Proc. Indian Acad. Sci. (Math. Sci.) Vol. 120, No. 4, September 2010, pp. 441–456.

Mixed norm estimate for Radon transform on weighted L^p spaces

ASHISHA KUMAR and SWAGATO K RAY

Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur 208 016, India
E-mail: ashishak@iitk.ac.in; skray@iitk.ac.in

MS received 9 February 2010; revised 8 April 2010

Abstract. We will discuss about the mapping property of Radon transform on L^p spaces with power weight. It will be shown that the Pitt's inequality together with the weighted version of Hardy–Littlewood–Sobolev lemma imply weighted inequality for the Radon transform.

Keywords. Radon transform; mixed norm; Pitt's inequality; Stein–Weiss potential.

1. Introduction

Given a measurable function f on $\mathbb{R}^n$ the Radon transform Rf of f is defined on the set of all affine hyperplanes. Note that any affine hyperplane with normal $\omega \in S^{n-1}$ and distance $|t|$ from the origin is given by the set $H_{\omega,t} = \{x \in \mathbb{R}^n: x \cdot \omega = t\}$, where $x \cdot \omega$ denotes the Euclidean inner product. Since $H_{\omega,t} = H_{-\omega,-t}$ it follows that the set of all affine hyperplanes can be parametrized by the set $S^{n-1} \times \mathbb{R}/\mathbb{Z}_2$. The Radon transform of a nice measurable function f at a point $(\omega, t) \in S^{n-1} \times \mathbb{R}$ is then defined by

$$Rf(\omega, t) = \int_{H_{\omega,t}} f. \tag{1.1}$$

The above integral is to be interpreted as the integral of the restriction of f on the affine hyperplane $H_{\omega,t}$ with respect to the $(n-1)$-dimensional Lebesgue measure on $H_{\omega,t}$. We note that the Radon transform obeys the symmetry relation $Rf(\omega, t) = Rf(-\omega, -t)$. We can make the above definition more explicit as follows. For $\omega \in S^{n-1}$ if $y \cdot \omega = t$ for $t \in \mathbb{R}$ then $y = t\omega + y'$ where $y' \in Sp_{\mathbb{R}}\{\omega\}^{\perp}$. Thus

$$Rf(\omega, t) = \int_{\mathbb{R}^{n-1}} f(t\omega + y')\mathrm{d}y'. \tag{1.2}$$

It now follows from Fubini's theorem that if $f \in L^1(\mathbb{R}^n)$ then the integral in (1.2) is well defined for each $\omega \in S^{n-1}$ and for almost every $t \in \mathbb{R}$. In fact,

$$\sup_{\omega \in S^{n-1}} \int_{\mathbb{R}} |Rf(\omega, t)|\mathrm{d}t \leq \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |f(t\omega + y')|\mathrm{d}y'\mathrm{d}t = \|f\|_{L^1(\mathbb{R}^n)}. \tag{1.3}$$

The situation is not so simple if $f \in L^p(\mathbb{R}^n)$ where $p > 1$. The best possible analogue of (1.3) was proved in [19].

Theorem 1.1. *There exists a positive constant C such that for all $f \in C_c^\infty(\mathbb{R}^n)$ the following inequality holds,*

$$\left(\int_{S^{n-1}}\left[\int_{\mathbb{R}}|Rf(\omega,t)|^q\mathrm{d}t\right]^{p'/q}\mathrm{d}\omega\right)^{1/p'} \leq C\|f\|_{L^p(\mathbb{R}^n)},$$

where $1 \leq p < \frac{n}{n-1}$ and $\frac{1}{q} = \frac{n}{p} - n + 1$.

It was also shown in [19] that there exists a function $f \in L^p(\mathbb{R}^n)$ with $p \geq \frac{n}{n-1}$ such that $Rf(\omega,t)$ is infinite for all $\omega \in S^{n-1}$ and for almost every $t \in \mathbb{R}$. So, apart from the mixed norm inequality, the above result also tells us that the lower dimensional integrals appearing in (1.2) are well defined if and only if $1 \leq p < \frac{n}{n-1}$. For analogues of Theorem 1.1 and discussion on related problems for certain operators which generalizes the Radon transform we refer the reader to [5], [7–10], [14], [29] and references therein.

One important problem regarding Radon transform is to describe the range and kernel of the operator. This problem has been extensively studied (see [16], [18] and references therein). In this connection the following weighted estimate was proved in [20].

Theorem 1.2. *For all $f \in C_c^\infty(\mathbb{R}^n)$, $n \geq 3$, there exists a positive constant C such that*

$$\int_{\mathbb{R}\times S^{n-1}}|Rf(\omega,t)|^2\mathrm{d}t\mathrm{d}\omega \leq C\int_{\mathbb{R}^n}|f(x)|^2\|x\|^{n-1}\mathrm{d}x.$$

One can now ask two natural questions. The first one is about validity of Theorem 1.2 for $n = 2$. The second one is about analogue of (1.3) for L^p spaces with power weight which will generalize Theorems 1.1 and 1.2. While the first question is easy to tackle, the second one poses considerable difficulty. Apart from answering the first question our goal in this paper is to prove inequalities of the form

$$\left(\int_{S^{n-1}}\left[\int_{\mathbb{R}}|Rf(\omega,t)|^q|t|^\beta\mathrm{d}t\right]^{r/q}\mathrm{d}\omega\right)^{1/r} \leq C\left(\int_{\mathbb{R}^n}|f(x)|^p\|x\|^\alpha\right)^{1/p}\mathrm{d}x \tag{1.4}$$

for appropriate values of α, β, p, q and r. We could prove inequalities of the form (1.4) only for a restricted range of α and β. Most of our results in this paper are valid only for $1 \leq p \leq 2$ and $r \leq p'$. It seems that these results are not best possible as far as the index r is concerned and perhaps a different technique is required to tackle the problem.

In this paper $L^p_\alpha(\mathbb{R}^n)$ will denote the space of measurable functions defined on $\mathbb{R}^n$ such that

$$\|f\|_{p,\alpha,n} = \left(\int_{\mathbb{R}^n}|f(x)|^p\|x\|^\alpha\mathrm{d}x\right)^{1/p} < \infty,$$

for $1 \leq p < \infty$. When $\alpha = 0$ then $L^p_\alpha(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $\|f\|_{p,\alpha,n} = \|f\|_{p,n}$. For a measurable function g defined on $S^{n-1} \times \mathbb{R}$ we will frequently use the following notation for mixed norms

$$\|g\|_{r,(q,\beta)} = \left(\int_{S^{n-1}}\left(\int_{\mathbb{R}}|g(\omega,t)|^q|t|^\beta\mathrm{d}t\right)^{r/q}\mathrm{d}\omega\right)^{1/r}.$$

We will also follow the standard practice of using the letter C for a constant whose value may change from one line to another. Occasionally the constant C will be suffixed to show its dependency on important parameters.

In §2 we will explain the relevant preliminaries and in §3 we will prove the main theorem.

2. Preliminaries

We first describe some well known properties of the Radon transform which we will be using. For the proofs of these results we refer the reader to [18] and [16]. For $f \in L^1(\mathbb{R}^n)$ we define the Fourier transform of f by

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x)\mathrm{e}^{-2\pi i x\cdot y}\mathrm{d}x, \qquad y \in \mathbb{R}^n. \tag{2.5}$$

The n-dimensional Fourier transform and the Radon transform are related by the so-called slice projection theorem

$$\widehat{Rf}(\omega, \lambda) = C_n \hat{f}(\lambda\omega), \tag{2.6}$$

for each fixed $\omega \in S^{n-1}$ and for all $f \in L^1(\mathbb{R}^n)$. Here

$$\widehat{Rf}(\omega, \lambda) = \int_{\mathbb{R}} Rf(\omega, t)\mathrm{e}^{-2\pi i\lambda t}\mathrm{d}t$$

is the one-dimensional Fourier transform in the t variable.

If f is a radial function then it follows easily from (1.2) that Rf is independent of $\omega \in S^{n-1}$. Using radiality of f and (1.2) one obtains the following explicit description of the Radon transform

$$Rf(\omega, t) = \int_{\mathbb{R}^{n-1}} f\left(\sqrt{t^2 + \|y'\|^2}\right)\mathrm{d}y' = C_n \int_{|t|}^{\infty} f(r)(r^2 - |t|^2)^{\frac{n-3}{2}} r\mathrm{d}r, \tag{2.7}$$

where $c_n = |S^{n-2}|$ and $f(r)$ stands for $f(re_n)$ with $e_n = (0, 0, \ldots, 0, 1)$. In particular, if $g(x) = \|x\|^\beta$ then

$$\begin{aligned} Rg(\omega, t) &= C \int_{|t|}^{\infty} r^\beta (r^2 - |t|^2)^{\frac{n-3}{2}} r\mathrm{d}r \\ &= C|t|^{\beta+n-1} \int_1^{\infty} u^{\beta+1}(u^2 - 1)^{\frac{n-3}{2}}\mathrm{d}u = C|t|^{\beta+n-1}, \end{aligned} \tag{2.8}$$

provided $\beta + n - 1 < 0$. Note that $g \notin L^p(\mathbb{R}^n)$ for any p, $1 \le p \le \infty$.

To proceed further we need the notion of spherical harmonics. We recall that the spherical harmonics on S^{n-1} are the restrictions to S^{n-1} of homogeneous harmonic polynomials defined on $\mathbb{R}^n$. Let $\{Y_{lm}\colon l = 0, 1, \ldots, m = 1, 2, \ldots, N(l)\}$ be an orthonormal basis of $L^2(S^{n-1})$ consisting of spherical harmonics. Here Y_{lm} is homogeneous of degree l. Now suppose that f is a function which can be written in polar coordinates as $f(r\omega) = g(r)Y_{lm}(\omega)$ where g is a function on $(0, \infty)$. By Hecke–Bochner identity one knows that $\hat{f}(\lambda\omega) = G(\lambda)Y_{lm}(\omega)$ (see p. 151 of [26]). From (2.6) it then follows that $Rf(\omega, t) = R_l g(t)Y_{lm}(\omega)$ for some function $R_l g$ defined on $(0, \infty)$. The following theorem, which can be considered as an analogue of the Hecke–Bochner identity for the Radon transform, gives an explicit expression for the function $R_l g$.

Theorem 2.1. *Let $f(x) = g(|x|)Y_{lm}(\omega)$ where Y_{lm} is a spherical harmonic of degree l and $g(r) = r^l v(r^2)$ where v is a compactly supported, smooth, radial function on $\mathbb{R}^n$. Then $Rf(\omega, t) = R_l g(t)Y_{lm}(\omega)$ where*

$$R_l g(t) = \frac{|S^{n-1}|}{G_l^{\left(\frac{n}{2}-1\right)}(1)} \left(\int_{|t|}^{\infty} G_l^{\left(\frac{n}{2}-1\right)}(s/r)g(r)\,(1-(s/r)^2)^{\frac{n-3}{2}}\mathrm{d}r \right). \tag{2.9}$$

Here $G_l^{\left(\frac{n}{2}-1\right)}$ denotes the Gegenbauer polynomials which form an orthonormal basis for the Hilbert space $L^2([-1,1], (1-t^2)^{(n-3)/2}\mathrm{d}t)$.

For proof of the above theorem we refer the reader to p. 71 of [17]. We will also need the following inequality for Gegenbauer polynomials (see Theorem 7.33.1 of [27])

$$|G_l^{\lambda}(t)/G_l^{\lambda}(1)| \leq 1, \qquad \text{for all } t \in [0,1]. \tag{2.10}$$

One important tool regarding reconstruction of a function from its Radon transform is the formal adjoint of R which is denoted by R^* and is defined by

$$R^*g(x) = \int_{S^{n-1}} g(\omega, x\cdot\omega)\mathrm{d}\omega, \quad x \in \mathbb{R}^n, \tag{2.11}$$

where g is a function defined on the set of all affine hyperplanes. If g is independent of ω and depends on $|t|$ alone then one has a simple formula for R^*g (see p. 159 of [18]),

$$R^*g(x) = C|x|^{2-n} \int_0^{|x|} g(s)(|x|^2 - s^2)^{\frac{n-3}{2}}\mathrm{d}s. \tag{2.12}$$

The following result regarding R^* is important for us (see p. 157 of [18]).

Lemma 2.2. *Let $f \in L^1(\mathbb{R}^n)$ or f be a nonnegative measurable function on $\mathbb{R}^n$ and F be a nonnegative measurable function on $S^{n-1}\times\mathbb{R}$. Then*

$$\int_{\mathbb{R}^n} f(x)R^*F(x)\mathrm{d}x = \int_{S^{n-1}\times\mathbb{R}} Rf(\omega,t)F(\omega,t)\mathrm{d}\omega\mathrm{d}t. \tag{2.13}$$

To prove weighted inequalities for the Radon transform we will need certain classical weighted inequalities. We state them here for readers benefit. The first one is Pitt's inequality.

Theorem 2.3. *If $1 < p \leq q < \infty$, $0 \leq \alpha < n(p-1)$, $0 \leq -\beta < n$ and $n = \frac{\beta+n}{q} + \frac{\alpha+n}{p}$ then there exists a positive constant C such that for all $f \in C_c^{\infty}(\mathbb{R}^n)$,*

$$\|\hat{f}\|_{q,\beta,n} \leq C\|f\|_{p,\alpha,n}. \tag{2.14}$$

For proof of the above theorem we refer the readers to [1], [22] and p. 569 of [11].

For $0 < \lambda < n$, let I_λ denote the Riesz potential of order λ, that is,

$$I_\lambda f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{\|x-y\|^{n-\lambda}}\mathrm{d}y.$$

We will need a weighted version of the Hardy–Littlewood–Sobolev lemma. In this direction the following theorem was proved in [25] (see also [1], [28]).

Theorem 2.4. *If* $0 < \lambda < n$, $1 < p \leq q < \infty$ *and* $\beta + n > 0$ *then there exists a constant* C *independent of* f *such that for all* $f \in C_c^\infty(\mathbb{R}^n)$ *the inequality*

$$\|I_\lambda f\|_{q,\beta,n} \leq C\|f\|_{p,\alpha,n}, \tag{2.15}$$

holds, provided $\alpha < n(p-1)$, $\frac{\alpha}{p} \geq \frac{\beta}{q}$ *and* $\frac{\alpha+n}{p} = \frac{\beta+n}{q} + \lambda$.

3. Main theorem

We first search for the necessary relation on p, q, α, β using dilation of a function. For $\lambda > 0$ we define $f_\lambda(x) = f(\lambda x)$ and hence

$$\|f_\lambda\|_{p,\alpha,n} = \lambda^{-(\alpha+n)/p}\|f\|_{p,\alpha,n}.$$

It is easy to see that $R(f_\lambda)(t,\omega) = \lambda^{-n+1}Rf(\lambda t,\omega)$ and hence it follows from (1.2) that

$$\|R(f_\lambda)\|_{r,(q,\beta)} = \lambda^{-(n-1+\frac{\beta+1}{q})}\|Rf\|_{r,(q,\beta)},$$

$1 \leq r \leq \infty$. So for the validity of (1.4) we must have the following relation between α, β, p, and q,

$$\frac{\alpha+n}{p} = n - 1 + \frac{\beta+1}{q}. \tag{3.16}$$

By modifying an example available in [19] we will show below that for $p > (\alpha+n)/(n-1)$ and $p = (\alpha+n)/(n-1) > 1$ the integral in (1.2) is infinite for large values of $|t|$. Hence a necessary condition for (1.4) is

$$1 \leq p < \frac{\alpha+n}{n-1}, \quad \text{or} \quad p = \frac{\alpha+n}{n-1} = 1. \tag{3.17}$$

Given $\gamma > 0$ we define

$$f(x) = \frac{1}{\|x\|^{n-1}(\log\|x\|)^\gamma}\chi_{\{x:\ \|x\|>e\}}(x), \qquad x \in \mathbb{R}^n.$$

We will first show that $f \in L_\alpha^p(\mathbb{R}^n)$ if $p > (\alpha+n)/(n-1)$ and $f \in L_\alpha^{(\alpha+n)/(n-1)}(\mathbb{R}^n)$ if $\gamma > (n-1)/(\alpha+n)$. Using the change of variable $u = \log\|x\|$ it follows that

$$\int_{\mathbb{R}^n} |f(x)|^p\|x\|^\alpha \mathrm{d}x = C\int_1^\infty \mathrm{e}^{u(\alpha+n-p+np)}u^{-\gamma p}\mathrm{d}u.$$

The above integral is certainly finite if $p > (\alpha+n)/(n-1)$. If $p = (\alpha+n)/(n-1)$ then we need the condition $\gamma p = \gamma(\alpha+n)/(n-1) > 1$ for finiteness of the above integral. Now we will show that Rf takes the value infinity on a set of positive measure. Since Rf is a radial function in the following we are going to treat it as a function on $(0,\infty)$. Now suppose that $n = 2$. By writing $r = \mathrm{e}^z$ and $|t| = \mathrm{e}^y$ it follows from (2.7) that for $y > 1$,

$$Rf(\mathrm{e}^y) = C\int_y^\infty (1-\mathrm{e}^{2(y-z)})^{-\frac{1}{2}}z^{-\gamma}\mathrm{d}z \geq \int_y^\infty \frac{1}{z^\gamma}\mathrm{d}z,$$

which is infinite if $\gamma \leq 1$. For $n \geq 3$ and $|t| > e$ we use the description of the function f and (2.7) to get

$$Rf(t) \geq \int_{2t}^{\infty} \frac{1}{(\log s)^{\gamma}} \left(1 - \frac{t^2}{s^2}\right)^{\frac{n-3}{2}} \frac{\mathrm{d}s}{s} \geq (3/4)^{\frac{(n-3)}{2}} \int_{2t}^{\infty} \frac{1}{s(\log s)^{\gamma}} \mathrm{d}s,$$

which is infinite if $\gamma \leq 1$. Conditions (3.16) and (3.17) now imply the following restrictions on α, β,

$$\alpha \geq -1, \qquad \beta \geq -1. \tag{3.18}$$

We first prove a mixed norm estimate for the Radon transform analogous to (1.3) for weighted L^1 spaces. This result brings out the role of the weight considered.

Lemma 3.1. *There exists a positive constant C such that for all $f \in C_c^{\infty}(\mathbb{R}^n)$ the following estimates hold.*

(1) *If* $\alpha \geq 0$, *then*

$$\|Rf\|_{\infty,(1,\alpha)} \leq C\|f\|_{1,\alpha,n}. \tag{3.19}$$

(2) *If* $-1 < \alpha < 0$, *then*

$$\|Rf\|_{r,(1,\alpha)} \leq C\|f\|_{1,\alpha,n}, \quad 1 \leq r < -\frac{1}{\alpha}. \tag{3.20}$$

Proof. For a fixed $\omega \in S^{n-1}$ and $\alpha \geq 0$ we have from (1.2)

$$\begin{aligned}
\int_{\mathbb{R}} |Rf(\omega, t)||t|^{\alpha} \mathrm{d}t &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}^{n-1}} f(t\omega + \omega') \mathrm{d}\omega' \right| |t|^{\alpha} \mathrm{d}t \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |f(t\omega + \omega')||t|^{\alpha} \mathrm{d}t \mathrm{d}\omega' \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |f(t\omega + \omega')||t\omega + \omega'|^{\alpha} \mathrm{d}t \mathrm{d}\omega' \\
&= \int_{\mathbb{R}^n} |f(x)|\|x\|^{\alpha} \mathrm{d}x.
\end{aligned}$$

This proves (1). For the second part of the lemma we proceed as follows:

$$\begin{aligned}
\|Rf\|_{r,(1,\alpha)} &= \left(\int_{S^{n-1}} \left(\int_{\mathbb{R}} |Rf(\omega, t)||t|^{\alpha} \mathrm{d}t \right)^r \mathrm{d}\omega \right)^{1/r} \\
&= \left(\int_{S^{n-1}} \left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}^{n-1}} f(t\omega + y') \mathrm{d}y' \right| |t|^{\alpha} \mathrm{d}t \right)^r \mathrm{d}\omega \right)^{1/r} \\
&\leq \left(\int_{S^{n-1}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |f(t\omega + y')||(t\omega + y') \cdot \omega|^{\alpha} \mathrm{d}y' \mathrm{d}t \right)^r \mathrm{d}\omega \right)^{1/r}
\end{aligned}$$

$$= \left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} |f(z)||z \cdot \omega|^\alpha \mathrm{d}z \right)^r \mathrm{d}\omega \right)^{1/r}$$

$$= \left(\int_{S^{n-1}} \left(\int_{\mathbb{R}^n} |f(z)|\|z\|^\alpha |\omega_z \cdot \omega|^\alpha \mathrm{d}z \right)^r \mathrm{d}\omega \right)^{1/r}$$

$$\leq \int_{\mathbb{R}^n} \left(\int_{S^{n-1}} \left(|f(z)|\|z\|^\alpha |\omega_z \cdot \omega|^\alpha \right)^r \mathrm{d}\omega \right)^{1/r} \mathrm{d}z$$

$$= \int_{\mathbb{R}^n} |f(z)|\|z\|^\alpha \left(\int_{S^{n-1}} |\omega_z \cdot \omega|^{\alpha r} \mathrm{d}\omega \right)^{1/r} \mathrm{d}z, \tag{3.21}$$

where $\omega_z = \frac{z}{\|z\|}$. Using rotation invariance of the measure on S^{n-1} we can choose $\omega_z = e_1$. Using spherical polar decomposition of S^{n-1} we now have

$$\int_{S^{n-1}} |e_1 \cdot \omega|^{\alpha r} \mathrm{d}\omega$$

$$= C \int_0^\pi \ldots \int_0^\pi \int_0^{2\pi} |\cos\theta|^{\alpha r} \sin^{n-2}\theta \sin^{n-3}\theta_2 \ldots \sin\theta_{n-2} \mathrm{d}\theta \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n.$$

Now it is easy to observe that $\int_{S^{n-1}} |\omega_z \cdot \omega|^{\alpha r} \mathrm{d}\omega$ is independent of z and is finite if and only if $\alpha r + 1 > 0$. This completes the proof of (2).

Remark 3.2. Although the case (2) above looks weaker than the case (1) yet it is best possible. If we consider a right circular, solid and symmetric (with respect to the coordinate hyper-planes) cylinder with base as $n-1$ dimensional solid sphere of radius 1 and height $2s > 2$ then it can be shown that for large s, $\|f\|_{p,\alpha,n} \leq C_1 s^{\frac{\alpha+1}{p}}$. On the other hand, the left-hand side of (1.4) is greater than or equal to $C_2 s^{\frac{1}{r'}}$. Hence for the validity of (1.4) we need $\frac{1}{r'} \leq \frac{\alpha+1}{p}$.

We can see it easily for the case $n = 2$. We consider a rectangle of width 2 and height $2s (> 2)$ with center as origin and sides parallel to axes. Let $H_{\theta,t}$ denote a line at a distance t from the origin and with normal at an angle θ with X-axis. Let f denote the characteristic function of the rectangle as described above. It can be easily seen that for $\cot^{-1} s < \theta < \pi/2$ and $0 < t < s\sin\theta - \cos\theta$ we have $Rf(\theta, t) = 2\csc\theta$. This shows that for large s the left-hand side of (1.4) is bigger than $C_1 s^{\frac{1}{r'}}$. Since $\|f\|_{p,\alpha,2} \leq C_2 s^{\frac{\alpha+1}{p}}$ it follows that for the validity of (1.4) the index r should satisfy the condition $\frac{1}{r'} \leq \frac{\alpha+1}{p}$.

Now we turn towards a generalization of Theorem 1.2 which in particular answers our first question. We observe from (3.16) that if $p = q = 2$ then α and β satisfy the relation $\alpha = \beta + n - 1$. We first show that Theorem 1.2 can be deduced from Theorem 2.3 and (2.6) for all $n \geq 2$ and $-1 < \beta \leq 0$ as follows:

$$\begin{aligned} \|Rf\|_{2,(2,\beta)} &\leq C\|\widehat{Rf}\|_{2,(2,-\beta)} \\ &= C\|\hat{f}\|_{2,-\beta+1-n,n} \\ &\leq C\|f\|_{2,\beta-1+n,n}. \end{aligned}$$

However, if $\beta > 0$ then this method of proof cannot be applied. We will show that the proof given in [20] can be modified to prove the most general case.

Lemma 3.3. *If* $n \geq 2$, $\beta > -1$ *and* $\alpha = \beta + n - 1$ *then there exists a positive constant* C *such that for all* $f \in C_c^\infty(\mathbb{R}^n)$,

$$\|Rf\|_{2,(2,\beta)} \leq C\|f\|_{2,\alpha,n}. \tag{3.22}$$

Proof. It suffices to prove the result for functions of the form

$$f(r\omega) = g(r)Y_{lm}(\omega), \tag{3.23}$$

where $g \in C_c(R)$ satisfies the symmetry condition $g(-r) = (-1)^l g(r),\ r \in \mathbb{R}$ and Y_{lm} is some fixed element of some fixed orthonormal basis of spherical harmonics. This is because the above collection of functions form an orthonormal basis of $L^2(R^n, \|x\|^\alpha \mathrm{d}x)$. We use Theorem 2.1 to write

$$Rf(\omega, t) = (R_l g(t))\, Y_{lm}(\omega), \tag{3.24}$$

where

$$R_l g(t) = \frac{\omega_{n-1}}{C_l^{(\frac{n}{2}-1)}(1)} \left(\int_{|t|}^\infty C_l^{(\frac{n}{2}-1)}(t/r) g(r)\, (1-(t/r)^2)^{\frac{n-3}{2}}\, \mathrm{d}r \right). \tag{3.25}$$

Thus using (2.10) we get

$$\begin{aligned}
\|Rf\|^2_{2,(2,\beta)} &= C\|R_l g\|^2_{L^2((0,\infty),|t|^\beta \mathrm{d}t)} \\
&= C\int_0^\infty \left| \frac{\omega_{n-1}}{C_l^{\left(\frac{n}{2}-1\right)}(1)} \int_{|t|}^\infty C_l^{\left(\frac{n}{2}-1\right)}(t/r) g(r) r^{n-2} \left(1 - \frac{t^2}{r^2}\right)^{\frac{n-3}{2}} \mathrm{d}r \right|^2 t^\beta \mathrm{d}t \\
&\leq C\int_0^\infty \left(\int_0^\infty |g(r)| r^{\frac{\alpha+n}{2}} \left(\frac{t}{r}\right)^{\frac{\beta+1}{2}} \chi_{(0,1)}(t/r) \left(1 - \frac{t^2}{r^2}\right)^{\frac{n-3}{2}} \frac{\mathrm{d}r}{r} \right)^2 \frac{\mathrm{d}t}{t} \\
&= C\|F * H\|^2_{L^2((0,\infty),\mathrm{d}t/t)},
\end{aligned} \tag{3.26}$$

where $F(r) = |g(r)|r^{\frac{\alpha+n}{2}}$ and $H(r) = r^{\frac{\beta+1}{2}}(1-r^2)^{\frac{n-3}{2}}\chi_{(0,1)}(r)$. Now using Young's inequality on the group $(0, \infty)$ we get

$$\|Rf\|_{2,(2,\beta)} \leq C\|F\|_{L^2((0,\infty),\mathrm{d}t/t)} \|H\|_{L^1((0,\infty),\mathrm{d}t/t)}. \tag{3.27}$$

This completes the proof just by observing that $\|F\|_{L^2((0,\infty),\mathrm{d}t/t)} = C\|f\|_{2,\alpha,n}$ and

$$\|H\|_{L^1((0,\infty),\mathrm{d}t/t)} = \int_0^1 r^{\frac{\beta+1}{2}} (1-r^2)^{\frac{n-3}{2}} \frac{\mathrm{d}r}{r} < \infty,$$

as $\beta > -1$ and $n \geq 2$.

The following result now follows by applying an interpolation argument (see 5.8.6 in page 130 of [3]) involving (3.19) and (3.22).

COROLLARY 3.4

For $1 < p \leq 2$, *there exists a positive constant* C *such that for all* $f \in C_c^\infty(\mathbb{R}^n)$,

$$\|Rf\|_{p',(p,\beta)} \leq C\|f\|_{p,\alpha,n}$$

provided $\beta = \alpha - (n-1)(p-1) > 1 - p$.

Using duality and Lemma 3.3 we can now prove the following $L^p - L^p$ boundedness of R^*.

Lemma 3.5. *For* $1 \leq p \leq 2$, *there exists a positive constant* C *such that for all* $f \in C_c^\infty(S^{n-1} \times \mathbb{R})$,

$$\|R^* f\|_{p,\beta,n} \leq C\|f\|_{p,(p,\alpha)},$$

provided $\alpha + 1 = \beta + n < p$.

Proof. We will prove the case $p = 1$ and $p = 2$ and then use an interpolation argument. To prove the case $p = 2$ we will use Lemma 2.2. For a nice non negative function F defined on $S^{n-1} \times \mathbb{R}$ we have

$$\begin{aligned}
\|R^*F\|_{2,\beta_2,n} &= \sup_{\{\|G\|_{2,\beta_2,n} \leq 1\}} \left| \int_{\mathbb{R}^n} R^*F(x)G(x)\|x\|^{\beta_2}\mathrm{d}x \right| \\
&= \sup_{\{\|G\|_{2,\beta_2,n} \leq 1\}} \left| \int_{S^{n-1}\times\mathbb{R}} F(\omega,t)R(G\|\cdot\|^{\beta_2})(\omega,t)\mathrm{d}t\mathrm{d}\omega \right| \\
&= \sup_{\{\|G\|_{2,\beta_2,n} \leq 1\}} \left| \int_{S^{n-1}\times\mathbb{R}} F(\omega,t)R(G\|\cdot\|^{\beta_2})(\omega,t)|t|^{-\beta_2+1-n}|t|^{\beta_2+n-1}\mathrm{d}t\mathrm{d}\omega \right| \\
&\leq \sup_{\{\|G\|_{2,\beta_2,n} \leq 1\}} \|F\|_{2,(2,\beta_2+n-1)} \\
&\quad \times \left(\int_{S^{n-1}\times\mathbb{R}} |R(G\|\cdot\|^{\beta_2})(\omega,t)|t|^{-\beta_2+1-n}|^2|t|^{\beta_2+n-1}\mathrm{d}t\mathrm{d}\omega \right)^{1/2} \\
&= \sup_{\{\|G\|_{2,\beta_2,n} \leq 1\}} \|F\|_{2,(2,\beta_2+n-1)}\|R(G\|\cdot\|^{\beta_2})\|_{2,(2,-\beta_2+1-n)}.
\end{aligned} \tag{3.28}$$

If $\beta_2 + n < 2$ then using Lemma 3.3 it follows from (3.28) that

$$\begin{aligned}
\|R^*F\|_{2,\beta_2,n} &\leq \sup_{\{\|G\|_{2,\beta_2,n} \leq 1\}} \|F\|_{2,(2,\beta_2+n-1)}\|G\|\cdot\|^{\beta_2}\|_{2,-\beta_2,n} \\
&\leq \|F\|_{2,(2,\beta_2+n-1)}
\end{aligned} \tag{3.29}$$

Now for $p = 1$ and $\beta_1 + n < 1$ we use Lemma 2.2 and (2.8) to get

$$\begin{aligned}
\int_{\mathbb{R}^n} R^*F(x)\|x\|^{\beta_1}\mathrm{d}x &= \int_{S^{n-1}\times\mathbb{R}} F(t,\omega)R(\|\cdot\|^{\beta_1})(\omega,t)\mathrm{d}t\mathrm{d}\omega \\
&= C\int_{S^{n-1}\times\mathbb{R}} F(t,\omega)|t|^{\beta_1+n-1}\mathrm{d}t\mathrm{d}\omega.
\end{aligned} \tag{3.30}$$

This gives the result for $p = 1$. Now we use an interpolation theorem with change in measures (see Theorem 2.11 of [24]) to complete the result.

Next we will prove the following $L^p - L^p$ boundedness for Radon transform.

Theorem 3.6. *For $1 \leq p < \infty$ there exists a positive constant C such that for all $f \in C_c^\infty(\mathbb{R}^n)$,*

$$\|Rf\|_{p,(p,\beta)} \leq C\|f\|_{p,\alpha,n}$$

provided $\beta = \alpha - (n-1)(p-1) > -1$.

Proof. The case $1 \leq p \leq 2$ follows from Lemmas 3.1 and 3.3 and [24]. For the range $2 < p < \infty$ we will use duality. Let p' denote the conjugate index of p. We have

$$\begin{aligned}\|Rf\|_{p,(p,\beta)} &= \sup_{\{\|g\|_{p',(p',\beta)}\leq 1\}} \left|\int_{S^{n-1}\times\mathbb{R}} Rf(\omega,t)g(\omega,t)|t|^\beta \mathrm{d}t\mathrm{d}\omega\right| \\ &= \sup_{\{\|g\|_{p',(p',\beta)}\leq 1\}} \left|\int_{\mathbb{R}^n} f(x)R^*(g|\cdot|^\beta)(x)\mathrm{d}x\right| \end{aligned} \tag{3.31}$$

But

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)R^*(g|\cdot|^\beta)(x)\mathrm{d}x &= \int_{\mathbb{R}^n} f(x)R^*(g|\cdot|^\beta)(x)\|x\|^{-\alpha}\|x\|^\alpha \mathrm{d}x \\ &\leq \|f\|_{p,\alpha,n}\left(\int_{\mathbb{R}^n} |R^*(g|\cdot|^\beta)(x)\|x\|^{-\alpha}|^{p'}\|x\|^\alpha \mathrm{d}x\right)^{1/p'} \\ &= \|f\|_{p,\alpha,n}\left(\int_{\mathbb{R}^n} |R^*(g|\cdot|^\beta)(x)|^{p'}\|x\|^{\alpha(1-p')}\mathrm{d}x\right)^{1/p'}. \end{aligned} \tag{3.32}$$

By using Corollary 3.5 we get

$$\int_{\mathbb{R}^n} f(x)R^*(g|\cdot|^\beta)(x)\mathrm{d}x \leq \|f\|_{p,\alpha,n}\left(\int_{S^{n-1}\times\mathbb{R}} |g(\omega,t)|t|^\beta|^{p'}|t|^{\alpha(1-p')+n-1}\mathrm{d}x\right)^{1/p'} \tag{3.33}$$

provided $1 < p' < 2$ and $\alpha(1-p') + n < p'$. Since the last inequality about α and p' follow from the hypothesis, the result now follows by (3.31) and (3.33).

To prove our main result we first address the following question: given α, β what is the range of p for which (1.4) can hold? This can be answered by considering the Radon transform of radial functions. By the formula of Radon transform for radial functions (see (2.7)) the inequality (1.4) takes the following form

$$\left(\int_0^\infty \left(\left|\int_t^\infty f(r)(r^2-t^2)^{\frac{n-3}{2}} r\mathrm{d}r\right|\right)^q t^\beta \mathrm{d}t\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty |f(r)|^p r^{\alpha+n-1}\mathrm{d}r\right)^{1/p},$$

which can be easily seen to be equivalent to

$$\|F_\alpha * H_\alpha\|_{L^q((0,\infty),\mathrm{d}t/t)} \leq C\|F_\alpha\|_{L^p((0,\infty),\mathrm{d}t/t)}, \tag{3.34}$$

where $F_\alpha(t) = f(t)t^{(\alpha+n)/p}$, $H_\alpha(t) = t^{\frac{\alpha+n}{p}-n+1}(1-t^2)^{\frac{n-3}{2}}\chi_{(0,1)}(t)$ and the convolution is on the multiplicative group $(0, \infty)$. Since convolution operators are always L^p improving (see [15]) it follows from (3.34) that a necessary condition on p, q for (1.4) to hold is given by $p \leq q$. This together with (3.16) now implies that $p \geq 1 + (\alpha - \beta)/(n-1)$. This is a nontrivial restriction on p only when $\alpha > \beta$.

The method of proof of our main result closely follows the idea of [19]. We will first embed the Radon transform into an analytic family of operators R_z. Construction of the above family goes exactly as in [19]. For $z \in \mathbb{C}$ we consider the distribution $g_z(x) = |x|^z$, $x \in \mathbb{R}$. It is known that $z \to g_z$ can be continued analytically to a meromorphic distribution valued function on $\mathbb{C}$ with simple poles at $z = -1, -3, -5, \ldots$ (see [19]). Since $\Gamma(z)^{-1}$ has simple zeros at $z = 0, -1, -2, \ldots$ it follows that

$$h_z = g_z / \Gamma\left(\frac{z+1}{2}\right) \tag{3.35}$$

is an entire distribution-valued function of z. We define

$$R_z f(\omega, t) = \int_{\mathbb{R}} Rf(\omega, t-s) h_z(s) \mathrm{d}s, \qquad \omega \in S^{n-1}, t \in \mathbb{R}. \tag{3.36}$$

It is known that (see [19] and [13])

$$\widehat{h_z}(t) = 2^{z/2+1}\pi^{1/2}|t|^{-z-1}\Gamma(-z/2), \qquad t \in \mathbb{R}. \tag{3.37}$$

For fixed $\omega \in S^{n-1}$ if $\widehat{R_z f}(\omega, \cdot)$ denotes the Fourier transform in the t variable (in the sense of distribution) then (3.37) shows that R_{-1} is a constant multiple of R. Now we are in a position to state and prove our main result.

Theorem 3.7. *If* $-1 < \beta \leq 0$, $\beta(2-n) < \alpha < \beta + n - 1$ *and* $1 + (\alpha - \beta)/(n-1) \leq p < (\alpha + n)/(n-1)$ *then for all* $f \in C_c^\infty(\mathbb{R}^n)$ *we have*

$$\|Rf\|_{p',(q,\beta)} \leq C\|f\|_{p,\alpha,n} \tag{3.38}$$

provided $\frac{\alpha+n}{p} = \frac{\beta+1}{q} + n - 1$ *and* $1 < p \leq 2$.

Proof. We will first prove the inequality

$$\|R_z f\|_{p',(p',\beta)} \leq C\|f\|_{p,\alpha,n} \tag{3.39}$$

for the operator R_z for certain values of z. Here α, β are as given in the hypothesis. Using dilation we observe that for (3.39) to be true it is necessary that α, β must satisfy the relation

$$\frac{\alpha+n}{p} = \frac{\beta+1}{p'} + n + \mathrm{Re}\,(z). \tag{3.40}$$

This follows as $\|R_z f_\lambda\|_{p',(p',\beta)} = \lambda^{-\frac{\beta+1}{p'}-n-\mathrm{Re}\,(z)}\|R_z f_\lambda\|_{p',(p',\beta)}$. In (3.40) if we put $p = 2$ then z for

$$\mathrm{Re}\,(z) = x_2 = \frac{\alpha - \beta - n - 1}{2}$$

satisfies (3.40). Similarly, if $p = 1 + (\alpha - \beta)/(n-1)$ then $p' = 1 + (n-1)/(\alpha - \beta)$ and hence z for

$$\mathrm{Re}\,(z) = x_1 = \frac{-2\alpha - \alpha\beta + n\beta + \beta^2 + \beta}{\alpha - \beta + n - 1}$$

satisfies (3.40). We now fix α and β as in the hypothesis. We will show that (3.39) holds for $z = x_1 + iy$ and $z = x_2 + iy$. By using Theorem 2.3 we have

$$\begin{aligned}\|R_{x+iy} f\|_{p',(p',\beta)} &= \left(\int_{S^{n-1}\times\mathbb{R}} |R_{x+iy} f(\omega, t)|^{p'} |t|^{\beta} \mathrm{d}t\mathrm{d}\omega\right)^{1/p'} \\ &\leq C\left(\int_{S^{n-1}\times\mathbb{R}} |\widehat{R_{x+iy} f}(\omega, \cdot)(s)|^{p'} |s|^{\beta_1} \mathrm{d}s\mathrm{d}\omega\right)^{1/p'} \qquad (3.41)\end{aligned}$$

provided $-1 < \beta \leq 0$, $1 < p'$, $0 \leq \beta_1 < p' - 1$ where

$$\beta_1 = p' - 2 - \beta. \qquad (3.42)$$

Here the first two conditions are in the hypothesis. Using (3.42) it can be seen that the third condition is equivalent to the relation $-1 < \beta \leq p' - 2$ which is true as $p' \geq 2$ and $\beta \leq 0$. Using (2.6) it follows from (3.41) that

$$\begin{aligned}\|R_{x+iy} f\|_{p',(p',\beta)} &\leq C\left(\int_{S^{n-1}\times\mathbb{R}} |\widehat{Rf(\omega,}\cdot)(s)|s|^{-1-x-iy}|^{p'} |s|^{\beta_1} \mathrm{d}s\mathrm{d}\omega\right)^{1/p'} \\ &= C\left(\int_{\mathbb{R}^n} |\hat{f}(z)|^{p'} \|z\|^{-p'(1+x)+\beta_1+1-n} \mathrm{d}z\right)^{1/p'}.\end{aligned}$$

Applying Theorem 2.3 to the previous inequality it now follows that

$$\|R_{x+iy} f\|_{p',(p',\beta)} \leq C\|f\|_{p,\alpha,n} \qquad (3.43)$$

provided

(a) $\frac{\alpha+n}{p} + \frac{\beta_1+1}{p'} - (1+x) = n$,
(b) $1 < p \leq p'$,
(c) $0 < -p'(1+x) + \beta_1 + 1 \leq n$,
(d) $0 \leq \alpha < n(p-1)$.

Since $\alpha > 0$ and $\beta \leq 0$ it follows that $1+\alpha/n < 1+(\alpha-\beta)/(n-1) \leq p$. This implies (d). Using (3.42) it follows that (a) is equivalent to (3.40). Since $1 < p \leq 2$, (b) holds. Using (3.42), (c) is equivalent to the inequality

$$0 < -xp' - 1 - \beta \leq n \qquad (3.44)$$

which holds for $x = x_1$, $p = 1 + (\alpha - \beta)/(n-1)$ and $x = x_2$, $p = 2$. Now by analytic interpolation (3.39) holds for all $x_2 \leq x \leq x_1$. Since $0 < \alpha - \beta < n - 1$ it follows that $x_2 < -1 < x_1$. Now by substituting $z = -1$ in (3.39) we get

$$\|Rf\|_{p_2',(p_2',\beta)} \leq C\|f\|_{p_2,\alpha,n} \qquad (3.45)$$

where $p_2 = 1 + (\alpha+1)/(\beta+n)$. Observe that the condition $\alpha > \beta(2-n)$ implies that $\beta = \alpha - (n-1)(p_1-1) > 1 - p_1$ where $p_1 = 1 + (\alpha-\beta)/(n-1)$. Hence from Corollary 3.4 we have

$$\|Rf\|_{p_1',(p_1,\beta)} \leq C\|f\|_{p_1,\alpha,n}. \tag{3.46}$$

Now by using the interpolation of mixed norm spaces (see [4]) it follows from (3.45) and (3.46) that

$$\|Rf\|_{p',(q,\beta)} \leq C\|f\|_{p,\alpha,n}. \tag{3.47}$$

where $1 + \frac{\alpha-\beta}{n-1} \leq p \leq 1 + \frac{\alpha+1}{\beta+n}$ and $\frac{\alpha+n}{p} = \frac{\beta+1}{q} + n - 1$.

Now to deal with the case

$$1 + \frac{\alpha+1}{\beta+n} < p < \frac{\alpha+n}{n-1}, \quad p \leq 2, \tag{3.48}$$

we will use the fact that $R_x f(\omega,t) = C_x I_{x+1} Rf(\omega,\cdot)(t)$ or equivalently $Rf(\omega,t) = C_x I_{-(x+1)}(R_x f(\omega,\cdot))(t)$ where $I_{-(x+1)}$ is the one-dimensional Reisz potential of order $-(x+1)$ with $-2 < x < -1$. For a given p in the range (3.48), let

$$x_0 = \alpha - \frac{\alpha+n+1}{p'}. \tag{3.49}$$

A direct calculation shows that p satisfies (3.48) if and only if x_0 satisfies the inequalities

$$-\frac{\alpha+1}{\alpha+n} < x_0 + 1 < \frac{\beta(\alpha+1)}{\alpha+n+\beta+1}, \qquad \frac{\alpha-n+1}{2} \leq x_0 + 1. \tag{3.50}$$

Since $-1 < \beta < 0$ it is clear from (3.50) that $-1 < x_0 + 1 < 0$. We now fix an $\omega \in S^{n-1}$ and use Theorem 2.4 to get

$$\begin{aligned}\|Rf(\omega,\cdot)\|_{q,\beta,1} &= \|I_{-(x_0+1)}(R_{x_0} f(\omega,\cdot))\|_{q,\beta,1}\\ &\leq C\|R_{x_0} f(\omega,\cdot)\|_{p'}\end{aligned} \tag{3.51}$$

provided $1 < p' \leq q$ and $\frac{1}{p'} = \frac{\beta+1}{q} - x_0 - 1$. Note that if p is as given in (3.48) it follows from (3.16) that $p' \leq q$. If we substitute the value of x_0 (see (3.49)) in the previous equation we see that p, q satisfy (3.16). Now from (3.51) we have

$$\begin{aligned}\|Rf\|_{p',(q,\beta)} &= \left(\int_{S^{n-1}} \|Rf(\omega,\cdot)\|_{q,\beta,1}^{p'} \mathrm{d}\omega\right)^{1/p'}\\ &\leq C\|R_{x_0} f\|_{p',(p',0)}.\end{aligned}$$

To complete the proof we just need to prove that

$$\|R_{x_0} f\|_{p',(p',0)} \leq C\|f\|_{p,\alpha,n}. \tag{3.52}$$

Since (3.39) holds for all values of α, β as in the hypothesis we can in particular choose $\beta = 0$, that is,

$$\|R_z f\|_{p',(p',0)} \leq C\|f\|_{p,\alpha,n}, \tag{3.53}$$

provided $0 < \alpha < (n-1)$, $1+\alpha/(n-1) \leq p < (\alpha+n)/(n-1)$ and $\frac{\alpha+n}{p} = \frac{1}{p'}+n+\mathrm{Re}\,(z)$. So (3.52) now follows from (3.53) provided α, x_0, and p satisfy

$$0 < \alpha < n-1,\ 1+\frac{\alpha}{n-1} \leq p < \frac{\alpha+n}{n-1},\ \frac{\alpha+n}{p} = \frac{1}{p'}+n+x_0. \tag{3.54}$$

Since $\beta < 0$ the first condition follows immediately from the hypothesis on α. Since p is now given by (3.48), it satisfies the second condition. The third condition is obvious from (3.49). This completes the proof.

Remark 3.8. The following remarks are in order.

(i) Since S^{n-1} is a finite measure space one can easily see that the above result also gives the following norm inequality

$$\|Rf\|_{L^q(\mathbb{R}\times S^{n-1},|t|^\beta \mathrm{d}t\mathrm{d}\omega)} \leq \|f\|_{p,\alpha,n}$$

provided $1+\frac{\alpha-\beta}{n-1} \leq p \leq 1+\frac{\alpha+1}{\beta+n}$ and α, β as in Theorem 3.7.

(ii) It is not known to us whether the index p' appearing in (1.4) is best possible for a given α, β and p. But if we consider a right circular, solid and symmetric (with respect to the co-ordinate hyper-planes) cylinder with base as $n-1$ dimensional solid sphere of radius $s > 1$ and height 2 then it can be shown that for large s, $\|f\|_{p,\alpha,n} \leq C_1 s^{\frac{\alpha+n-1}{p}}$. On the other hand, it turns out that the left-hand side of (1.4) is greater than or equal to $C_2 s^{\frac{n-1}{r'}}$. It follows that the index r has to satisfy the following condition

$$\frac{\alpha+n-1}{p} \geq \frac{n-1}{r'}. \tag{3.55}$$

Note that, if $\alpha = 0$ then this is a necessary condition on r (see [19]). The proof of this fact for the case $n = 2$ is similar to the proof given in Remark 3.2. To illustrate the problem we consider the particular case $n = 2$, $\alpha = 1$, $\beta = 0$ and $p = 2$. Then according to (3.55) the expected range of r is given by $1 \leq r \leq \infty$. So far we know the result only in the range $1 \leq r \leq 2$. In the following we will show that r cannot take the value infinity. We will see that the following inequality

$$\sup_{\omega\in S^1}\left(\int_{\mathbb{R}} |Rf(\omega,t)|^2 \mathrm{d}t\right)^{1/2} \leq C\|f\|_{2,1,2}$$

does not hold. We first note that by using rotations the above inequality is equivalent to

$$\left(\int_{\mathbb{R}} |Rf(\omega,t)|^2 \mathrm{d}t\right)^{1/2} \leq C\|f\|_{2,1,2}, \tag{3.56}$$

for each fixed $\omega \in S^1$ and for all nice functions f. So we can choose $\omega = (1,0) \in S^1$. By choosing $f(x,y) = g(x)h(y)$ it now follows from (3.56) and (2.6) that h must satisfy the inequality

$$\|h\|_1 \leq C(\|h\|_2 + \|h\|_{2,1,1}).$$

By choosing $h(y) = (y \log y)^{-1}\chi_{(\gamma,\infty)}$, $\gamma > 1$ it follows that the above does not hold. By using (2.6) we can see that in this particular case the inequality (1.4) is equivalent to

$$\left(\int_{S^1}\left(\int_{\mathbb{R}}|\hat{f}(\lambda,\omega)|^2 d\lambda\right)^{r/2}\mathrm{d}\omega\right)^{1/r} \leq C\|f\|_{2,1,2}.$$

It is not known to us whether any such inequality can be true for any r with $2 < r < \infty$.

Acknowledgement

The authors would like to thank Parasar Mohanty for many clarifying discussions. The authors would also like to thank the referee for several useful comments.

References

[1] Beckner W, Pitt's inequality with sharp convolution estimates, *Proc. Am. Math. Soc.* **136** (2008) 1871–1885, MR2373619 (2009d:42025)

[2] Benedetto J J and Heinig H P, Weighted Fourier inequalities: new proofs and generalizations, *J. Fourier Anal. Appl.* **9** (2003) 1–37, MR1953070 (2004a:42007)

[3] Bergh J and Löfström J, Interpolation spaces: An introduction, Grundlehren der Mathematischen Wissenschaften, No. 223 (Berlin-New York: Springer-Verlag) (1976) MR0482275 (58 # 2349)

[4] Benedek A and Panzone R, The space L^p, with mixed norm, *Duke Math. J.* **28** (1961) 301–324, MR0126155 (23 # A3451)

[5] Christ M, Estimates for the k-plane transform, *Indiana Univ. Math. J.* **33** (1984) 891–910, MR0763948 (86k:44004)

[6] Coifman R R, Remarks on weak type inequalities for operators commuting with translations, *Bull. Am. Math. Soc.* **74** (1968) 710–714 MR0231132 (37 # 6687)

[7] Drury S W, An endpoint estimate for certain k-plane transforms, *Canad. Math. Bull.* **29** (1986) 96–101, MR0824891 (87i:44002)

[8] Drury S W, A survey of k-plane transform estimates, Commutative harmonic analysis (NY: Canton) (1987) pp. 43–55, Contemp. Math. 91 (Amer. Math. Soc., Providence, RI) (1989) MR1002587 (92b:44002)

[9] Duoandikoetxea J, Naibo V and Oruetxebarria O, k-plane transforms and related operators on radial functions, *Michigan Math. J.* **49** (2001) 265–276, MR1852303 (2002g:42017)

[10] Duoandikoetxea J and Oruetxebarria O, Mixed norm inequalities for directional operators associated to potentials, *Potential Anal.* **15(3)** (2001) 273–283, MR1837268 (2002c:47107)

[11] García-Cuerva J and Rubio de Francia J L, Weighted norm inequalities and related topics. North-Holland Mathematics Studies 116, Notas de Matemática 104, (Amsterdam: North-Holland Publishing Co.) (1985) MR0807149 (87d:42023)

[12] Grafakos L, Classical and modern Fourier analysis (2004) (NJ: Pearson Education, Inc.) MR2449250

[13] Gel'fand I M and Shilov G E, Generalized functions. Vol. I: Properties and operations, Translated by Eugene Saletan (New York-London: Academic Press) (1964) MR0166596 (29 # 3869)

[14] Gressman P T, Sharp L^p–L^q estimates for generalized k-plane transforms, *Adv. Math.* **214(1)** (2007) 344–365, MR2348034 (2008m:47047)

[15] Hörmander L, Estimates for translation invariant operators in L^p spaces, *Acta Math.* **104** (1960) 93–140, MR0121655 (22 # 12389)

[16] Helgason S, The Radon transform. Second edition. Progress in Mathematics, 5 (Boston, MA: Birkhäuser Boston Inc.) (1999) MR0573446 (83f:43012)

[17] Ludwig D, The Radon transform on Euclidean space, *Comm. Pure Appl. Math.* **19** (1966) 49–81, MR0172311 (30 # 2530)

[18] Markoe A, Analytic tomography, Encyclopedia of Mathematics and its Applications, **106** (Cambridge: Cambridge University Press) (2006) MR2220852 (2007c:44001)

[19] Oberlin D M and Stein E M, Mapping properties of the Radon transform, *Indiana Univ. Math. J.* **31** (1982) 641–650, MR0667786 (84a:44002)

[20] Quinto E T, Null spaces and ranges for the classical and spherical Radon transforms, *J. Math. Anal. Appl.* **90** (1982) 408–420, MR0680167 (85e:44004)

[21] Rubin B, Reconstruction of functions from their integrals over k-planes, *Israel J. Math.* **141** (2004) 93–117, MR2063027 (2005b:44004)

[22] Stein E M, Interpolation of linear operators, *Trans. Am. Math. Soc.* **83** (1956) 482–492, MR0082586 (18,575d)

[23] Strichartz R S, L^p estimates for Radon transforms in Euclidean and non-Euclidean spaces, *Duke Math. J.* **48** (1981) 699–727, MR0782573 (86k:43008)

[24] Stein E M and Weiss G, Interpolation of operators with change of measures, *Trans. Am. Math. Soc.* **87** (1958) 159–172, MR0092943 (19,1184d)

[25] Stein E M and Weiss G, Fractional integrals on n-dimensional Euclidean space, *J. Math. Mech.* **7** (1958) 503–14, MR0098285 (20 # 4746)

[26] Stein E M and Weiss G, Introduction to Fourier analysis on Euclidean spaces, Princeton Mathematical Series, No. 32 (Princeton, N.J.: Princeton University Press) (1971) MR0304972 (46 # 4102)

[27] Szegö G, Orthogonal polynomials, American Mathematical Society Colloquium Publications, vol. 23, revised ed. (American Mathematical Society, Providence, R.I.) (1959) MR0106295 (21 # 5029)

[28] Walsh T, On L^p estimates for integral transforms, *Trans. Am. Math. Soc.* **155** (1971) 195–215, MR0284880 (44 # 2104)

[29] Wolff T, A mixed norm estimate for the X-ray transform, *Rev. Mat. Iberoamericana* **14** (1998) 561–600, MR1681585 (2000j:44006)