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Power cocentralizing generalized derivations on prime rings

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Abstract. Let R be a prime ring, U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, L a non-central Lie ideal of R, H and G non-zero generalized derivations of R. Suppose that there exists an integer $n \geq 1$ such that $(H(u)u - uG(u))^n = 0$, for all $u \in L$, then one of the following holds: (1) there exists $c \in U$ such that $H(x) = xc$, $G(x) = cx$; (2) R satisfies the standard identity s_4 and $\mathrm{char}(R) = 2$; (3) R satisfies s_4 and there exist $a, b, c \in U$, such that $H(x) = ax + xc$, $G(x) = cx + xb$ and $(a - b)^n = 0$.

Keywords. Prime rings; differential identities; generalized derivations.

1. Introduction

Let R be a prime ring with center $Z(R)$ and extended centroid C. Many results in literature indicate that the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R. A well-known result of Posner [13] states that if d is a derivation of R such that $[d(x), x] \in Z(R)$, for any $x \in R$, then either $d = 0$ or R is commutative. Later in [2], Bresar proves that if d and δ are derivations of R such that $d(x)x - x\delta(x) \in Z(R)$, for all $x \in R$, then either $d = \delta = 0$ or R is commutative. In [9], Lee and Wong extended the Bresar's result to the Lie case. They prove that if $d(x)x - x\delta(x) \in Z(R)$, for all x in some non-central Lie ideal L of R then either $d = \delta = 0$ or R satisfies s_4, the standard identity of degree 4. These facts in a prime ring are natural tests which evidence that the set $\{d(u)u - u\delta(u), u \in L\}$ is rather large in R.

Here we will consider the same situation in case the derivations d and δ are replaced respectively by the generalized derivations H and G. More specifically an additive map $G\colon R \longrightarrow R$ is said to be a generalized derivation if there is a derivation d of R such that, for all $x, y \in R$, $G(xy) = G(x)y + xd(y)$. A significative example is a map of the form $G(x) = ax + xb$, for some $a, b \in R$; such generalized derivations are called inner. Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting (see for example [10]). Here our purpose is to prove the following theorem:

Theorem. *Let R be a prime ring, U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, L a non-central Lie ideal of R, H and G non-zero generalized derivations of R. Suppose that there exists an integer $n \geq 1$ such that $(H(u)u - uG(u))^n = 0$, for all $u \in L$, then one of the following holds*:

(1) *there exists* $c \in U$ *such that* $H(x) = xc$, $G(x) = cx$;
(2) R *satisfies the standard identity* s_4 *and* $\mathrm{char}(R) = 2$;
(3) R *satisfies* s_4 *and there exist* $a, b, c \in U$, $\alpha \in C$ *such that* $H(x) = ax + xc$, $G(x) = cx + xb$ *and* $(a - b)^n = 0$.

Before starting the proofs, we fix some well-known facts. In what follows let R be a non commutative prime ring, U its Utumi quotient ring and $C = Z(U)$ the center of U. We refer the reader to [1] for the definitions and the related properties of these objects. Moreover we denote by s_4 the standard polynomial in four non-commuting variables. In particular we make use of the following:

Fact 1. If I is a two-sided ideal of R, then R, I and U satisfies the same generalized polynomial identities with coefficients in U [3].

Fact 2. Every derivation d of R can be uniquely extended to a derivation of U (see Proposition 2.5.1 in [1]).

Fact 3. We denote by $\mathrm{Der}(U)$ the set of all derivations on U. By a derivation word we mean an additive map Δ of the form $\Delta = d_1 d_2 \dots d_m$, with each $d_i \in \mathrm{Der}(U)$. Then a differential polynomial is a generalized polynomial, with coefficents in U of the form $\Phi({}^{\Delta_j} x_i)$ involving noncommutative indeterminates x_i on which the derivations words Δ_j act as unary operations. The differential polynomial $\Phi({}^{\Delta_j} x_i)$ is said to be a differential identity on a subset T of U if it vanishes for any assignment of values from T to its indeterminates x_i.

Let D_{int} be the C-subspace of $\mathrm{Der}(U)$ consisting of all inner derivations on U and let d be a non-zero derivation on R. By Theorem 2 in [7] we have the following result (see also Theorem 1 in [11]): If $\Phi(x_1, \dots, x_n, {}^d x_1, \dots, {}^d x_n)$ is a differential identity on R, then one of the following holds:

(1) either $d \in D_{\mathrm{int}}$;
(2) or R satisfies the generalized polynomial identity $\Phi(x_1, \dots, x_n, y_1, \dots, y_n)$.

Fact 4. If I is a two-sided ideal of R, then R, I and U satisfies the same differential identities [11].

We refer the reader to Chapter 7 in [1] for a complete and detailed description of the theory of generalized polynomial identities involving derivations.

Fact 5. If one assumes that either R does not satisfy s_4 or $\mathrm{char}(R) \neq 2$, then there exists a non-zero two-sided ideal I of R such that $0 \neq [I, R] \subseteq L$. In particular, if R is a simple ring it follows that $[R, R] \subseteq L$.

This follows from pp. 4–5 in [6], Lemma 2 and Proposition 1 in [4] and Theorem 4 in [8].

2. The case of inner generalized derivations on prime rings

We dedicate this section to prove the theorem in case both the generalized derivations H and G are inner, that is there exist $b, c, p, q \in U$ such that $H(x) = bx + xc$ and $G(x) = px + xq$, for all $x \in R$.

In all that follows we suppose that if R satisfies s_4 then $\mathrm{char}(R) \neq 2$. In light of Fact 5 there exists a non-central ideal I of R such that $[I, I] \subseteq L$. This implies that $(b[r_1, r_2]^2 + [r_1, r_2](c - p)[r_1, r_2] - [r_1, r_2]^2 q)^n = 0$ for all $r_1, r_2 \in I$. Moreover by Fact 1, I and R satisfy the same generalized polynomial identities, thus $(b[r_1, r_2]^2 + [r_1, r_2](c - p)[r_1, r_2] - [r_1, r_2]^2 q)^n = 0$ for all $r_1, r_2 \in R$. Hence we assume that R satisfies the following generalized polynomial identity

$$P(x_1, x_2) = (b[x_1, x_2]^2 + [x_1, x_2](c - p)[x_1, x_2] - [x_1, x_2]^2 q)^n$$

and $P(x_1, x_2)$ is a generalized polynomial in the free product $U *_C C\{x_1, x_2\}$ of the C-algebra U and the free C-algebra $C\{x_1, x_2\}$.

We first prove the following:

Lemma 1. *If R does not satisfy any non trivial generalized polynomial identity then $q, b \in C$ and $b + c = p + q$.*

Proof. Let $T = U *_C C\{X\}$ be the free product over C of the C-algebra U and the free C-algebra $C\{X\}$, with X the set consisting of non-commuting indeterminates x_1, x_2.

For brevity, we write $P(X)$ instead of $P(x_1, x_2)$ and $f(X)$ instead of $[x_1, x_2]$.

Now consider the generalized polynomial $P(X) \in U *_C C\{X\}$. By our hypothesis, R does not satisfy any non-trivial generalized polynomial identity and

$$\begin{aligned} P(X) = {} & bf(X)^2(bf(X)^2 + f(X)(c - p)f(X) - f(X)^2 q)^{n-1} \\ & + (f(X)(c - p)f(X) - f(X)^2 q)(bf(X)^2 \\ & + f(X)(c - p)f(X) - f(X)^2 q)^{n-1} = 0 \in T. \end{aligned}$$

Suppose that $\{b, 1\}$ are linearly C-independent. By [3], it follows that

$$bf(X)^2(bf(X)^2 + f(X)(c - p)f(X) - f(X)^2 q)^{n-1} = 0 \in T$$

which means, again since R is not a GPI-ring,

$$(bf(X)^2 + f(X)(c - p)f(X) - f(X)^2 q)^{n-1} = 0 \in T.$$

Continuing this process we get

$$(bf(X)^2 + f(X)(c - p)f(X) - f(X)^2 q) = 0 \in T$$

and this implies $f(X)^2 = 0 \in T$, since $\{b, 1\}$ are linearly C-independent. Of course this is a contradiction. Therefore $\{b, 1\}$ must be linearly C-dependent, that is $b \in C$. A similar argument shows that suppose $\{q, 1\}$ are linearly C-independent, then

$$\begin{aligned} & (bf(X)^2 + f(X)(c - p)f(X) - f(X)^2 q)^{n-1}(bf(X)^2 + f(X)(c - p)f(X)) \\ & \quad + (bf(X)^2 + f(X)(c - p)f(X) - f(X)^2 q)^{n-1}(-f(X)^2 q) = 0 \in T \end{aligned}$$

and as a consequence

$$(bf(X)^2 + f(X)(c - p)f(X) - f(X)^2 q)^{n-1}(-f(X)^2 q) = 0 \in T.$$

As above $(bf(X)^2 + f(X)(c - p)f(X) - f(X)^2 q)^{n-1} = 0 \in T$ and continuing the process we finally have $(bf(X)^2 + f(X)(c - p)f(X) - f(X)^2 q) = 0 \in T$, which again implies the contradiction $f(X)^2 = 0 \in T$. Therefore $q \in C$, as b does. Hence R satisfies $f(X)(b - q + c - p)f(X))^n = 0 \in T$, which is a trivial generalized polynomial identity only in the case $b - q + c - p = 0$, as required. □

Lemma 2. *Let R be a dense ring of linear transformations over a vector space V over the field C.*

(1) *If* $\dim_C(V) \geq 3$ *then* $b, q \in C$ *and* $b + c = p + q$;
(2) *If* $\dim_C(V) = 2$ *then either* $\mathrm{char}(R) = 2$ *or* $c - p = \alpha \in C$ *and* $(b - q + \alpha)^n = 0$.

Proof.

(1) Let $\dim_C(V) \geq 3$.

Our first aim is to show that for any $v \in V$, v and vb are linearly C-dependent.

By contradiction let v, vb be C-independent. There exists $w \in V$ such that v, vb, w are linearly independent. By the density of R, there exist $x, y \in R$ such that

$$vx = 0, vy = w, vbx = v, vby = 0, wx = 0, wy = -vb.$$

By calculation we obtain:

$$v[x, y] = 0, vb[x, y] = w, w[x, y] = v.$$

Then

$$0 = v(b[x, y]^2 + [x, y](c - p)[x, y] - [x, y]^2 q)^n = v \neq 0$$

a contradiction. Thus $\{v, vb\}$ are linearly C-dependent, for all $v \in V$. In this case it is well-known that $b \in C$. Hence R satisfies

$$([x_1, x_2](c - p)[x_1, x_2] + [x_1, x_2]^2(b - q))^n.$$

Now suppose there exists $v \in V$ such that $\{v, v(c - p)\}$ are linearly C-independent. Also in this case there exists $w \in V$ such that v, $v(c - p)$, w are linearly independent. By the density of R, there exist $x, y \in R$ such that

$$vx = 0, vy = w, v(c - p)x = v, v(c - p)y = 0, wx = 0, wy = -v(c - p).$$

By calculation we obtain:

$$v[x, y] = 0, v(c - p)[x, y] = w, w[x, y] = v.$$

Then

$$0 = w([x, y](c - p)[x, y] + [x, y]^2(b - q))^n = w \neq 0,$$

a contradiction. Thus $\{v, v(c - p)\}$ are linearly C-dependent for all $v \in V$, and this means that $c - p \in C$. Therefore R satisfies $([x_1, x_2]^2(c - p + b - q))^n$. Finally assume that there exists $v \in V$ such that $\{v, v(c - p + b - q)\}$ are linearly C-independent. Let $w \in V$ such that v, $v(c - p + b - q)$, w are linearly independent. By the density of R, there exist $x, y \in R$ such that

$$vx = v, vy = w, v(c - p + b - q)x = v, v(c - p + b - q)y = 0,$$
$$wx = 0, wy = -v(c - p + b - q).$$

Then

$$v[x,y]=w,\quad v(c-p+b-q)[x,y]=w,\quad w[x,y]=v$$

and

$$0=v([x,y]^2(c-p+b-q))^n=v(c-p+b-q)\neq 0$$

a contradiction. It follows that $(c-p+b-q)\in C$ and $q\in C$. So R satisfies $[x_1,x_2]^{2n}(c-p+b-q)^n$, that is $(c-p+b-q)=0$, since R cannot be commutative in this case.

(2) Let $\dim_D(V)=2$.

In this case $U=M_2(C)$, the ring of 2×2 matrices over C. In particular the polynomial $[x_1,x_2]^2$ is central-valued on U, as in $M_2(C)$. Hence U satisfies $([x_1,x_2]^2(b-q)+[x_1,x_2](c-p)[x_1,x_2])^n$. Of course if $\mathrm{char}(R)=2$ we are done. Thus we assume that $\mathrm{char}(R)\neq 2$.

Here we denote $A=b-q=\sum a_{ij}e_{ij}$, $B=c-p=\sum b_{ij}e_{ij}$, where e_{ij} is the usual matrix unit with 1 in the (i,j) entry and zero elsewhere, and any a_{ij}, b_{ij} is an element of C.

Since the matrix $M=[x_1,x_2]^2(b-q)+[x_1,x_2](c-p)[x_1,x_2]$ is nilpotent in $M_2(C)$, it is not invertible and its determinant is zero. We use this condition a number of times to prove our result in this case.

For $[x_1,x_2]=[e_{12}+e_{21},e_{22}]=e_{12}-e_{21}$ we have

$$M=\begin{bmatrix}-a_{11}-b_{22} & -a_{12}+b_{21}\\ -a_{21}+b_{12} & -a_{22}-b_{11}\end{bmatrix}$$

and

$$0=\det(M)=(a_{11}+b_{22})(a_{22}+b_{11})-(b_{21}-a_{12})(b_{12}-a_{21}). \tag{1}$$

Again for $[x_1,x_2]=[e_{12}-e_{21},e_{22}]=e_{12}+e_{21}$ we have

$$M=\begin{bmatrix}a_{11}+b_{22} & a_{12}+b_{21}\\ a_{21}+b_{12} & a_{22}+b_{11}\end{bmatrix} \tag{2}$$

and

$$0=\det(M)=(a_{11}+b_{22})(a_{22}+b_{11})-(b_{21}+a_{12})(b_{12}+a_{21}). \tag{3}$$

By (1) and (3) we get $2(a_{12}b_{12}+a_{21}b_{21})=0$ and since $\mathrm{char}(R)\neq 2$ it follows that

$$a_{12}b_{12}+a_{21}b_{21}=0. \tag{4}$$

Let $[x_1,x_2]=[e_{12}+2e_{21},-e_{22}]=-e_{12}+2e_{21}$. Hence

$$M=\begin{bmatrix}-2a_{11}-2b_{22} & -2a_{12}+b_{21}\\ -2a_{21}+4b_{12} & -2a_{22}-2b_{11}\end{bmatrix}$$

and by calculation

$$0=\det(M)=2(a_{11}+b_{22})(a_{22}+b_{11})+(2a_{12}-b_{21})(2b_{12}-a_{21}). \tag{5}$$

By replacing (3) in (5) and using (4) we have

$$b_{12}(a_{12} - b_{21}) = 0. \tag{6}$$

Our aim is to prove that either $b_{12} = 0$ or $b_{21} = 0$. To do this, we suppose that both $b_{12} \neq 0$ and $b_{21} \neq 0$ and we will prove that this assumption leads to a contradiction.

By (6) we get $a_{12} = b_{21}$. By using this last in (4) it follows that $b_{21}(b_{12} + a_{21}) = 0$, that is $b_{12} = -a_{21}$. In light of this, the matrix in (2) is

$$M = \begin{bmatrix} a_{11} + b_{22} & 2b_{21} \\ 0 & a_{22} + b_{11} \end{bmatrix}$$

and since it is nilpotent it follows that $(a_{11}+b_{22})^n = (a_{22}+b_{11})^n = 0$, that is $a_{11}+b_{22} = a_{22} + b_{11} = 0$. Therefore, if both b_{12} and b_{21} are not zero, then

$$a_{12} - b_{21} = 0, \quad a_{21} + b_{12} = 0, \quad a_{11} + b_{22} = 0, \quad a_{22} + b_{11} = 0. \tag{7}$$

Let $\varphi(x) = (1 + e_{12})x(1 - e_{12})$ and $\chi(x) = (1 - e_{12})x(1 + e_{12})$ be inner automorphisms of R. Of course

$$\varphi(([x_1, x_2]^2(b - q) + [x_1, x_2](c - p)[x_1, x_2])^n) = 0$$

and

$$\chi(([x_1, x_2]^2(b - q) + [x_1, x_2](c - p)[x_1, x_2])^n) = 0$$

that is, the matrices $\varphi(A), \varphi(B), \chi(A), \chi(B)$ satisfy the same properties of A and B. Denote by $\varphi(A)_{ij}$ the (i, j)-entry of $\varphi(A)$, $\varphi(B)_{ij}$ the (i, j)-entry of $\varphi(b)$, $\chi(A)_{ij}$ the (i, j)-entry of $\chi(A)$ and $\chi(B)_{ij}$ the (i, j)-entry of $\chi(B)$.

Therefore $\varphi(B)_{21} = \varphi(A)_{12}$ and by (7) it follows that

$$a_{12} = b_{21} = \varphi(B)_{21} = \varphi(A)_{12} = a_{12} + a_{22} - a_{11} - a_{21}$$

that is,

$$a_{22} - a_{11} - a_{21} = 0. \tag{8}$$

On the other hand, $\chi(B)_{21} = \chi(A)_{12}$ and by (7) it follows that

$$a_{12} = b_{21} = \chi(B)_{21} = \chi(A)_{12} = a_{12} - a_{22} + a_{11} - a_{21},$$

that is,

$$-a_{22} + a_{11} - a_{21} = 0. \tag{9}$$

By (8) and (9) and since $\mathrm{char}(R) \neq 2$, we get the contradiction $b_{12} = -a_{21} = 0$.

This argument proves that one of b_{12} and b_{21} must be zero. Without loss of generality we may assume that $b_{12} = 0$. As above, consider the following inner automorphisms of R:

$$\varphi(x) = (1 + e_{12})x(1 - e_{12}) \qquad \text{and} \qquad \chi(x) = (1 - e_{12})x(1 + e_{12}).$$

If either $0 = \varphi(B)_{21} = b_{21}$ or $0 = \chi(B)_{21} = b_{21}$ then B is a diagonal matrix. On the other hand, if both $\varphi(B)_{21} = b_{21} \neq 0$ and $\chi(B)_{21} = b_{21} \neq 0$, then by the previous argument we have that $\varphi(B)_{12} = 0$ and also $\chi(B)_{12} = 0$. By calculation one has

$$0 = \varphi(B)_{12} = b_{22} - b_{11} - b_{21}$$

and

$$0 = \chi(B)_{12} = -b_{22} + b_{11} - b_{21}$$

and the last two equalities imply $b_{21} = 0$, since $\mathrm{char}(R) \neq 2$. Thus we may conclude that in any case B must be a diagonal matrix in $M_2(C)$. Using this, we may repeat the same above argument and consider the matrix $\varphi(B)$: it must be a diagonal one. Hence $0 = \varphi(B)_{12} = b_{22} - b_{11}$, which implies $b_{11} = b_{22} = \alpha \in C$. Therefore, if we denote by I_2 the identity matrix in $M_2(C)$, we have that $B = \alpha \cdot I_2$ is a central matrix in $M_2(C)$. Therefore R satisfies $([x_1, x_2]^2(b - q + \alpha))^n = 0$ and since $[x_1, x_2]^2$ is central-valued on $M_2(C)$, it follows that $(b - q + \alpha)^n = 0$ as required. □

PROPOSITION 1

Let R be a prime ring, and b, c, p, q elements of U such that $(b[r_1, r_2]^2 + [r_1, r_2](c - p)[r_1, r_2] - [r_1, r_2]^2 q)^n = 0$ for all $r_1, r_2 \in R$. Then one of the following holds:

(1) *$b, q \in C$ and $b + c = p + q$*;
(2) *R satisfies s_4 and* $\mathrm{char}(R) = 2$;
(3) *R satisfies s_4, $c - p = \alpha \in C$ and $(b - q + \alpha)^n = 0$.*

Proof. By Lemma 1 we may assume that R satisfies the non-trivial generalized polynomial identity

$$P(x_1, x_2) = (b[x_1, x_2]^2 + [x_1, x_2](c - p)[x_1, x_2] - [x_1, x_2]^2 q)^n.$$

Since U and R satisfy the same generalized polynomial identities with coefficients in U (see Fact 1), then $P(x_1, x_2)$ is also a generalized identity for U. Hence we may now suppose that U satisfies some non-trivial generalized polynomial identity. By [12] U is primitive having a non-zero Socle $\mathrm{Soc}(U)$ with C as the associated division ring and by Jacobson's theorem (p. 75 in [5]) U is isomorphic to a dense ring of linear transformations of some vector space V over C. Thus we may conclude by Lemma 2. □

3. The general case on prime rings

We consider now the more general situation and prove the main theorem of the paper. Let L be a non-central Lie ideal of R, H and G non-zero generalized derivations of R and suppose that there exists an integer $n \geq 1$ such that $(H(u)u - uG(u))^n = 0$, for all $u \in L$.

We suppose that if R satisfies s_4 then $\mathrm{char}(R) \neq 2$. Therefore, as in §1, by Fact 5 we may assume that there exists a non-zero ideal I of R such that

$$(H([r_1, r_2])[r_1, r_2] - [r_1, r_2]G([r_1, r_2]))^n = 0$$

for all $r_1, r_2 \in I$. Under these assumptions we have the following.

Theorem 1. *One of the following holds*:

(1) *there exists* $c \in U$ *such that* $H(x) = xc$, $G(x) = cx$;
(2) R *satisfies the standard identity* s_4 *and* $\mathrm{char}(R) = 2$;
(3) R *satisfies* s_4 *and there exist* $a', b', c' \in U$, *such that* $H(x) = a'x + xc'$, $G(x) = c'x + xb'$ *and* $(a' - b')^n = 0$.

Proof. By Theorem 3 in [10] every generalized derivation g on a dense right ideal of R can be uniquely extended to the Utumi quotient ring U of R, and thus we can think of any generalized derivation of R to be defined on the whole U and to be of the form $g(x) = bx + d(x)$ for some $b \in U$ and d a derivation on U. Thus we may assume that there exist $a, b \in U$ and d, δ derivations on U such that

$$H(x) = ax + d(x) \quad \text{and} \quad G(x) = bx + \delta(x).$$

Since I, R and U satisfy the same differential identities [11], without loss of generality, in order to prove our results we may assume that

$$(H([r_1, r_2])[r_1, r_2] - [r_1, r_2]G([r_1, r_2]))^n = 0$$

for all $r_1, r_2 \in U$. Hence U satisfies

$$((a[x_1, x_2] + d([x_1, x_2]))[x_1, x_2] - [x_1, x_2](b[x_1, x_2] + \delta([x_1, x_2]))^n,$$

that is,

$$\begin{aligned}((a[x_1, x_2] &+ [d(x_1), x_2] + [x_1, d(x_2)])[x_1, x_2] \\ &- [x_1, x_2](b[x_1, x_2] + [\delta(x_1), x_2] + [x_1, \delta(x_2)]))^n.\end{aligned} \tag{10}$$

Case 1. Let d be the inner derivation induced by $c \in U$ and δ the inner one induced by $q \in U$, that is $d(x) = [c, x]$ and $\delta(x) = [q, x]$, so that $H(x) = ax + [c, x]$ and $G(x) = bx + [q, x]$. In this last case, again by Fact 3 and (10), U satisfies

$$((a + c)[x_1, x_2]^2 - [x_1, x_2](c + b + q)[x_1, x_2] + [x_1, x_2]^2 q)^n. \tag{11}$$

By Lemma 1 and Proposition 1 we have:

- either $a + c, q \in C$, with $a + c + q = c + b + q$, that is $a = b$ (in this case we obtain the conclusion $H(x) = xa$ and $G(x) = ax$),
- or $U = M_2(C)$, $c + b + q = \alpha \in C$ and by (14) it follows that U satisfies

  $$((a - b)[x_1, x_2]^2)^n.$$

  Since $[x_1, x_2]^2$ is central valued on U, we have that $(a - b)^n = 0$, that is, $(a - \alpha + c + q)^n = 0$. In this case we have the conclusion $H(x) = a'x + xc'$, $G(x) = c'x + xq'$, where $a' = a + c$, $c' = -c$, $q' = \alpha - q$ and $(a' - q')^n = 0$.

Case 2. Let d and δ be C-linearly independent modulo D_{int}. In this case, by Fact 3 and (10), U satisfies the generalized polynomial identity

$$\begin{aligned}((a[x_1, x_2] &+ [x_3, x_2] + [x_1, x_4])[x_1, x_2] \\ &- [x_1, x_2](b[x_1, x_2] + [x_5, x_2] + [x_1, x_6]))^n,\end{aligned} \tag{12}$$

in particular, U satisfies the blended component $(a[x_1, x_2]^2 - [x_1, x_2]b[x_1, x_2])^n$. By Lemma 1 and Proposition 1, we have two subcases.

- either $a = b \in C$, then by (12) we have

$$(([x_3, x_2] + [x_1, x_4])[x_1, x_2] - [x_1, x_2]([x_5, x_2] + [x_1, x_6]))^n = 0$$

and for $x_3 = x_5 = x_6 = 0$, $x_2 = x_4$ we also have $[x_1, x_2]^{2n} = 0$ which implies that U is commutative, a contradiction,
- or $U = M_2(C)$ and $b \in C$. In this last case, by (12) we have

$$((a - b)[x_1, x_2]^2 + [x_3, x_2][x_1, x_2] + [x_1, x_4][x_1, x_2] \\ - [x_1, x_2][x_5, x_2] - [x_1, x_2][x_1, x_6])^n = 0$$

and for $x_1 = e_{11}$, $x_2 = x_6 = e_{22}$, $x_3 = e_{21}$, $x_4 = x_5 = e_{11}$ we have the contradiction $e_{22} = 0$.

Case 3. Let now $\alpha d + \beta\delta = ad(p)$, the inner derivation induced by the element $p \in U$.

In case $\alpha = 0$ then $\delta = ad(q)$ is the inner derivation induced by the element $q = \beta^{-1}p$. In the light of Case 1, we may assume that d is not an inner derivation of U. In this case, by Fact 3 and (10), U satisfies the generalized polynomial identity

$$((a[x_1, x_2] + [x_3, x_2] + [x_1, x_4])[x_1, x_2] \\ - [x_1, x_2](b[x_1, x_2] + q[x_1, x_2] - [x_1, x_2]q))^n, \tag{13}$$

in particular, U satisfies the blended component $(a[x_1, x_2]^2 - [x_1, x_2](b + q)[x_1, x_2] + [x_1, x_2]^2 q)^n$. By Lemma 1 and Proposition 1, we have two subcases:

- either $a, q \in C$, with $a + q = b + q$, that is $a = b \in C$. In this case by (13) U satisfies $(([x_2, x_3] + [x_1, x_4])[x_1, x_2])^n$, that is U is a PI-ring. Hence there exists a suitable field K such that U and the matrix ring $M_t(K)$ satisfies the same polynomial identities. For $t \geq 2$ and $x_1 = e_{11}$, $x_2 = e_{12}$, $x_3 = e_{22}$ and $x_4 = e_{21}$, we have the contradiction $(-e_{22})^n = 0$.
- or $U = M_2(C)$, $b + q \in C$ and by (13) it follows that

$$(a[x_1, x_2]^2 + [x_3, x_2][x_1, x_2] + [x_1, x_4][x_1, x_2] - [x_1, x_2]^2 b)^n = 0$$

and for $x_1 = e_{12}$, $x_2 = e_{22}$, $x_3 = e_{21}$ and $x_4 = e_{22}$, we have again the contradiction $(-e_{22})^n = 0$.

On the other hand, if $\beta = 0$, then $d = ad(c)$ is the inner derivation induced by the element $c = \alpha^{-1}p$ and again by Case 1 we may assume that δ is not an inner derivation of U. Hence by Fact 3 and (10), U satisfies

$$((a[x_1, x_2] + c[x_1, x_2] - [x_1, x_2]c)[x_1, x_2] \\ - [x_1, x_2](b[x_1, x_2] + [x_3, x_2] + [x_1, x_4]))^n. \tag{14}$$

By Lemma 1 and Proposition 1, we have

- either $a + c = b + c \in C$, with $a = b$. In this case by (14) U satisfies $(-[x_1, x_2][x_3, x_2] - [x_1, x_2][x_1, x_4])^n$, that is U is a PI-ring. Hence there exists a suitable field K such that U and the matrix ring $M_t(K)$ satisfies the same polynomial identities. For $t \geq 2$ and $x_1 = e_{11}$, $x_2 = e_{12}$, $x_3 = e_{11}$ and $x_4 = e_{21}$, we have the contradiction $(e_{22})^n = 0$,

- or $U = M_2(C)$, $b + c \in C$ and by (14) it follows as above $(-[x_1, x_2][x_3, x_2] - [x_1, x_2][x_1, x_4])^n = 0$ and we get again a contradiction.

Hence we now consider the case when both $\alpha \neq 0$ and $\beta \neq 0$. Therefore $\delta = \gamma d + ad(q)$, for $\gamma = -\alpha\beta^{-1} \neq 0$ and $ad(q)$ is the inner derivation induced by $q = \beta^{-1}p$. Moreover by Case 1 we again may assume that d is not an inner derivation in U. By Fact 3 and (10), U satisfies

$$((a[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)])[x_1, x_2] - [x_1, x_2](b[x_1, x_2] \\ + \gamma[d(x_1), x_2] + \gamma[x_1, d(x_2)] + [q, [x_1, x_2]]))^n$$

and so U satisfies the generalized polynomial identity

$$((a[x_1, x_2] + [y_1, x_2] + [x_1, y_2])[x_1, x_2] - [x_1, x_2](b[x_1, x_2] \\ + \gamma[y_1, x_2] + \gamma[x_1, y_2] + [q, [x_1, x_2]]))^n,$$

in particular, U satisfies the blended component $([y_1, x_2][x_1, x_2] - \gamma[x_1, x_2][y_1, x_2])^n$. For $y_1 = x_1$, U satisfies $(1-\gamma)^n[x_1, x_2]^{2n}$. This means that either $[x_1, x_2]^{2n}$ is an identity for R, or $\gamma = 1$. In the first case it is easy to see that U is commutative. In the second one, U satisfies the polynomial identity

$$([y_1, x_2][x_1, x_2] - [x_1, x_2][y_1, x_2])^n \tag{15}$$

As above, since U is a PI-ring, there exists a suitable field K such that U and the matrix ring $M_t(K)$ satisfies the same polynomial identities. For $t \geq 2$ and $y_1 = e_{12} - e_{21}$, $x_1 = e_{12}$, $x_2 = e_{22}$ in (15), we have the contradiction $(e_{22} - e_{11})^{2n} = 0$. Hence $t = 1$ and R must be commutative. □

As a reduction of the previous Theorem, we may also prove the following:

Theorem 2. *Let R be a prime ring of characteristic different from 2, U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, H and G non-zero generalized derivations of R. Suppose that there exists an integer $n \geq 1$ such that $(H(x)x - xG(x))^n = 0$, for all $x \in R$, then either R is commutative and $H = G$ or there exists $c' \in U$ such that $H(x) = xc'$, $G(x) = c'x$.*

Proof. Suppose first that R is not commutative. By Theorem 1, either there exists $c \in U$ such that $H(x) = xc$ and $G(x) = cx$, or R satisfies s_4 and there exist $a, b, c \in U$, such that $H(x) = ax + xc$, $G(x) = cx + xb$ and $(a - b)^n = 0$. In the first case we are done. Thus we assume the second one: hence $U = M_2(C)$ the 2×2 matrix ring over C. Since $H(x) = ax + xc$, $G(x) = cx + xb$, by the main assumption we have $0 = (H(x)x - xG(x))^n = (ax^2 - x^2b)^n$ for all $x \in R$. We denote $a = \sum a_{ij}e_{ij}$, $b = \sum b_{ij}e_{ij}$, where any a_{ij}, b_{ij} is an element of C.

Also here in order to prove our result, since the matrix $(ax^2 - x^2b)$ is not invertible in $M_2(C)$, we use the condition that its determinant is zero.

For $x = e_{11} = x^2$, it follows that

$$X = (ae_{11} - e_{11}b) = \begin{bmatrix} a_{11} - b_{11} & -b_{12} \\ a_{21} & 0 \end{bmatrix}$$

and

$$\det(X) = 0 \Longrightarrow a_{21}b_{12} = 0, \tag{16}$$

and from this

$$X^n = 0 \Longrightarrow (a_{11} - b_{11})^n = 0 \Longrightarrow a_{11} = b_{11} = \alpha. \tag{17}$$

Analogously for $x = e_{22} = x^2$ it follows

$$Y = (ae_{22} - e_{22}b) = \begin{bmatrix} 0 & a_{12} \\ -b_{21} & a_{22} - b_{22} \end{bmatrix}$$

and

$$\det(Y) = 0 \Longrightarrow a_{12}b_{21} = 0 \tag{18}$$

and from this

$$Y^n = 0 \Longrightarrow (a_{22} - b_{22})^n = 0 \Longrightarrow a_{22} = b_{22} = \beta. \tag{19}$$

Let $x = e_{22} + e_{21} = x^2$. It follows that

$$Z = (ae_{22} + ae_{21} - e_{22}b - e_{21}b) = \begin{bmatrix} a_{12} & a_{12} \\ \beta - \alpha - b_{21} & -b_{12} \end{bmatrix}$$

and using (17)–(19)

$$\det(Z) = 0 \Longrightarrow a_{12}(-\beta + \alpha - b_{12}) = 0. \tag{20}$$

Our first aim is to prove that if b is not a diagonal matrix then either a is a diagonal one or $a + b$ is a central matrix. To do this, we suppose that both b and a are not diagonal and then we will prove that $a + b$ is a central matrix. In particular assume $b_{12} \neq 0$. Thus by (16) we have $a_{21} = 0$. Moreover since we suppose that a is not diagonal, $a_{12} \neq 0$ and by (18) it follows $b_{21} = 0$.

Let $x = e_{11} + e_{21} = x^2$, by using (17) it follows that

$$M = (ae_{11} + ae_{21} - e_{11}b - e_{21}b) = \begin{bmatrix} a_{12} & -b_{12} \\ \beta - \alpha & -b_{12} \end{bmatrix}$$

and

$$\det(M) = 0 \Longrightarrow b_{12}(\beta - \alpha - a_{12}) = 0. \tag{21}$$

Since $a_{12} \neq 0$ and $b_{12} \neq 0$, by (20) and (21) we have

$$\begin{aligned} -b_{12} + \beta - \alpha &= 0, \\ a_{12} + \beta - \alpha &= 0, \end{aligned}$$

therefore $b_{12} = -a_{12} = \gamma$ and

$$a = \begin{bmatrix} \alpha & \gamma \\ 0 & \beta \end{bmatrix}, \qquad b = \begin{bmatrix} \alpha & -\gamma \\ 0 & \beta \end{bmatrix}.$$

This means that if b is not diagonal and also a is not diagonal, then $a + b$ is a diagonal matrix.

Let $\varphi(x) = (1 + e_{21})x(1 - e_{21})$ be an inner automorphism of $M_2(C)$. Of course,

$$\varphi(ax^2 - x^2b)^n = 0$$

and the matrices $\varphi(a)$, $\varphi(b)$ satisfy the same properties of a and b. Denote by $\varphi(a)_{ij}$ the (i, j)-entry of $\varphi(a)$, $\varphi(b)_{ij}$ the (i, j)-entry of $\varphi(b)$. Notice that $\varphi(a)_{12} = a_{12} \neq 0$ and $\varphi(b)_{12} = b_{12} \neq 0$, thus by the above assertion $\varphi(a + b)$ is a diagonal matrix. Remark that the (2, 1)-entry of $\varphi(a + b)$ must be zero, and it is $0 = 2\alpha - 2\beta$. Hence the matrix $a + b$ is central in $M_2(C)$, in other words, if b is not a diagonal matrix and a is not a diagonal one, then

$$a = \begin{bmatrix} \alpha & \gamma \\ 0 & \alpha \end{bmatrix}, \qquad b = \begin{bmatrix} \alpha & -\gamma \\ 0 & \alpha \end{bmatrix}.$$

In this situation, let $x = e_{11} + e_{21} + e_{12} + e_{22}$ so that $x^2 = 2x$. It follows that

$$N = (ax^2 - x^2b) = 2(ax - xa) = \begin{bmatrix} 2\beta & 4\beta \\ 0 & 2\beta \end{bmatrix}$$

and $\det(N) = 0$ implies $b_{12} = \beta = 0$ which is a contradiction.

This says that if b is not a diagonal matrix then a must be a diagonal one, say $a = a_{11}e_{11} + a_{22}e_{22}$.

Again assume $b_{12} \neq 0$. As above let $\varphi(x) = (1 + e_{21})x(1 - e_{21})$ be an inner automorphism of $M_2(C)$. Notice that $\varphi(b)_{12} = b_{12} \neq 0$, thus by the above assertion $\varphi(a)$ is a diagonal matrix. Remark that the (2, 1)-entry of $\varphi(a)$ must be zero, and it is $0 = a_{11} - a_{22}$. Hence the matrix a is central in $M_2(C)$, and if b is not a diagonal matrix then a is a central one, say $a = \alpha(e_{11} + e_{22})$. From (17) and (19), we get

$$a = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \qquad b = \begin{bmatrix} \alpha & b_{12} \\ b_{21} & \alpha \end{bmatrix}$$

and if $(a - b)^n = 0$, we get

$$\begin{bmatrix} 0 & -b_{12} \\ -b_{21} & 0 \end{bmatrix}^n = 0,$$

that is, $b_{12}b_{21} = 0$ and so $b_{21} = 0$. Again for $x = e_{22} + e_{21} = x^2$, it follows

$$T = (ae_{22} + ae_{21} - e_{22}b - e_{21}b) = \begin{bmatrix} 0 & 0 \\ 0 & -b_{12} \end{bmatrix}$$

and using $T^n = 0$ we have $b_{12}^n = 0$, which is again a contradiction.

Therefore b must be a diagonal matrix and analogously one may prove that a also must be a diagonal one and from (16) and (18), we get

$$a = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \qquad b = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}.$$

Of course, for $\varphi(x) = (1 + e_{12})x(1 - e_{12})$, $\varphi(a) = \varphi(b)$ is a diagonal matrix. In particular, the $(1, 2)$-entry of $\varphi(a)$ must be zero, and it is $0 = \beta - \alpha$. Hence $a = b$ is central in $M_2(C)$. Therefore $H(x) = x(c + a)$ and $G(x) = (c + a)x$ and we are done.

Consider finally the case when R is a commutative prime ring. This means that $U = C$ is a field. Since U does not contain any non-zero nilpotent element and any non-zero zero-divisor, by easy calculations we have that $(H - G)(x) = 0$ for all $x \in R$ and then $H = G$. $\square$

Remark 1. Recall that any H generalized derivation of R has the following form: $H(x) = ax + d(x)$ for a suitable $a \in U$ and d a derivation of R. We point out that in the case the conclusion $H(x) = xc$ occurs for some $c \in U$, one may easily conclude that d is an inner derivation of R induced by some element $q \in U$, $c = a$ and $[a, R] + d(R) = (0)$.

As consequences of the previous results, we also have the following:

Theorem 3. *Let R be a non-commutative prime ring of characteristic different from* 2, *U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, G a non-zero generalized derivation of R, L a non-central Lie ideal of R. Suppose that there exists an integer $n \geq 1$ such that $[G(x), x]^n = 0$, for all $x \in L$, then either there exists $c \in C$ such that $G(x) = cx$ or R satisfies s_4 and there are $c \in U$, $\alpha \in C$ such that $G(x) = cx + xc + \alpha x$, for all $x \in R$.*

Theorem 4. *Let R be a non-commutative prime ring of characteristic different from* 2, *U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, G a non-zero generalized derivation of R. Suppose that there exists an integer $n \geq 1$ such that $[G(x), x]^n = 0$, for all $x \in R$, then there exists $c \in C$ such that $G(x) = cx$.*

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