambda_\omega^+$ respectively.

The limit of the above quantity, when it exits, gives the ILAC.

We consider a correlation measure (mentioned also in [12]) ρ and identify the ILAC as the distribution function of a marginal of the measure ρ in a diagonal direction. This identification enables us to prove theorems on the Lifshitz tails behaviour of the ILAC more easily since it involves only comparing the marginal of ρ with the density of states of either of the operators $H_\omega^\pm$. We also do not need to approximate to define the ILAC, but can obtain the function directly.

In the next section, we present an abstract version of the correlation measure ρ and the density of states n for a pair of random covariant operators and obtain relations between the two.

2. General covariant operators

We start with a definition of a random family of self-adjoint operators which are covariant under a group action.

Hypotheses 1.

(1) $\mathcal{H}$ is a (separable, complex) Hilbert space, $(\Omega, \mathbb{F}, \mathbb{P})$ a probability space.
(2) There is a locally compact abelian group G and $\{U_x\}_{x\in G}$ is a group of unitary operators on $\mathcal{H}$, i.e. the U_x are unitary and $U_{x+y} = U_x U_y$, $U_0 = \mathrm{Id}$, $U_{-x} = U_x^{-1} = U_x^*$.
(3) There is a discrete subgroup L of G and an orthogonal projection P on $\mathcal{H}$ such that $\{U_n^* P U_n\}_{n\in L}$, $\{U_n P U_n^*\}_{n\in L}$ are orthogonal partitions of unity on $\mathcal{H}$. We set $P_n = U_n^* P U_n$, $\tilde{P}_n = U_n P U_n^*$.
(4) $\{T_n\}_{n\in L}$ is a group of probability preserving transformations on Ω.

DEFINITION 1

A family $\{H_\omega\}_{\omega\in\Omega}$ of self-adjoint operators on $\mathcal{H}$ is called measurable if the family $\{(H_\omega + i)^{-1}\}_{\omega\in\Omega}$ is weakly measurable.

It is a fact that bounded measurable functions of measurable families of operators are weakly measurable and the product of weakly measurable families is weakly measurable (see [3], [2] and §2.4 of [26] for measurability issues).

DEFINITION 2

A weakly measurable family H_ω of bounded operators is called *covariant* (with respect to U_x, T_x) if

$$H_{T_x\omega} = U_x^* H_\omega U_x, \quad \text{for all } x \in G$$

Also, a measurable family H_ω of self-adjoint operators is called *covariant* (with respect to U_x, T_x) if

$$H_{T_x\omega} = U_x^* H_\omega U_x, \quad \text{for all } x \in G.$$

If H_ω is a covariant family of self-adjoint operators and f is a bounded measurable function, then the family $f(H_\omega)$ is covariant (also in [3], [2]). Moreover, if both H_ω and K_ω are covariant families of bounded operators, then $H_\omega K_\omega$ is a covariant family. We denote by $\|\cdot\|_2$ the Hilbert–Schmidt norm and by $\|\cdot\|_1$ the trace norm when operator arguments are involved.

PROPOSITION 1

Let H_ω and K_ω be covariant families of bounded operators such that $H_\omega P$, $H_\omega^* P$, $K_\omega P$, $K_\omega^* P$ are Hilbert Schmidt for almost every ω and

$$\mathbb{E}(\|H_\omega^* P\|_2),\ \mathbb{E}(\|H_\omega P\|_2),\ \mathbb{E}(\|K_\omega^* P\|_2) \text{ and } \mathbb{E}(\|K_\omega P\|_2) \tag{1}$$

are finite. Then $PH_\omega K_\omega P$ and $PK_\omega H_\omega P$ are trace class and

$$\mathbb{E}(\mathrm{Tr}(PH_\omega K_\omega P)) = \mathbb{E}(\mathrm{Tr}(PK_\omega H_\omega P)). \tag{2}$$

Proof. Without loss of generality we may and will assume that H_ω and K_ω are self-adjoint since the proof extends to the non self-adjoint case by linearity. In view of the assumptions of the proposition we have

$$\begin{aligned}\mathbb{E}\mathrm{Tr}(PH_\omega K_\omega P) &= \mathbb{E}\sum_n \mathrm{Tr}(PH_\omega K_\omega P)\\ &= \mathbb{E}\sum_n \mathrm{Tr}(PK_{T_n^{-1}\omega}\tilde{P}_n H_{T_n^{-1}\omega}P),\end{aligned}$$

where the covariance of H_ω, K_ω and the definitions of P_n, $\tilde{P}_n$ were used. Now an estimate using Cauchy–Schwarz inequality shows that

$$\mathbb{E}\|PH_\omega K_\omega P\|_1 \le \sqrt{\left[\mathbb{E}\left(\sum_n \|\tilde{P}_n H_\omega\|_2^2\right)\mathbb{E}\left(\sum_n \|\tilde{P}_n K_\omega\|_2^2\right)\right]} \tag{3}$$

$$\le \sqrt{\mathbb{E}(\mathrm{Tr}(PH_\omega^2 P)\mathbb{E}(PK_\omega^2 P)}. \tag{4}$$

In the above, by Fubini's theorem, the interchange of expectation and the summation is allowed and therefore the result follows. □

Remark 2. If we take covariant families of bounded operators $H_\omega, K_\omega, L_\omega$ satisfying condition (1) then the above Proposition gives:

$$\mathbb{E}(\mathrm{Tr}(PH_\omega K_\omega L_\omega P)) = \mathbb{E}(\mathrm{Tr}(PL_\omega H_\omega K_\omega P)) \tag{5}$$

If H_ω, K_ω are covariant families of bounded, positive (i.e. ≥ 0) operators satisfying the conditions (1) then

$$\mathbb{E}(\mathrm{Tr}PH_\omega K_\omega P)) \ \geq \ 0 \tag{6}$$

Hypotheses 2. Let H_ω be family of self-adjoint operators, which are bounded below, on a Hilbert space $\mathcal{H}$. Let $E_{H_\omega}(\cdot)$ be the (projection-valued) spectral measure of H_ω such that for any bounded Borel set A, the operators $PE_{H_\omega}(A)$, $E_{H_\omega}(A)P$ are trace class for a.e. ω and form a *covariant* family of operators.

For operators H_ω satisfying the above hypothesis, it is clear that for any finite x, the spectral measure $E_{H_\omega}((-\infty, x]) = E_{H_\omega}([c, x])$, with c finite and smaller than the infimum of the spectrum of H_ω. Therefore the hypothesis implies that for any finite x, the operators $PE_{H_\omega}((-\infty, x])$, $E_{H_\omega}((-\infty, x])P$ are trace class. Therefore we can now define the density of states for such operators.

DEFINITION 3

Let H_ω be a family of self-adjoint operators satisfying Hypothesis 2. Then the *density of states* of this family is defined to be the unique σ-finite measure n associated with the monotone right continuous function F,

$$F(x) = \mathbb{E}(\mathrm{Tr}(PE_{H_\omega}((-\infty, x])P)),$$

via $n((a, b]) = F(b) - F(a)$, $a, b \in \mathbb{R}$.

Thus for any bounded Borel set A, $n(A)$ agrees with the right-hand side of the above relation with A replacing $(-\infty, x]$.

In the above framework we define another measure that is used to define the ILAC. To do this we need a pair $H_\omega^\pm$ of self-adjoint operators as in Hypothesis 2 and consider the associated projection-valued measures $E_{H_\omega^\pm}(\cdot)$. We then define the density of states of these operators by

$$n_\pm(A) = \mathbb{E}(\mathrm{Tr}(PE_{H_\omega^\pm}(A)P)). \tag{7}$$

Consider the semi algebra $\mathcal{I} \times \mathcal{I}$ of subsets of $\mathbb{R}^2$ where

$$\mathcal{I} = \mathbb{R} \cup \{(a, b]: a, b \in \mathbb{R}\} \cup \{(a, \infty): a \in \mathbb{R}\} \cup \{(-\infty, a]: a \in \mathbb{R}\}.$$

We define the correlation measure ρ on $\mathcal{I} \times \mathcal{I}$ as

$$\rho(A \times B) = \mathbb{E}(\mathrm{Tr}(PE_{H_\omega^+}(A)E_{H_\omega^-}(B)P)), \tag{8}$$

where ρ is set to be ∞ if either A or B is an unbounded element of $\mathcal{I}$.

Using Hypothesis 2 and Proposition 1 we see that the following is valid.

PROPOSITION 3

Consider the operators $H_\omega^\pm$ satisfying Hypothesis 2 *and let $n_\pm$ and ρ be as in eq.* (8). *Then for any $B, C \in \mathcal{I}$ bounded*,

(1) $\rho(B \times C) = \mathbb{E}(\mathrm{Tr}(PE_{H_\omega^-}(C)E_{H_\omega^+}(B)E_{H_\omega^-}(C)P))$
(2) $\rho(B \times C) = \mathbb{E}(\mathrm{Tr}(PE_{H_\omega^+}(B)E_{H_\omega^-}(C)E_{H_\omega^+}(B)P))$
(3) *The following inequalities are valid*:

$$\rho(B \times C) \le n_+(B), \quad \rho(B \times C) \le n_-(C).$$

Proof. Since the subsets B, C are bounded the operators $PE_{H_\omega^-}(C)$, $PE_{H_\omega^+}(B)$ are covariant trace class operators (and hence Hilbert–Schmidt) satisfying inequality (1). Therefore the result follows by an application of Proposition 1 and Remark 2. □

The set function ρ takes values in $[0, \mathrm{Tr}(P)]$ if P is a trace class and in $[0, \infty]$, if $PE_{H_\omega^\pm((a,b])}$ are trace class only for bounded intervals $(a, b]$, in view of Proposition 3. We set

$$\rho(A) = \sum_{i=1}^{\infty} \rho(A_i \times B_i), \text{ if } A = \coprod_{i=1}^{\infty} A_i, A_i \in \mathcal{I}.$$

It is a simple exercise to see that this ρ is well defined on $\mathcal{I} \times \mathcal{I}$ and (via standard Caratheodory extension) extends as a σ-finite measure to the whole Borel σ-algebra of $\mathbb{R}^2$.

We collect the arguments about ρ in a proposition.

PROPOSITION 4

Consider a pair of covariant operators $H_\omega^\pm$ satisfying the Hypothesis 2 *and consider the correlation measures ρ extended to the Borel σ-algebra on $\mathbb{R}^2$ from that given by eq.* (8). *Then the following are valid*:

(1) *If P is trace class, then ρ is a finite measure on $\mathbb{R}^2$, with support in the closure of $\cup_\omega \sigma(H_\omega^+) \times \sigma(H_\omega^-)$.*
(2) *If P is not trace class but, $PE_{H_\omega^\pm}((a,b])P$ is trace class, for bounded intervals $(a,b]$, then ρ is a positive σ-finite measure on $\mathbb{R}^2$, with support in the closure of $\cup_\omega \sigma(H_\omega^+) \times \sigma(H_\omega^-)$.*

Remark 5. Typically the first case occurs (often with ρ being a probability) for operators on $\ell^2(\mathbb{Z}^d)$ and the second case occurs in $L^2(\mathbb{R}^d)$.

We take the transformation T on $\mathbb{R}^2$ given by

$$T\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \begin{pmatrix}\frac{\lambda_1+\lambda_2}{\sqrt{2}}\\ \frac{\lambda_1-\lambda_2}{\sqrt{2}}\end{pmatrix}.$$

Using this T we define the ILAC A as the distribution function,

$$\mathcal{A}(\lambda) - \mathcal{A}(\lambda') = \nu\left(\frac{1}{\sqrt{2}}(\lambda',\lambda]\right), \text{ where } \nu(B) = \rho \circ T^{-1}(B \times \mathbb{R}). \tag{9}$$

In the above equation the factor $\frac{1}{\sqrt{2}}$ is because of the normalization we used for T, so that this definition of ILAC agrees with the standard one in the case of finite box operators. We also note that since the operators $H_\omega^\pm$ are assumed to be bounded below $\mathcal{A}(-\infty) = 0$.

In the case when $\mathbb{P}$ in Hypothesis 1 is ergodic with respect to the action of G on Ω, then, the spectra $\sigma(H_\omega^\pm)$ of covariant families of operators $H_\omega^\pm$ are almost everywhere constant sets. In such a case we can talk about the infimum of spectra of $H_\omega^\pm$ without reference to ω. In this context we have the following theorem.

Theorem 2.1. *Suppose $H_\omega^\pm$ are a pair of random families of self-adjoint operators satisfying Hypothesis* 1. *Assume further that $\mathbb{P}$ is ergodic with respect to the action of G on Ω.*

(1) *Let $E_\pm = \inf\sigma(H_\omega^\pm)$. Then $\mathcal{A}(E_+ + E_- + a) - \mathcal{A}(E_+ + E_- - a) \leq n_\pm((E_\pm - 2a, E_\pm + 2a)), a > 0$.*
(2) *Let $E'_\pm = \sup\sigma(H_\omega^\pm)$. Then $\mathcal{A}(E'_+ + E'_- + a) - \mathcal{A}(E'_+ + E'_- - a) \leq n_\pm((E'_\pm - 2a, E'_\pm + 2a)),\ a > 0$.*

Proof. We shall prove the first case, the other proof is similar (where one has to use the fact that $\lambda_1 \leq E'_+, \lambda_2 \leq E'_-$ respectively for the other case and work it out). Let E_+, E_- be the infima of the spectra $\sigma(H_\omega^+), \sigma(H_\omega^-)$ of H_ω^+, H_ω^-. We consider Cartesian product $\Sigma = \sigma(H_\omega^+) \times \sigma(H_\omega^-)$ of the spectra of $H_\omega^\pm$, which is the support of the measure ρ. Therefore if we denote points of Σ by (λ_1,λ_2), so that $\lambda_1 \geq E_+, \lambda_2 \geq E_-$, then the possible values of $\lambda_1 + \lambda_2$ have a lower bound $E_- + E_+$, so $\lambda_1 + \lambda_2 \in (E_- + E_+, E_- + E_+ + a)$ implies $\lambda_1 \in (E_+ - 2a, E_+ + 2a)$ and $\lambda_2 \in (E_- - 2a, E_- + 2a)$ (see figure 1). This immediately implies the inclusions (the first inclusion is clear and the second one uses the above):

$$\begin{aligned}&\{(\lambda_1,\lambda_2)\colon \lambda_2 \in (E_-, E_- + (a/2)) \text{ and } \lambda_1 \in (E_+, E_+ + (a/2))\}\\ &\quad\subset \{(\lambda_1,\lambda_2)\colon \lambda_1 + \lambda_2 \in (E_- + E_+ - a, E_- + E_+ + a)\}\\ &\quad\subset \{(\lambda_1,\lambda_2)\colon \lambda_2 \in (E_-, E_- + 2a) \text{ and } \lambda_1 \in (E_+, E_+ + 2a)\}.\end{aligned}$$

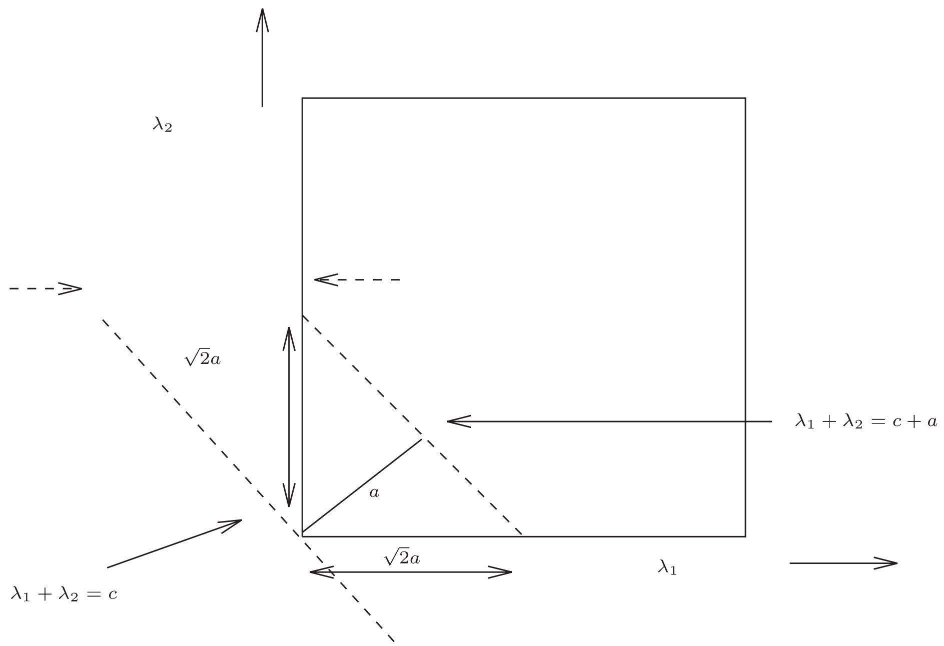

Figure 1. A corner of a rectangle.

This then would lead to the inequalities that

$$\begin{aligned}
&\mathcal{A}(E_+ + E_- + a) - \mathcal{A}(E_+ + E_-)\\
&\quad = \rho \circ T^{-1}\left(\frac{1}{\sqrt{2}}(E_+ + E_-, E_+ + E_- + a] \times \mathbb{R}\right)\\
&\quad = \rho(\{(\lambda_1, \lambda_2)\colon E_- + E_+ \le \lambda_1 + \lambda_2 \le E_- + E_+ + a\})\\
&\quad \le \rho((E_-, E_- + 2a) \times (E_+, E_+ + 2a))\\
&\quad \le \min\{\rho((E_-, E_- + 2a) \times \mathbb{R}), \rho((E_+, E_+ + 2a) \times \mathbb{R})\}\\
&\quad \le \min\{n_-((E_- - 2a, E_- + 2a)), n_+((E_+ - 2a, E_+ + 2a))\},
\end{aligned}$$

where the last inequality comes from Proposition 3(3) and enlarging the intervals slightly, which only increases the bound since $n_\pm$ are measures. □

Remark 6. When the asymptotic behaviour of the density of states $n_\pm$ is given as:

$$\limsup_{a>0}(\exp(C(2a)^\alpha)n_\pm((E_\pm - a, E_\pm + a))),$$

for some constants $C > 0, \alpha > 0$ we say that $n_\pm$ have Lifshitz tails behaviour. Setting $h(a) = \mathrm{e}^{-C(2a)^\alpha}$ and using the above inequalities, it follows that

$$\limsup_{a>0} \frac{1}{h(2a)} (A(E_- + E_+ + a) - A(E_- + E_+ - a))$$

$$\leq \limsup_{a>0} \frac{1}{h(2a)} n_+((E_+ - a, E_+ + 2a)) < \infty.$$

If the spectrum of an operator H has gaps, e.g. $\sigma(H) = \cup[a_n, b_n]$ with $b_n < a_n$ we call the band edges $a_n, n = 2, \ldots$ and $b_n, n = 1, \ldots$ *internal band edges* (as opposed to the band edge a_1). Under certain conditions, the density of states of random operators shows Lifshitz behaviour also at internal band edges (see [13], [14] and [16]). In the case when the density of states $n_\pm$ have Lifshitz tails behaviour at internal band edges, the same behaviour is valid for ILAC under some conditions. Suppose the spectra of $H_\omega^\pm$ consist of bands $\cup_{i=1}^N [a_i^\pm, b_i^\pm]$. Then the product of the spectra is $\cup_{i=1,j}^N [a_i^+, b_i^+] \times [a_j^-, b_j^-]$. Let us denote $R_{ij} = [a_i^+, b_i^+] \times [a_j^-, b_j^-]$. Then, the measure ρ is supported on the set $\cup_{i=1,j}^N R_{ij}$.

We index the pairs (ij) by β and use R_β to denote a rectangle forming part of Σ henceforth. So we have $\Sigma = \cup_\beta R_\beta$.

The central point in the proof of Theorem 2.1 is that if (c, d) is a corner of the rectangle formed by the lowest bands of the spectra of $H_\omega^\pm$, then the strip $\{(\lambda_1, \lambda_2) : c+d \leq \lambda_1+\lambda_2 \leq c+d+a\}$ intersected with the support of ρ is a triangle of side length $\sqrt{2}a$ (if a is not too big, see figure 1), hence its ρ measure is smaller than that of the square with the corner (c, d) and side length $2a$, as can be seen in figure 1. As we see in figure 2, there may be some rectangles in the support of ρ, with this property. Those rectangles in figure 2, where

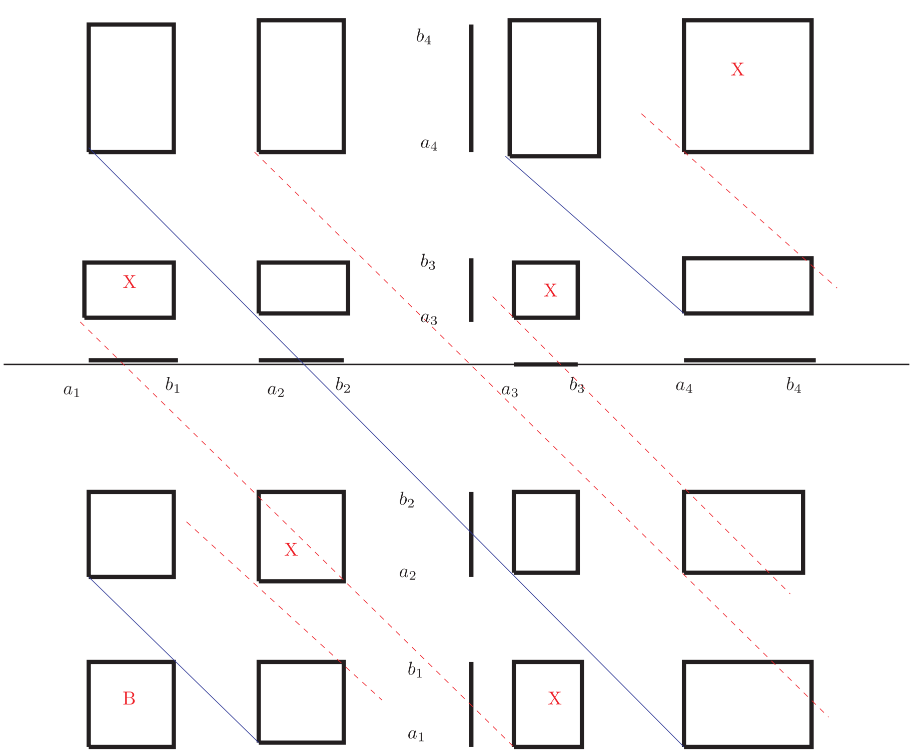

Figure 2. Products of spectra.

this is not true are marked by X and the solid lines are those lines $\lambda_1 + \lambda_2 = \text{const}$ for which this feature is valid and the dashed lines are those for which this is not true.

In the definition below the sets $R_\beta \subset \mathbb{R}^2$ and we denote the coordinates of $\mathbb{R}^2$ by (λ_1, λ_2).

DEFINITION 4

Let the support of ρ be $\Sigma = \cup_\beta R_\beta$, with $R_\beta = [a_i^+, b_i^+] \times [a_j^-, b_j^-]$, $\beta = (ij)$. Then we call a corner (c, d) of a rectangle R_β *good*, if the intersection of the line $\lambda_1 + \lambda_2 = c + d$ with Σ consists of finitely many points and all of them are corners of rectangles forming Σ. Given a corner (c, d) in Σ we shall denote by $\mathcal{K}_{c,d}$ the set of corners that lie on the line $\lambda_1 + \lambda_2 = c + d$.

Theorem 2.2. *Let the spectra of $H_\omega^\pm$ be as in Theorem* 2.1 *and let Σ be the support of the measure ρ given in eq.* (8). *Let $\mathcal{A}$, as given in eq.* (9) *be the corresponding ILAC. If* (c, d) *is a good corner in Σ. Denote the elements of $\mathcal{K}_{c,d}$ by $\{(c_\gamma, d_\gamma)\}$. Then we have*

$$\begin{aligned}
&\mathcal{A}(c + d + a) - \mathcal{A}(c + d - a) \\
&\quad \leq \sum_{(c_\gamma, d_\gamma) \in \mathcal{K}_{c,d}} \min\{n_+((c_\gamma - 2a, c_\gamma + 2a)), n_-((d_\gamma - 2a, d_\gamma + 2a))\}.
\end{aligned}$$

Proof. Firstly we note that if we take a rectangle, R_β, then only the lower-left and the top-right corners are candidates of being *good* corners, since for the other two corners, the line $\lambda_1 + \lambda_2 = \text{const}$ that contains the said corner will pass through the rectangle and hence has infinitely many points. We will prove the theorem for a good corner (c, d) which is a lower left corner of a rectangle, the proof for the case of a top-right good corner is similar. In this case we see immediately that if (c, d) is a good corner in Σ, then the intersection of the strip $S_a((c, d)) = \{(\lambda_1, \lambda_2): c + d \leq \lambda_1 + \lambda_2 \leq c + d + a\}$ with Σ is contained in finitely many rectangles R_β forming Σ. Further $S_a((c, d)) \cap R_\beta$ is contained in a square of side length $2a$ contained in R_β and having one corner common with a corner of R_β. Given a good corner (c, d) and the associated strip $S_a((c, d))$, let $(c_\gamma, d_\gamma) \in \mathcal{K}_{c,d}$ denote the corner of rectangle R_γ that has nonempty intersection with it. (Note that this corner satisfies $c_\gamma + d_\gamma = c + d$.)

Then whenever (c, d) is a good corner we have the inequality, with γ ranging over a finite set,

$$S_a((c, d)) \cap \Sigma \subset \cup_{(c_\gamma, d_\gamma) \in \mathcal{K}_{c,d}} [c_\gamma, c_\gamma + 2a] \times [d_\gamma, d_\gamma + 2a]. \tag{10}$$

This inequality implies immediately that:

$$\begin{aligned}
&\mathcal{A}(c + d + a) - \mathcal{A}(c + d - a) \\
&\quad \leq \mathcal{A}(c + d + a) - \mathcal{A}(c + d) = \rho(S_a((c, d)) \cap \Sigma) \\
&\quad \leq \sum_{(c_\gamma, d_\gamma) \in \mathcal{K}_{c,d}} \rho([c_\gamma, c_\gamma + 2a) \times [d_\gamma, d_\gamma + 2a)) \\
&\quad \leq \sum_{(c_\gamma, d_\gamma) \in K_{c,d}} \min\{n_+((c_\gamma - 2a, c_\gamma + 2a)), n_-((d_\gamma - 2a, d_\gamma + 2a))\},
\end{aligned} \tag{11}$$

where in the last inequality we enlarged the sets using the fact that $n_\pm$ are measures.

This shows that ILAC has the same continuity property as the density of states at the band edges. □

In the theorem below we identify good corners for a simple case of spectra having two bands.

Theorem 2.3. *Consider a pair of self-adjoint operators $H_\omega^\pm$ as in Theorem* 2.1. *Suppose a.e. ω, the spectra of H_ω^+, H_ω^- are given by $\cup_{i=1}^2[a_i^+, b_i^+]$ and $\cup_{i=1}^2[a_i^-, b_i^-]$, respectively, where $a_i^\pm$, $b_j^\pm$ are listed in the increasing order. Then the corners*

$$\{(a_1^+, a_1^-), (b_1^+, b_1^-), (a_2^+, a_2^-), (b_2^+, b_2^-)\}$$

are good *whenever $a_i^\pm$, $b_i^\pm$ satisfy,*

$$a_1^+ + a_1^- < b_1^+ + b_1^- < \max(a_2^+ + a_1^-, a_1^+ + a_2^-)$$
$$< \max(b_2^+ + b_1^-, b_1^+ + b_2^-) < a_2^+ + a_2^- < b_2^+ + b_2^-.$$

In the case $a_i^+ = a_i^-$, $b_i^+ = b_i^-$, $i = 1, 2$, even the corners

$$\{(a_2^+, a_1^-), (a_1^+, a_2^-), (b_2^+, b_1^-), (b_1^+, b_2^-)\}$$

are good.

Proof. This is a direct verification to see that the diagonal lines $\lambda_1 + \lambda_2 = \text{const}$ passing through the respective corners do not intersect any other rectangle. In the latter case when the spectra are the same, we have $a_2^+ + a_1^- = a_1^+ + a_2^-$ and $b_2^+ + b_1^- = b_1^+ + b_2^-$, hence the stated result. □

Remark 7. In the symmetric case $a_i^\pm = a_i$, $b_j^\pm = b_j$, however, the rectangles R_β, $R_\gamma \subset S_\beta$ if $\beta = (ij)$, $\gamma = (ji)$. In this case the above assumption still ensures that the lower-left and top-right corners of the rectangles are good.

3. Discrete models

Consider $\ell^2(\mathbb{Z}^d)$ and the discrete Laplacian $(\Delta u)(n) = \sum_{|n-i|=1} u(i)$. Consider real-valued i.i.d random variables $\{q(n)\}$ with common distribution μ. Let V_ω denote the operator of multiplication by the sequence $q_\omega(n)$. Consider the operators

$$H_\omega^\pm = \Delta \pm q_\omega.$$

Taking $G = L = \mathbb{Z}^d$, it is known that operators $E_{H_\omega}(A)$ are covariant. The projection P is taken to be the projection $|\delta_0\rangle\langle\delta_0|$ onto the subspace generated by the vector δ_0, which is an element of the standard basis for $\ell^2(\mathbb{Z}^d)$.

Then the density of states in these models are given by

$$n_\pm((a,b)) = \mathbb{E}(\mathrm{Tr}(PE_{H_\omega^\pm}((a,b))) = \mathbb{E}(\langle\delta_0, E_{H_\omega^\pm}((a,b))\delta_0\rangle)$$

and the correlation measure ρ is given by

$$\rho((a,b)\times(c,d)) = \mathbb{E}(\langle\delta_0, E_{H_\omega^+}((a,b))E_{H_\omega^-}((c,d))\delta_0\rangle)$$

and is a probability measure as per Proposition 4(1), since P is trace class in this case.

In this model the density of states of $H_\omega^\pm$ are shown to have Lifshitz tails behaviour at the bottom of the spectra [22], under the condition that μ satisfies $\mu((a, a+\epsilon)) \geq C\epsilon^N$, where a is the infimum of the support of μ.

In the case when the support of μ has two closed intervals $[a_1, b_1] \cup [a_2, b_2]$, ($a_i, b_i$ arranged in an increasing order so that $a_{i+1} > a_i$ for all i) and such that $b_i + 2d < a_{i+1} - 2d$, Simon [23] proves the Lifshitz tails behaviour at the internal band edges, if μ satisfies $\mu((a_i, a_i+\epsilon)) \geq C\epsilon^N$ and $\mu((b_i - \epsilon, b_i)) \geq C\epsilon^N$ for all i. When $[a_1 - 2d, b_1 + 2d]$ and $[a_2 - 2d, b_2 + 2d]$ are disjoint, Lifshitz tails behaviour at the band edges is also shown for the associated density of states n. That is at any of the band edges one has $n((E-\delta, E+\delta)) = O(\mathrm{e}^{-C\delta^{-\frac{d}{2}}})$ as $\delta \to 0$.

An application of Theorems 2.1 and 2.2 shows that the results are true for the ILAC $\mathcal{A}$, namely

Theorem 3.4. *Consider the Anderson models as above on $\ell^2(\mathbb{Z}^d)$. If $[a_1^\pm, b_1^\pm] \cup [a_2^\pm, b_2^\pm]$, are $\pm(\mathrm{supp}\,(\mu))$. Then, for some $C > 0$,*

- *External band edge case. For $E \in \{a_1^+ + a_1^-, b_2^+ + b_2^-\}$, one has*

$$\mathcal{A}(E+\delta) - \mathcal{A}(E-\delta) = o(\mathrm{e}^{-C\delta^{-\frac{d}{2}}}), \ \text{as } \delta \to 0.$$

- *Internal band edge case. If the gap between the intervals $[a_1^\pm - 2d, b_1^\pm + 2d]$ and $[a_2^\pm - 2d, b_2^\pm + 2d]$ is large enough, then*

$$\mathcal{A}(E+\delta) - \mathcal{A}(E-\delta) = o(\mathrm{e}^{-C\delta^{-\frac{d}{2}}}), \ \text{as } \delta \to 0,$$

for any $E \in \{b_1^+ + a_1^-, a_2^+ + a_1^-, b_2^+ + a_1^-, a_1^+ + a_2^-, b_1^+ + b_2^-, a_2^+ + a_2^-\}$.

Remark 8. We gave a simple example of a discrete model, however there are many more, those with periodic backgrounds [11], those which are unbounded [17] and so on. We refer to the review [7] for the numerous cases where the Lifshitz tails for the density of states is proved and for which our theorem applies to the ILAC.

4. Continuous models

Let us start by stating a theorem which is essentially a very weak version of the uncertainty principle.

We take $H_0 = -\Delta = -\sum_{i=1}^d D_j^2$, $D_j = i\frac{\partial}{\partial x_j}$, is self-adjoint on its natural domain in $L^2(\mathbb{R}^d)$ and its spectrum is $[0, \infty)$.

We start with a couple of lemmas. We recall the definition of the trace ideal $\mathscr{I}_p$ to be those bounded operators K with the property $|K|^p$ which is a trace class. Recall that elements of $\mathscr{I}_1$ are called trace class operators.

Lemma 9. *Consider $L^2(\mathbb{R}^d)$ and the operator $M = |-i\nabla|$. Then the operator $(|x|+i)^{-1}$ $(M+i)^{-1} \in \mathscr{I}_{d+1}$.*

Proof. Since the function $f(x) = (|x|+i)^{-1}$ is $L^{d+1}(\mathbb{R}^d)$ and the operator in question is just $f(x)f(-i\nabla)$, the result follows by an application of Theorem 4.1 in [24], which gives an estimate

$$\|f(x)g(-i\nabla)\|_{\mathscr{I}_p} \leq 2\pi^{-\frac{d}{p}} \|f\|_p \|g\|_p.$$

□

Let V be an operator of multiplication by a function $V(x)$ on $L^2(\mathbb{R}^d)$ on its natural domain such that V is bounded with respect to H_0 having relative bound smaller than 1. This implies that the operator $V(H+i)^{-1}$ is bounded. Then $H = H_0 + V$ is also self-adjoint (Kato–Rellich theorem) on the domain of H_0 and its spectrum is also bounded below. Writing $(H_0+i)(H+i)^{-1} = I - V(H+i)^{-1}$, we see that $(H_0+i)(H+i)^{-1}$ is also bounded. Let P denote the operator of multiplication by the indicator function χ_Λ of a bounded region $\Lambda \subset \mathbb{R}^d$ on $L^2(\mathbb{R}^d)$. Let $E_H(A)$ denote the spectral measure of a bounded Borel set A, with respect to the (projection valued) spectral measure of H. Then we have the following.

Theorem 4.1. *Consider $L^2(\mathbb{R}^d)$ and the operator $H_0 = -\Delta$. Let V be an operator of multiplication by a function $V(x)$, such that V is relatively bounded with respect to H_0 with relative bound $c < 1$ and consider $H = H_0 + V$. Suppose either*

(1) $d \leq 3$, *then $PE_H(A)$ and $E_H P$ are Hilbert–Schmidt, so $PE_H(A)P$ is a trace class for any bounded Borel set A.*

(2) *Suppose $d \geq 1$ and suppose V is bounded or $\frac{\partial}{\partial x_j}V$, $j = 1, \ldots, d$ are relatively bounded with respect to H_0. Then $PE_H(A)$ and $E_H(A)P$ are trace classes.*

Proof.
(1) Writing $PE_H(A) = P(|x|+i)^d(|x|+i)^{-d}(H_0+i)^{-1}(H_0+i)(H+i)^{-1}(H+i)E_h(A)$, we see that since all the factors are bounded, it is enough to show that $(|x|+i)^{-d}(H_0+i)^{-1}$ is Hilbert–Schmidt. The operator $(H_0+i)^{-1}$ is multiplication by $(|\xi|^2+i)^{-1}$ after taking Fourier transforms and hence is in $L^2(\mathbb{R}^d)$, $d \leq 3$. Therefore an application of Lemma 9 shows that the product is Hilbert–Schmidt.

(2) We will prove that $PE_H(A) \in \mathscr{I}_1$, the proof for $E_H(A)P$ is similar. By taking a compactly supported smooth function ϕ which is value 1 on the closure of A, we have $\phi(H)E_H(A) = E_H(A)$. We will therefore show that $P\phi(H)$ is a trace class for any compactly supported smooth function ϕ. We also note that the function $H\phi(H)$ is again a function of the same type as ϕ.

Further since P is a multiplication by compactly supported function of x, $P(x^2+i)^d$ is bounded. Therefore we will show that $(x^2+i)^{-d}\phi(H) \in \mathscr{I}_1$.

We prove this by induction. Before we start, we note that if $M \in \mathscr{I}_p$ and N is a bounded operator then $MN \in \mathscr{I}_p$.

First consider $(x^2+i)^{-1}\phi(H)$. We write this product as $(x^2+i)^{-1}(H+i)^{-1}(H+i)\phi(H)$ and consider (recalling $M = |-i\nabla|$),

$$(x^2+i)^{-1}(H+i)^{-1} = (x^2+i)^{-1}(M+i)^{-1}(M+i)(H_0+i)^{-1}(H_0+i)(H+i)^{-1}. \tag{12}$$

The product of the first two factors is in $\mathscr{I}_{d+1}$ (since $(|\xi|+i){-d-1}$ is integrable), by Lemma 9, the next two factors form a bounded operator (which can be seen by taking Fourier transforms). The final two factors form a bounded operator as argued before the lemma. Therefore the entire product is in $\mathscr{I}_{d+1}$. Since $(H+i)\phi(H)$ is bounded also, we get that $(x^2+i)^{-1}\phi(H) \in \mathscr{I}_{d+1}$.

Now assume that $(x^2+i)^{-n}\phi(H) \in \mathscr{I}_{\frac{d+1}{n}}$, and show that $(x^2+i)^{-n-1}\phi(H) \in \mathscr{I}_{\frac{d+1}{n+1}}$. We write $\psi(H) = (H+i)\phi(H)$, then

$$
\begin{aligned}
&(x^2+i)^{-d-1}\phi(H) \\
&\quad = (x^2+i)^{-d-1}(H+i)^{-1}(H+i)\phi(H)(x^2+i)^{-1}[(x^2+i)^{-n},(H+i)^{-1}]\psi(H) \\
&\qquad + (x^2+i)^{-1}(H+i)^{-1}(x^2+i)^{-n}\psi(H).
\end{aligned} \tag{13}
$$

Using Theorem 2.8(2.5b) in [24] (which says $M \in \mathscr{I}_q, N \in \mathscr{I}_r \implies MN \in \mathscr{I}_p$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$), and using the induction hypothesis and the already proved fact that $(x^2+i)^{-1}(H+i)^{-1} \in \mathscr{I}_{d+1}$, the last term is seen to be in $\mathscr{I}_{\frac{d+1}{n+1}}$.

So we concentrate on the first term

$$(x^2+i)^{-1}[(x^2+i)^{-n},(H+i)^{-1}]\psi(H) \tag{14}$$

$$
\begin{aligned}
&= (x^2+i)^{-1}(H+i)^{-1}[H,(x^2+i)^{-n}](H+i)^{-1}\psi(H) \\
&= (x^2+i)^{-1}(H+i)^{-1}[H_0,(x^2+i)^{-n}](H+i)^{-1}\psi(H) \\
&= (x^2+i)^{-1}(H+i)^{-1} \\
&\quad \times \left(-4ni\sum_{j=1}^{d} P_j x_j (x^2+i)^{-1} - 2d(x^2+i)^{-1} + 4d(n+1)x^2(x^2+i)^{-2}\right) \\
&\quad \times (x^2+i)^{-n}\psi_1(H)
\end{aligned} \tag{15}
$$

where we set $(H+i)^{-1}\psi(H) = \psi_1(H)$, where $P_j = -i\nabla_j$.

$$
\begin{aligned}
&(x^2+i)^{-1}(H+i)^{-1}P_j \\
&\quad = (x^2+i)^{-1}(H_0+1)^{-\frac{1}{2}}(H_0+1)^{\frac{1}{2}}(H+i)^{-1}(H_0+1)^{\frac{1}{2}}(H_0+1)^{-\frac{1}{2}}P_j,
\end{aligned}
$$

and using Lemmas 9 and 10 below, we see that this expression is in $\mathscr{I}_{d+1}$. Induction hypothesis gives $(x^2+i)^{-n}\psi_1(H) \in \mathscr{I}_{\frac{d+1}{n}}$. Therefore combining these two facts we see that $(x^2+i)^{-n-1}\phi(H) \in \mathscr{I}_{\frac{d+1}{n+1}}$. □

Lemma 10. *Suppose either V is bounded or $\left(\frac{\partial}{\partial x_j}V\right)(H+i)^{-1}$, $j = 1,\ldots,d$ are bounded. Then $(H_0+1)^{\frac{1}{2}}(H+i)^{-1}P_j$ is a bounded operator for each $j = 1,\ldots,d$.*

Proof. Consider the case when $\frac{\partial}{\partial x_j}V(H+i)^{-1}$ is bounded for each j. Then writing the expression using commutators

$$
\begin{aligned}
&(H_0+1)^{\frac{1}{2}}(H+i)^{-1}P_j \\
&\quad = (H_0+1)^{\frac{1}{2}}P_j(H+i)^{-1} + (H_0+1)^{\frac{1}{2}}(H+i)^{-1}[P_j,H](H+i)^{-1} \\
&\quad = (H_0+1)^{\frac{1}{2}}P_j(H+i)^{-1} + (H_0+1)^{\frac{1}{2}}(H+i)^{-1}\left(\frac{\partial}{\partial x_j}V\right)(H+i)^{-1}.
\end{aligned}
$$

The boundedness of the first term was seen before since $(H_0+1)^{\frac{1}{2}}P_j(H_0+1)^{-1}$ and $(H_0+1)(H+1)^{-1}$ are bounded. The second term is bounded by the assumption on V and the boundedness of $(H_0+1)^{-\frac{1}{2}}$.

Now consider the case when V is bounded. Then taking f, g in the domain of H_0, we have

$$\langle f, (H_0 + 1)f\rangle = \langle g, (H_0 + V + c + 1)f\rangle - \langle g, (V + c)f\rangle,$$

where c is a positive constant such that $H + c + 1$ is a positive operator (which is possible since H is bounded below). Since $H + c + 1$ is positive it has a unique square root, so using the boundedness of V and the above inequality, we obtain, for some finite C,

$$\|(H_0 + 1)^{\frac{1}{2}} f\|^2 \leq \|(H + c)^{\frac{1}{2}} f\|^2 + C\|f\| \leq D\|(H + c)^{\frac{1}{2}} f\|^2.$$

Taking $f = (H_0 + 1)^{\frac{1}{2}} g$, $\|g\| = 1$, for a set of g coming from $C_0^\infty(\mathbb{R}^d)$, we see that

$$C \leq \|(H + c)^{\frac{1}{2}}(H_0 + 1)^{-\frac{1}{2}} g\|^2, \ C > 0,$$

K independent of g. This shows that $(H + c)^{\frac{1}{2}}(H_0 + 1)^{-\frac{1}{2}}$ has a bounded inverse and that its inverse $(H_0 + 1)^{-\frac{1}{2}}(H + c)^{\frac{1}{2}}$ and $(H + c)^{\frac{1}{2}}(H_0 + 1)^{-\frac{1}{2}}$ are both bounded (since M bounded implies M^* is also bounded). Therefore writing

$$\begin{aligned}
&(H_0 + 1)^{\frac{1}{2}}(H + i)^{-1} P_j \\
&\quad = (H_0 + 1)^{\frac{1}{2}}(H + c)^{-\frac{1}{2}}(H + c)(H + i)^{-1}(H + c)^{-\frac{1}{2}}(H_0 + 1)^{\frac{1}{2}}(H_0 + 1)^{-\frac{1}{2}} P_j,
\end{aligned}$$

we see that the left-hand side is bounded. □

We are now ready to present examples where the theorems of the previous section are applicable. We first give a few examples of models on the lattice.

Consider $L^2(\mathbb{R}^d)$, $H_0 = \Delta$, $q(n), n \in \mathbb{Z}^d$, i.i.d random variables with distribution μ having compact support. Let Λ denote the unit cube centred at $0 \in \mathbb{R}^d$ and $\Lambda(n)$ denote the unit cube centred at the point $n \in \mathbb{Z}^d$. Let $V_\omega = \sum_{n\in\mathbb{Z}^d} q^\omega(n)\chi_{\Lambda(n)}$, where χ_A is the operator of multiplication by the indicator function of A. Then taking

$$H_\omega^\pm = \Delta \pm V_\omega,$$

we see that, since V_ω is bounded for each ω, the conditions of Theorem 4.1 are satisfied. Further taking $G = \mathbb{R}^d$ and $L = \mathbb{Z}^d$, $(U_x f)(y) = f(y - x)$, on $L^2(\mathbb{R}^d)$, $q_{Tm_\omega}(n) = q_\omega(n + m)$, in Hypothesis 1, the spectral projections $E_{H_\omega^\pm}((a, b))$ are covariant families in the sense of Definition 1. Theorem 4.1 shows that $\chi_{\Lambda(0)} E_{H_\omega^\pm}((a, b))\chi_{\Lambda(0)}$ is a trace class whenever (a, b) is a bounded interval. Hence we can define the density of states and the ILAC as in eqs (7) and (8), by taking P to be multiplication by $\chi_{\Lambda(0)}$. Therefore Theorems 2.1 and 2.2 are valid in this case.

Our theorem covers models where the random potential has the following forms.

- $V^\omega(x) = \sum_{n\in\mathbb{Z}^d} q^\omega(n)u(x - n)$, $\{q(n)\}$ i.i.d. random variables whose distribution has compact support and u a nice function with $u(x - n)$ summable.
- An addition of a periodic background potential W to the random potential above.
- Addition of magnetic fields.

If in all these cases the density of states have Lifshitz tails behaviour at the band edges the same is acquired by the ILAC at an appropriate energy level.

Remark 11. Let us remark that in the above examples we can even replace the Laplacian $-\Delta$ with a real polynomial function Q of $-i\nabla$ and the results go through, if for some $R > 0$, the polynomial satisfies

$$c_1\|\xi\|^{2n} \leq Q(\xi) \leq c_2\|\xi\|^{2n}, |\xi| > R, c_1, c_2 > 0.$$

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