Proc. Indian Acad. Sci. (Math. Sci.) Vol. 120, No. 4, September 2010, pp. 457–464.

Rigidity of minimal submanifolds with flat normal bundle

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MS received 19 June 2009; revised 12 April 2010

Abstract. Let M^n $(n \geq 3)$ be an n-dimensional complete immersed $\frac{n-2}{n}$-superstable minimal submanifold in an $(n+p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$ with flat normal bundle. We prove that if the second fundamental form of M satisfies some decay conditions, then M is an affine plane or a catenoid in some Euclidean subspace.

Keywords. Catenoid; minimal submanifolds; flat normal bundle.

1. Introduction

Let M^n be an n-dimensional complete minimal immersed submanifold in $\mathbb{R}^{n+p}$. Denote by $|A|$ the norm of the second fundamental form of M.

When $p = 1$, M is said to be stable if

$$0 \leq \int_M (|\nabla f|^2 - |A|^2 f^2), \quad \forall f \in C_0^\infty(M). \tag{1.1}$$

For some number $0 < \delta \leq 1$, it is defined that M is δ-stable if

$$0 \leq \int_M (|\nabla f|^2 - \delta |A|^2 f^2), \quad \forall f \in C_0^\infty(M). \tag{1.2}$$

Obviously, given $\delta_1 > \delta_2$, δ_1-stable implies δ_2-stable. So M is stable implies that M is δ-stable. There are some works on δ-stable complete minimal hypersurfaces $\mathbb{R}^{n+1}$. It is known that a complete stable minimal surface in $\mathbb{R}^3$ must be a plane, which was proved by do Carmo and Peng, and Fischer-Cobrie and Schoen independently [3, 6]. Do Carmo and Peng [4] showed that if M is a stable complete minimal hypersurface in $\mathbb{R}^{n+1}$ and

$$\lim_{R\to\infty} \frac{1}{R^{2+2q}} \int_{B(2R)\setminus B(R)} |A|^2 = 0, q < \sqrt{\frac{2}{n}},$$

then M is a hyperplane. Shen and Zhu [9] showed that if M is a complete stable minimal hypersurface in $\mathbb{R}^{n+1}$ with finite total curvature, that is,

$$\int_M |A|^n < +\infty,$$

then M is a hyperplane. Kawai proved that a $\delta\left(>\frac{1}{8}\right)$-stable complete minimal surface in $\mathbb{R}^3$ must be a plane [8]. For higher dimension $n \geq 3$, Tam and Zhou [11] showed that if M is an $\frac{n-2}{n}$-stable complete minimal hypersurface in $\mathbb{R}^{n+1}$ and

$$\lim_{R\to\infty} \frac{1}{R^2} \int_{B(2R)\setminus B(R)} |A|^{\frac{2(n-2)}{n}} = 0,$$

then M is either a hyperplane or a catenoid. In [1], Cheng and Zhou proved that if M is an $\frac{n-2}{n}$-stable complete minimal hypersurface in $\mathbb{R}^{n+1}$ and has bounded norm of the second fundamental form, then M must either have only one end or be a catenoid.

Wang [12] introduced the concept of super stability for minimal submanifolds when $p \geq 1$. M is said to be super-stable if

$$0 \leq \int_M (|\nabla f|^2 - |A|^2 f^2), \quad \forall f \in C_0^\infty(M). \tag{1.3}$$

When $p = 1$, the definition of super stability is exactly the same as that of stability and the normal bundle is trivially flat. Wang [12] proved that a complete super-stable minimal submainfold in $\mathbb{R}^{n+p}$ with finite total curvature is an affine plane. Seo [10] showed that if M is a complete super-stable minimal submainfold in $\mathbb{R}^{n+p}$ with flat normal bundle and $\int_M |A|^2 < +\infty$, then M is an affine plane. Recently, the author [7] proved that if M^n ($n \leq 7$) is a complete super-stable minimal submainfold in $\mathbb{R}^{n+p}$ with flat normal bundle and $\int_M |A|^3 < +\infty$, then M is an affine plane. For some number $0 < \delta \leq 1$, it is defined that M is δ-super-stable if

$$0 \leq \int_M (|\nabla f|^2 - \delta |A|^2 f^2), \quad \forall f \in C_0^\infty(M). \tag{1.4}$$

Now we study δ-super-stable minimal submanifolds in $\mathbb{R}^{n+p}$ with flat normal bundle. Our main results in this paper are stated as follows.

Theorem 1.1. *Let M^n ($n \geq 3$) be a δ-super-stable complete immersed minimal submanifold in $\mathbb{R}^{n+p}$ with flat normal bundle.*

(1) *If M is $\delta\left(>\frac{n-2}{n}\right)$-super-stable and*

$$\lim_{R\to\infty} \frac{1}{R^{2+2q\delta}} \int_{B(2R)\setminus B(R)} |A|^{2\delta} = 0, q < \sqrt{\left(\frac{2}{n}-1\right)\delta^{-1}+1},$$

then M is an affine plane.

(2) *If M is $\delta\left(>\frac{n-2}{n}\right)$-super-stable and*

$$\lim_{R\to\infty} \frac{1}{R^2} \int_{B(2R)\setminus B(R)} |A|^{\frac{2(n-2)}{n}} = 0,$$

then M is an affine plane.

(3) *If M is $\frac{n-2}{n}$-super-stable and*

$$\lim_{R\to\infty} \frac{1}{R^2} \int_{B(2R)\setminus B(R)} |A|^{\frac{2(n-2)}{n}} = 0,$$

then M is a catenoid in some Euclidean subspace or an affine plane.

Remark 1.2. Theorem 1.1 can be regarded as generalization of the above theorems due to Seo and Tam and Zhou.

2. Proof of the theorem

We follow the notations of Chern, do Carmo and Kobayashi [2].

Let M^n be an n-dimensional minimal immersed submanifold in $\mathbb{R}^{n+p}$. We choose an orthonormal frame $e_1, e_2, \ldots, e_{n+p}$ in $\mathbb{R}^{n+p}$ such that, restricted to M, the vectors $e_1, e_2, \ldots, e_n$ are tangent to M. And we shall denote the second fundamental form by h^α_{ij}. Then we have $|A|^2 = \sum (h^\alpha_{ij})^2$ and

$$2|A|\Delta|A| + 2|\nabla|A||^2 = \Delta|A|^2 = 2\sum (h^\alpha_{ijk})^2 + 2\sum (h^\alpha_{ij})\Delta h^\alpha_{ij}. \tag{2.1}$$

By eq. (2.23) of [2], we have

$$\sum (h^\alpha_{ij})\Delta h^\alpha_{ij} = -\sum (h^\alpha_{ik}h^\beta_{jk} - h^\alpha_{jk}h^\beta_{ik})(h^\alpha_{il}h^\beta_{jl} - h^\alpha_{jl}h^\beta_{il}) - \sum h^\alpha_{ij}h^\alpha_{kl}h^\beta_{ij}h^\beta_{kl}. \tag{2.2}$$

Before the proof of Theorem 1.1, we need to prove the following lemma.

Lemma 2.1. *Let M be an n-dimensional immersed minimal submanifold in $\mathbb{R}^{n+p}$ with flat normal bundle. Then*

$$|A|\Delta|A| + |A|^4 \geq \frac{2}{n}|\nabla|A||^2 + E, \tag{2.3}$$

where $E = \sum_{\alpha\neq\beta}(\sum_{i,j}(h^\alpha_{ij})^2)(\sum_{i,j}(h^\beta_{ij})^2)$ is nonnegative.

Proof. Since M has flat normal bundle, we have $h^\alpha_{ik}h^\beta_{jk} - h^\alpha_{jk}h^\beta_{ik} = 0$. Therefore, from (2.2) we obtain

$$\sum (h^\alpha_{ij})\Delta h^\alpha_{ij} = -\sum h^\alpha_{ij}h^\alpha_{kl}h^\beta_{ij}h^\beta_{kl}. \tag{2.4}$$

For each α, let H_α denote the symmetric matrix (h^α_{ij}), and set $S_{\alpha\beta} = \sum h^\alpha_{ij}h^\beta_{ij}$. Then the $(p \times p)$ matrix $(S_{\alpha\beta})$ is symmetric and can be assumed to be diagonal for a suitable choice of $e_{n+1}, \ldots, e_{n+p}$. Thus we have from (2.4)

$$\sum (h^\alpha_{ij})\Delta h^\alpha_{ij} = -\sum S^2_{\alpha\alpha} = -\sum_\alpha \left(\sum_{i,j}(h^\alpha_{ij})^2\right)^2. \tag{2.5}$$

On the other hand,

$$|A|^4 = (|A|^2)^2 = \left(\sum_\alpha \sum_{i,j}(h^\alpha_{ij})^2\right)^2 = \sum_\alpha \left(\sum_{i,j}(h^\alpha_{ij})^2\right)^2 + E. \tag{2.6}$$

Hence from (2.1), (2.5) and (2.6) we have

$$2|A|\Delta|A| + 2|\nabla|A||^2 = 2\sum (h^\alpha_{ijk})^2 + 2E - 2|A|^4.$$

Since $\sum (h^\alpha_{ijk})^2 = |\nabla A|^2$, from the above equality we get

$$|A|\Delta|A| + |\nabla|A||^2 = |\nabla A|^2 + E - |A|^4. \tag{2.7}$$

For M, Xin [13] gave the following estimate:

$$|\nabla A|^2 - |\nabla |A||^2 \geq \frac{2}{n}|\nabla |A||^2.$$

Combining with (2.7), we obtain

$$|A|\Delta |A| + |A|^4 \geq \frac{2}{n}|\nabla |A||^2 + E.$$

This completes the proof of Lemma 2.1. □

Proof of Theorem 1.1. For any $\epsilon > 0$ and $a > 0$, let $u = (|A|^2 + \epsilon)^{\frac{a}{2}}$. Then

$$\begin{aligned}
\Delta u &= u(\Delta \log u + |\nabla \log u|^2) \\
&= \frac{au}{2}\left(\frac{\Delta |A|^2}{|A|^2+\epsilon} - \frac{|\nabla |A|^2|^2}{(|A|^2+\epsilon)^2}\right) + \frac{ua^2}{4}\frac{|\nabla |A|^2|^2}{(|A|^2+\epsilon)^2} \\
&= au\left(\frac{\frac{1}{2}\Delta |A|^2}{|A|^2+\epsilon} + (a-2)\frac{|A|^2|\nabla |A||^2}{(|A|^2+\epsilon)^2}\right) \\
&\geq au\left(\frac{\left(1+\frac{2}{n}\right)|\nabla |A||^2 - |A|^4 + E}{|A|^2+\epsilon} + (a-2)\frac{|A|^2|\nabla |A||^2}{(|A|^2+\epsilon)^2}\right) \\
&= -au\frac{(|A|^2+\epsilon)^2}{|A|^2+\epsilon} + au\frac{2\epsilon|A|^2}{|A|^2+\epsilon} + au\frac{2\epsilon^2}{|A|^2+\epsilon} + au\frac{E}{|A|^2+\epsilon} \\
&\quad + au\left(\frac{\left(1+\frac{2}{n}\right)(|A|^2+\epsilon)|\nabla |A||^2}{(|A|^2+\epsilon)^2} + \frac{(a-2)|A|^2|\nabla |A||^2}{(|A|^2+\epsilon)^2}\right) \\
&= -au(|A|^2+\epsilon) + au\frac{E}{|A|^2+\epsilon} + \left(\frac{2}{n}+a-1\right)au\frac{|A|^2|\nabla |A||^2}{(|A|^2+\epsilon)^2} \\
&\quad + au\frac{2\epsilon|A|^2}{|A|^2+\epsilon} + au\frac{2\epsilon^2}{|A|^2+\epsilon} + au\frac{\left(1+\frac{2}{n}\right)\epsilon|\nabla |A||^2}{(|A|^2+\epsilon)^2} \\
&\geq -au(|A|^2+\epsilon) + au\frac{E}{|A|^2+\epsilon} + \left(\frac{2}{n}+a-1\right)au\frac{|A|^2|\nabla |A||^2}{(|A|^2+\epsilon)^2}, \quad (2.8)
\end{aligned}$$

where we apply (2.3) in Lemma 2.1. It follows from (2.8) by directly computing that

$$u\Delta u \geq -au^{2+\frac{2}{a}} + au^{2-\frac{2}{a}}E + \left(\frac{2}{n}+a-1\right)a^{-1}|\nabla u|^2. \quad (2.9)$$

Let $q \geq 0$ and $f \in C_0^\infty(M)$. Multiplying (2.9) by $u^{2q}f^2$ and integrating over M, we obtain

$$\begin{aligned}
&a\int_M u^{2(1+q)-\frac{2}{a}}f^2E + \left(\frac{2}{n}+a-1\right)a^{-1}\int_M |\nabla u|^2u^{2q}f^2 \\
&\leq a\int_M u^{2(1+q)+\frac{2}{a}}f^2 + \int_M u^{2q+1}f^2\Delta u
\end{aligned}$$

$$= a \int_M u^{2(1+q)+\frac{2}{a}} f^2 - 2 \int_M u^{2q+1} f \langle \nabla f, \nabla u \rangle$$
$$- (2q+1) \int_M u^{2q} f^2 |\nabla u|^2,$$

which gives

$$a \int_M u^{2(1+q)-\frac{2}{a}} f^2 E + \left(2(q+1) - \frac{n-2}{na}\right) \int_M |\nabla u|^2 u^{2q} f^2$$
$$\leq a \int_M u^{2(1+q)+\frac{2}{a}} f^2 - 2 \int_M u^{2q+1} f \langle \nabla f, \nabla u \rangle. \tag{2.10}$$

Using the Cauchy–Schwarz inequality, we can rewrite (2.10) as

$$\left(2(q+1) - \frac{n-2}{na} - \epsilon'\right) \int_M u^{2q} f^2 |\nabla u|^2 + a \int_M u^{2(1+q)-\frac{2}{a}} f^2 E$$
$$\leq a \int_M u^{2(1+q)+\frac{2}{a}} f^2 + \frac{1}{\epsilon'} \int_M u^{2(q+1)} |\nabla f|^2. \tag{2.11}$$

On the other hand, replacing f by $u^{(1+q)} f$ in the δ-super-stability inequality (1.4), we have

$$\delta \int_M (u^{\frac{2}{a}} - \epsilon) u^{2(1+q)} f^2 \leq \delta \int_M |A|^2 u^{2(1+q)} f^2$$
$$\leq (1+q)^2 \int_M u^{2q} f^2 |\nabla u|^2 + \int_M u^{2(1+q)} |\nabla f|^2$$
$$+ 2(1+q) \int_M u^{(2q+1)} f \langle \nabla f, \nabla u \rangle, \tag{2.12}$$

which gives

$$\delta \int_M (u^{\frac{2}{a}} - \epsilon) u^{2(1+q)} f^2$$
$$\leq (1+q)(1+q+\epsilon') \int_M u^{2q} f^2 |\nabla u|^2 + \left(1 + \frac{1+q}{\epsilon'}\right) \int_M u^{2(1+q)} |\nabla f|^2. \tag{2.13}$$

(1) *When* $\delta > \frac{n-2}{n}$. If $2(q+1) - \frac{n-2}{na} - \epsilon' > 0$, then introducing (2.11) to (2.13), we obtain

$$\left[\left(2(q+1) - \frac{n-2}{na} - \epsilon'\right)\delta - (1+q)(1+q+\epsilon')a\right] \int_M u^{2(1+q)+\frac{2}{a}} f^2$$
$$\leq \left[\left(2(q+1) - \frac{n-2}{na} - \epsilon'\right)\left(1 + \frac{1+q}{\epsilon'}\right) + \frac{(1+q)(1+q+\epsilon')}{\epsilon'}\right]$$
$$\times \int_M u^{2(1+q)} |\nabla f|^2 + \epsilon \left(2(q+1) - \frac{n-2}{na} - \epsilon'\right) \delta \int_M u^{2(1+q)} f^2. \tag{2.14}$$

(i) If $a=\delta$, taking $q<\sqrt{\left(\frac{2}{n}-1\right)\delta^{-1}+1}$, we see that

$$\left(2(q+1)-\frac{n-2}{n\delta}\right)\delta-(1+q)^2\delta>0,$$

and then we can choose $\epsilon'>0$ sufficiently small so that

$$\left(2(q+1)-\frac{n-2}{n\delta}-\epsilon'\right)\delta-(1+q)(1+q+\epsilon')\delta>0.$$

Let $\epsilon\to 0$, it follows from (2.14) that for $q<\sqrt{\left(\frac{2}{n}-1\right)\delta^{-1}+1}$ the following inequality holds:

$$\int_M u^{2(1+q)+\frac{2}{\delta}}f^2\leq C_1\int_M u^{2(1+q)}|\nabla f|^2, \tag{2.15}$$

where C_1 is a constant that depends on n,ϵ' and q. We now try to transform (2.15) the right hand side only involved u in the power two. For that, we use Young's inequality:

$$ab\leq\frac{\beta^s a^s}{s}+\frac{\beta^{-t}b^t}{t},\ \frac{1}{s}+\frac{1}{t}=1, \tag{2.16}$$

where $\beta>0$ is arbitrary and $1<s<\infty$, $1<t<\infty$. Let r, $0<r<2+2q$, be a number yet to be determined. By using (2.16), we obtain

$$\begin{aligned}u^{2+2q}|\nabla f|^2&=f^2\left(u^{2+2q}\frac{|\nabla f|^2}{f^2}\right)=f^2\left(u^{2+2q-r}u^r\frac{|\nabla f|^2}{f^2}\right)\\&\leq f^2\left(\frac{\beta^s}{s}u^{s(2+2q-r)}+\frac{\beta^{-t}}{t}\left(u^r\frac{|\nabla f|^2}{f^2}\right)^t\right).\end{aligned} \tag{2.17}$$

We now choose r to satisfy the following equations:

$$s(2+2q-r)=2+2q+\frac{2}{\delta},\ rt=2,\ \frac{1}{s}+\frac{1}{t}=1.$$

This is indeed possible, and the solution is

$$r=\frac{2}{1+q\delta},\ t=1+q\delta,\ s=\frac{1+q\delta}{q\delta}.$$

Using these values and the fact that β may be made small, from (2.15) and (2.17) we obtain

$$\int_M u^{2+2q+\frac{2}{\delta}}f^2\leq C_2\int_M u^2\frac{|\nabla f|^{2+2q\delta}}{f^{2q\delta}}, \tag{2.18}$$

where C_2 is a constant that depends on n,ϵ',β and q. Now we use the arbitrariness of f to replace f by $f^{1+q\delta}$ in (2.18) and obtain

$$\int_M u^{2+2q+\frac{2}{\delta}}f^{2+2q\delta}\leq C_3\int_M u^2|\nabla f|^{2+2q\delta}. \tag{2.19}$$

Let f be a smooth function on $[0, \infty)$ such that $f \geq 0$, $f = 1$ on $[0, R]$ and $f = 0$ in $[2R, \infty)$ with $|f'| \leq \frac{2}{R}$. Then considering $f \circ r$, where r is the function in the definition of $B(R)$, we have from (2.19)

$$\int_{B(R)} u^{2+2q+\frac{2}{\delta}} \leq \frac{4C_3}{R^{2+2q\delta}} \int_{B(2R)\setminus B(R)} u^2. \tag{2.20}$$

Let $R \to +\infty$, by assumption that $\lim_{R\to\infty} \frac{1}{R^{2+2q\delta}} \int_{B(2R)\setminus B(R)} |A|^{2\delta} = 0$, from (2.20) we conclude $|A| = 0$, i.e., M is an affine plane.

(ii) If $a = \frac{n-2}{n}$, taking $q = 0$, we see that

$$\left(2(q+1) - \frac{n-2}{na}\right)\delta - (1+q)^2 a > 0,$$

and then we can choose $\epsilon' > 0$ sufficiently small so that

$$\left(2(q+1) - \frac{n-2}{na} - \epsilon'\right)\delta - (1+q)(1+q+\epsilon')a > 0.$$

Let $\epsilon \to 0$, it follows from (2.14) that the following inequality holds:

$$\int_M u^{2+\frac{2n}{n-2}} f^2 \leq C_4 \int_M u^2 |\nabla f|^2, \tag{2.21}$$

where C_4 is a constant that depends on n, ϵ'. Under the assumption that $\lim_{R\to\infty} \frac{1}{R^2} \int_{B(2R)\setminus B(R)} |A|^{\frac{2(n-2)}{n}} = 0$, from (2.21) we conclude that $|A| = 0$, i.e., M is an affine plane.

(2) *When* $\delta = \frac{n-2}{n}$. Multiplying (2.10) by $(1+q)$ and adding it up with (2.12), we obtain

$$\begin{aligned}(1+q)\left((1+q) - \frac{n-2}{na}\right)\int_M u^{2q} f^2 |\nabla u|^2 + a(1+q)\int_M u^{2(1+q)-\frac{2}{a}} f^2 E \\ \leq (a(1+q)-\delta)\int_M u^{2(1+q)+\frac{2}{a}} f^2 + \int_M u^{2(1+q)} |\nabla f|^2 \\ + \epsilon\delta \int_M u^{2(1+q)} f^2.\end{aligned} \tag{2.22}$$

If $(1+q)a = \frac{n-2}{n}$, from (2.22) we obtain

$$\frac{n-2}{n}\int_M u^{2(1+q)-\frac{2}{a}} f^2 E \leq \int_M u^2 |\nabla f|^2 + \epsilon\delta \int_M u^2 f^2. \tag{2.23}$$

Taking f as before, we have from (2.23)

$$\frac{n-2}{n}\int_{B(R)} u^{2(1+q)-\frac{2}{a}} E \leq \frac{4}{R^2}\int_{B(2R)\setminus B(R)} u^2 + \epsilon\delta \int_{B(2R)} u^2. \tag{2.24}$$

If $|A| > 0$, let $\epsilon \to 0$ and then $R \to +\infty$, by assumption that $\lim_{R\to\infty} \frac{1}{R^2} \int_{B(2R)\setminus B(R)} |A|^{\frac{2(n-2)}{n}} = 0$. From (2.24) we conclude that $E = 0$. Then we obtain $\sum_{i,j}(h_{ij}^\alpha)^2, \alpha = n+1, n+2, \ldots, n+p$ at least have $p-1$ zeros by (2.6). So it is easy to see from a theorem of [5] that M lies in a totally geodesic $\mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+p}$. Hence M is an $\frac{n-2}{n}$-stable complete minimal hypersurface in $\mathbb{R}^{n+1}$. Therefore we obtain that M is a catenoid in some Euclidean subspace or an affine plane according to Theorem 3.1 in [11]. □

Acknowledgements

The author would like to thank the referee for some helpful suggestions. This work is supported by National Natural Science Foundation of China (10671087) and Jiangxi Province Natural Science Foundation of China (2008GZS0060, 2009GZS0017).

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