Proc. Indian Acad. Sci. (Math. Sci.) Vol. 120, No. 2, April 2010, pp. 141–148.

On s-semipermutable subgroups of finite groups and p-nilpotency

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MS received 5 October 2009; revised 13 November 2009

Abstract. A subgroup H of a group is said to be s-semipermutable in G if it is permutable with every Sylow p-subgroup of G with $(p, |H|) = 1$. Using the concept of s-semipermutable subgroups, some new characterizations of p-nilpotent groups are obtained and several results are generalized.

Keywords. s-Semipermutable subgroups; p-nilpotent groups; finite groups.

1. Introduction

All groups considered in this paper are finite groups. Most of the notations are standard and can be found in [4] and [3].

Recall that a group H is said to be s-permutable (s-quasinormal) [6] in G if $HP = PH$ for all Sylow subgroups P of G. A subgroup H of a group G is called s-semipermutable [2] in G if it is permutable with every Sylow p-subgroup of G with $(p, |H|) = 1$. Many authors have investigated the structure of a group under the assumption that some subgroups are well situated in the group. Buckly [1] proved that a group of odd order is supersolvable if every minimal subgroup of G is normal in the group. Srinivasan [10] showed that a group G is supersolvable if all maximal subgroups of every Sylow subgroup of G are normal. Later, several authors obtain the same conclusion if normality is replaced by some weaker normality (see [8]).

In this paper, we obtain some new characterizations of p-nilpotent groups and generalize several interesting results on the basis of s-semipermutability.

Let H be a subgroup of a group G. Then we say that H is weakly s-permutable [9] in G if G has a subnormal subgroup T such that $HT = G$ and $T \cap H \leq H_{sG}$, where H_{sG} denotes the subgroup of H generated by all those subgroups of H which are s-permutable in G. It is clear that every s-permutable subgroup is both s-semipermutable subgroup and weakly s-permutable subgroup. The following two examples show that, in general, a weakly s-permutable subgroup may not be a s-semipermutable subgroup and a s-semipermutable subgroup may not be a weakly s-permutable subgroup.

Example 1.1. Let $G = Z_5 \wr \langle a \rangle = K \rtimes \langle a \rangle$, where K is the base group of the regular wreath product G and the order of a is four. Clearly $\langle a^2 \rangle$ is s-semipermutable in G but is not weakly s-permutable in G.

Example 1.2. Let $G = S_4$, H is a subgroup of order 2 which is not contained in the maximal subgroup A_4 of G. Then H is a weakly s-permutable subgroup of G but not a s-semipermutable subgroup of G.

2. Preliminaries

Lemma 2.1 (*Lemma* 1, *Lemma* 2 *of* [2]). *Let H and K be subgroups of a group G with $H \leqslant K$.*

(1) *If H is s-semipermutable in G, then H is s-semipermutable in K.*
(2) *Suppose that H is a p-group for some prime p and N is normal in G. Then NH/N is s-semipermutable in G/N, if H is s-semipermutable in G.*
(3) *If N is normal in G, then the subgroup HN/N is s-semipermutable in G/N for every s-semipermutable subgroup H in G satisfying $(|H|, |N|) = 1$.*

Lemma 2.2 (*Lemma* 2 *of* [14]). *Let N be a solvable minimal normal subgroup of a group G such that N is not contained in the Frattini subgroup $\Phi(G)$ of G. If every maximal subgroup of N is s-semipermutable in G, then N is a cyclic subgroup of G with prime order.*

Lemma 2.3 (*Lemma* 3 *of* [14]). *Let H be a subnormal s-semipermutable subgroup of and H be a p-group for some prime p, then P is s-quasinormal in G.*

Lemma 2.4 (*Lemma* 2.2 *of* [7]). *Assume that H is s-quasinormal in a group G and H is a p-group for some prime p, then $O^p(G) \leq N_G(H)$.*

Lemma 2.5. *Let N be an elementary abelian normal subgroup of a group G. If there exists a subgroup D in N satisfying $1 < |D| < |N|$ and every subgroup H of N with $|H| = |D|$ is s-semipermutable in G, then there exists a maximal subgroup L of N which is normal in G.*

Proof. Let $\{L_1, L_2, \ldots, L_s\}$ be a set of all maximal subgroups of N such that L_i and L_j are not conjugate in G for some natural number i which is not equal to j. Since L_i is a product of some subgroup of N whose order is equal to $|D|$, L_i is s-quasinormal in G. It follows from Lemmas 2.3 and 2.4 that $O^p(G)$ is contained in $N_G(L_i)$ for every natural number i contained in $\{1, 2, \ldots, s\}$. Then we have that $|G\colon N_G(L_i)| = p^{\alpha_i}$, where α_i is a natural number. By Chapter III, Theorem 8.5(d) of [5], we know

$$\sum |G\colon N_G(L_i)| \equiv 1 (\mathrm{mod}\, p),$$

i.e., there exists some $t \in \{1, 2, \ldots, s\}$ satisfying $\alpha_t = 0$, and hence L_t is normal in G.

Lemma 2.6 (*Lemma* 2.20 *of* [9]). *Suppose that p be an old prime and Q be a p'-group of the automorphisms of the p-group P. If every subgroup of P with prime order is Q-invariant, then Q is a cyclic group.*

Lemma 2.7. *Let p be a prime dividing the order of a group G satisfying $(|G|, p - 1) = 1$, and P be a Sylow p-group of G. Suppose that every subgroup of p with order p or* 4 *(if $p = 2$) is s-semipermutable in G. Then G is a p-nilpotent group.*

Proof. Suppose that this theorem is false, and if G be a counterexample with smallest order. It is easy to see that the hypotheses of this theorem is subgroup inherited and hence G is a minimal non-p-nilpotent group (the group whose proper subgroups are p-nilpotent but G itself is not p-nilpotent). By Chapter III, Theorem 5.2 of [5], we know $G = PQ$, where P is a normal p-subgroup of G, Q is a q-subgroup of G such that $q \neq p$, and $\exp P \leq 4$ (where $p = 2$). Let $x \in P \backslash \Phi(P)$, then $o(x) = p$ or 4, and therefore $\langle x \rangle \Phi(P)/\Phi(P)$ is s-semipermutable in $G/\Phi(P)$ by the hypotheses. Thus $\langle x \rangle \Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$ by Lemmas 2.3 and 2.4. Since $P/\Phi(P)$ is a chief factor of $G/\Phi(P)$, we have $P = \langle x \rangle \Phi(P) = \langle x \rangle$. Thus G is a p-nilpotent group by [13], a contradiction.

The following lemma is due to Dr Wei Xianbiao.

Lemma 2.8. *Let P be a Sylow p-subgroup of a group G for some prime p and N be a normal subgroup of G with order p. If $N_G(P)$ and G/N are p-nilpotent groups, then G is also a p-nilpotent group.*

Proof. If $G' \cap N = 1$, then it follows from $(G/N)' = G'N/N \cong G'/G' \cap N = G'$ that G' and $N_G(P)$ are both p-nilpotent. Since $G'P$ is normal in G, we have $G = N_G(G'P) = O_{p'}(G')N_G(P)$ is p-nilpotent. So we may assume that $G' \cap N \neq 1$. The minimality of N implies that $N \leq G'$. Moreover we may assume that $N \nleq \Phi(G)$. Thus there exists a maximal subgroup M of G such that $G = MN$ and $M \cap N = 1$. Since $G/C_G(N)$ is isomorphic to some subgroups of Aut(N), we have $G' \leq C_G(N)$ and therefore $G' \cap M$ is normal in G. Thus $G' = G' \cap MN = N \times (G' \cap M)$ is p-nilpotent since both N and $M \cong G/N$ are p-nilpotent. It follows that G is a p-nilpotent group.

3. The main results

Theorem 3.1. *Let p be an odd prime dividing the order of a group G and P be a Sylow p-subgroup of G. Suppose that $N_G(P)$ is a p-nilpotent group and there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ is s-semipermutable in G. Then G is a p-nilpotent group.*

Proof. Assume that the theorem is false and let G be a counterexample with smallest order.

(1) $O_{p'}(G) = 1$. If $O_{p'}(G) \neq 1$, then every subgroup of $PO_{p'}(G)/O_{p'}(G)$ with order $|D|$ is s-semipermutable in $G/O_{p'}(G)$ which is p-nilpotent by the minimality of G, and hence G is p-nilpotent, a contradiction.

(2) If M is a proper subgroup of G with $P \leq M$, then M is p-nilpotent. It is trivial to see $N_M(P)$ is p-nilpotent. By Lemma 2.1, we have that M satisfies the hypotheses of our theorem. The minimality of G implies that M is a p-nilpotent group.

(3) $O_p(G) \neq 1$, $G/O_p(G)$ is a p-nilpotent group and $C_G(O_p(G)) \leq O_p(G)$. Indeed, since G is not p-nilpotent, there exists a non-trivial characteristic subgroup K of P by Corollary of [11], such that $N_G(K)$ is not a p-nilpotent group. Hence, we can choose a characteristic subgroup K of P such that $N_G(K)$ is not a p-nilpotent group, but for every characteristic subgroup L satisfying $K < L \leq P$, we have $N_G(L)$ is a p-nilpotent group. Now (2) implies $N_G(K) = G$, hence $KO_p(G) < P$.

Let $\bar{G} = G/O_p(G)$, $\bar{P} = P/O_p(G)$, $\bar{Z} = Z(J(\bar{P}))$ and $G_1/O_p(G) = N_{\bar{G}}(\bar{Z})$, where $J(\bar{P})$ is the Thompson subgroup of P. Then $\bar{Z}$ is not normal in $\bar{G}$, hence $N_G(P) \leq G_1 < G$. By (2), G_1 is a p-nilpotent group and therefore $N_{\bar{G}}(\bar{Z})$ is a p-nilpotent group. So $\bar{G}$ is a

p-nilpotent group by Glauberman–Thompson theorem. Moreover, we have G is a p-solvable group. Since $O_{p'}(G) = 1$ by (1), we have $C_G(O_p(G)) \leq O_p(G)$.

(4) $G = PQ$. Since $\bar{G}$ is p-nilpotent by (1), we may let $\bar{G} = \overline{PT}$, where $\bar{T}$ is the normal p-complement of $\bar{P}$ in $\bar{G}$. By Frattini argument, we have $\bar{G} = \overline{PT} = \bar{T}N_{\bar{G}}(\overline{Q_1})$, where $\overline{Q_1}$ is a Sylow q-subgroup of $\bar{T}$. Without loss of generalization, we may assume that $\bar{P} \leq N_{\bar{G}}(\overline{Q_1})$, since $N_{\bar{G}}(\overline{Q_1})$ normalizes $Z(\overline{Q_1})$, $\bar{P}Z(\overline{Q_1})$ is a subgroup of $N_{\bar{G}}(\overline{Q_1})$. Let G_2 be the inverse image of $\bar{P}Z(\overline{Q_1})$ in G. Then $G_2 = PQ$, where Q is a q-subgroup of G. If $G_2 < G$, then G_2 is a p-nilpotent group by (2) and hence $Q < C_G(O_p(G)) \leq O_p(G)$, a contradiction. Thus $G_2 = G = PQ$.

(5) $|D| > p$. Assume that $|D| = p$. Let $x \in O_p(G)$, where $o(x) = p$, then $\langle x \rangle$ is s-semipermutable in G by hypotheses. Applying Lemmas 2.3 and 2.4, we have $Q \leq N_G(\langle x \rangle)$. Since $\langle x \rangle$ is subnormal in G, if $Q = C_Q(\langle x \rangle)$ for all such element x, then Q acts on $\Omega_1(O_p(G))$ trivially. Hence $Q \leq C_G(O_p(G)) \leq O_p(G)$, a contradiction. Thus there exists an element $x \in P$ with order p satisfying $Q > C_Q(\langle x \rangle)$. Since $Q/C_G(x)$ is isomorphic to a subgroup of $\mathrm{Aut}(\langle x \rangle)$, we have $q|(p-1)$ and hence $q < p$. On the other hand, if $C_Q(\Omega_1(O_p(G))) > 1$, then $C_Q(\Omega_1(O_p(G)))$ acts on $O_p(G)$ triviality. Therefore $C_Q(\Omega_1(O_p(G))) \leq C_G(O_p(G)) \leq O_p(G)$, a contradiction. So $C_Q(\Omega_1(O_p(G))) = 1$ and hence we can assume $Q \leq \mathrm{Aut}(\Omega_1(O_p(G)))$. Now by Lemma 2.6, we have Q is cyclic and hence G is a q-nilpotent group. This leads to $G = N_G(P)$ is a p-nilpotent group, which contradicts the choice of G.

(6) Suppose that N is a minimal normal subgroup of G contained in $O_p(G)$, then N is the unique minimal normal subgroup of G contained in $O_p(G)$ and $\bar{G} = G/N$ is a p-nilpotent group.

Indeed, if $|N| = p$, it is easy to see that $N_{\bar{G}}(\bar{P}) = \overline{N_G(P)}$ is p-nilpotent. Hence by Lemma 2.1, $\bar{G}$ satisfies the hypotheses of our theorem. The choice of G implies that $\bar{G}$ is p-nilpotent and therefore G is p-nilpotent by Lemma 2.8, a contradiction. Hence we can assume $|N| > p$. By Lemma 2.5 we have $|D| \geq |N|$. If $|D| > |N|$, then $\bar{G}$ clearly satisfies the hypotheses of our theorem and hence $\bar{G}$ is a p-nilpotent group by the choice of G. If $|D| = |N|$, then we can take a subgroup S of P such that N is a maximal subgroup of S. Since N is not cyclic, there exists a maximal subgroup R of S such that $R \neq N$. By the hypotheses R is s-semipermutable in G, and hence $S = RN$ is also s-semipermutable in G. Therefore S/N is s-semipermutable in G/N by Lemma 2.1. This implies that $\bar{G}$ is a p-nilpotent group by the choice of G and Lemma 2.8. Moreover since the class of all p-nilpotent group is a saturated formation, we may assume that N is the unique minimal normal subgroup of G contained in $O_p(G)$.

(7) $|P : D| > p$. Assume that $|P : D| = p$. Then all maximal subgroups of P are s-semipermutable in G by hypotheses. Now we claim that G is a p-nilpotent group. Let N be a minimal normal subgroup of G contained in $O_p(G)$. Then by (4) and (6), we know N is contained in $O_p(G)$ but not contained in $\Phi(G)$. Hence there exists a maximal subgroup M of G such that $G = MN$ and $M \cap N = 1$. By Lemma 2.4 of [13], $O_p(G) \cap M$ is normal in G. Therefore we have $N = O_p(G)$. Obviously $P = N(P \cap M)$. Let $P^* = P \cap M$, then P^* is a Sylow p-subgroup of M. Let P_1 be a maximal subgroup of P such that $P^* \leq P_1$, then $P_1 Q = Q P_1$ by hypotheses. By (4) we may assume $M = P_1 Q$. Hence $P_1 = P^*$ and $|N| = p$. If $p < q$, then NQ is p-nilpotent and so $Q \leq N_G(O_p(G)) = O_p(G)$, a contradiction. If $p > q$, then $M \cong G/N = G/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$. This leads to M is cyclic. Hence Q is cyclic. Thus G is a q-nilpotent group. Applying the hypotheses, we have $G = N_G(P)$ is a p-nilpotent group, a contradiction.

(8) *Final contradiction*. Since G is solvable, there exists a normal maximal subgroup M of G such that $|G : M| = r$, where r is a prime dividing the order of G.

(i) If $r = p$, then $P \cap M$ is a Sylow p-subgroup in M. Assume that $N_G(P \cap M) < G$. Then $N_G(P \cap M)$ is a p-nilpotent group by (2), and hence $N_M(P \cap M)$ is also a p-nilpotent group. It follows from (7) that M satisfies the hypotheses of our theorem and hence M is a p-nilpotent group. Thus Q is normal in G. If $N_G(P \cap M) = G$, then $P \cap M = O_p(G)$ since $P \cap M$ is maximal in P. Again by Lemma 2.5, we know $|D| \geq |N|$ and therefore $|P : D| \leq |P : N| = p$, which contradicts (5).

(ii) If $r = q$, then P is contained in M. By (2), M is a p-nilpotent group. Let T be the p' Hall subgroup of M. Assume that $T \neq 1$, then we have $T \leq C_G(O_p(G)) \leq O_p(G)$, a contradiction. If $T = 1$, then $P = M$ is normal in G, and hence $G = N_G(P)$ is a p-nilpotent group, a contradiction too. This contradiction completes the proof of this theorem.

Remark. This theorem generalizes Theorem 3.1 of [12].

COROLLARY 3.1

Let N be a normal subgroup of a group G such that G/N is a p-nilpotent group, where p is an odd prime dividing the order of N. Suppose that $N_G(P)$ is a p-nilpotent group and there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with $|H| = |D|$ is s-semipermutable in G, where $P \in \mathrm{Syl}_p(N)$. Then G is a p-nilpotent group.

Proof. Obviously $N_N(P)$ is a p-nilpotent group and every subgroup H of P with $|H| < |D|$ is s-semipermutable in N. By Theorem 3.1, we have N is a p-nilpotent group. Let T be the Hall p'-subgroup of N, then T is normal in G. If $T \neq 1$, then every subgroup HT/T of PT/T with $|HT/T| = |D|$ is s-semipermutable in G/T by Lemma 2.1. Since $(G/T)/(N/T) \cong G/N$ is p-nilpotent, G/T is p-nilpotent by induction and hence G is p-nilpotent as desired. If $T = 1$, then $N = P$ by hypotheses and $G = N_G(P)$ is p-nilpotent.

The following example shows that in Theorem 3.1 and Corollary 3.1, the assumption that $N_G(P)$ is p-nilpotent is essential.

Example 3.1. Suppose that $n > 1$, p be an old prime and $G = \langle a, b | a^{P^n} = b^2 = 1, b^{-1} ab = a^{-1} \rangle$. Let $A = \langle a \rangle$ and $B = \langle b \rangle$, then $G = A \rtimes B$. Clearly every subgroup of A with order p^m is s-semipermutable in G, where $1 < m < n$, but G is not p-nilpotent.

However, if $p - 1$ is coprime to the order of G, then we can get the following result.

Theorem 3.2. *Let p be a priming dividing the order of a group G satisfying $(|G|, p-1) = 1$ and P be a Sylow p-group of G. Suppose there exists a nontrivial subgroup D of P such that $1 < |D| < |P|$ and every subgroup H with order $|D|$ and $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-semipermutable in G, then G is a p-nilpotent group.*

Proof. Assume that the theorem is not true, and let G be a counterexample with smallest order.

(1) $O_{p'}(G) = 1$. If $O_{p'}(G) \neq 1$, then every subgroup of $PO_{p'}(G)/O_{p'}(G)$ with order $|D|$ and $2|D|$ is s-semipermutable in $G/O_p(G)$ by Lemma 2.1. Hence $G/O_p(G)$ is a p-nilpotent group by the choice of G and therefore G is a p-nilpotent group, a contradiction.

(2) $O_p(G) \neq 1$. Indeed, assume that $O_p(G) = 1$. Let H be a subgroup of P with order $|D|$, and Q be a Sylow q-subgroup of G, where $q \neq p$. Let x be any element of G. Then by the hypotheses $HQ^x = Q^x H$. Obviously we have $HQ < G$. By Chapter VI, Lemma 4.10 of [5], we know G is not a nonabelian simple group. Let N be a minimal normal subgroup of G, then $N < G$ and N is direct of some nonabelian simple groups which are isomorphic to each other.

If $P \leq N$, then it is easy to see that N satisfies the hypotheses of our theorem and hence N is a p-nilpotent group, which contradicts (1). So we may assume that $P \nleq N$. Let $N_p = N \cap P$. If $NP < G$, then the hypotheses is still true for NP and hence NP is a p-nilpotent group, a contradiction too. Thus we may assume that $G = NP$. If $|D| < |N_p|$, then every subgroup of N with order $|D|$ and $2|D|$ (when $p = 2$) of N is s-semipermutable in G by Lemma 2.1, which implies that N is a p-nilpotent group. If $|D| \geq |N_p|$, then we may choose a subgroup H of P with order $|D|$ or $2|D|$ (when $p = 2$ and $|P : D| > 2$) such that $N_p \leq H$. Let N_q be any Sylow q-subgroup of N, where $q \neq p$, then N_q is also a Sylow q-subgroup of G. Let x be any element of G, then we have $HN_q^x = N_q^x H$ by the hypothesis and $HN_q^x \cap N = (H \cap N)N_q^x = N_p N_q^x$. Since $N_p N_q^x < N$, there exists a non-trivial subgroup L of N such that $N_p < L$ or $N_q^x < L$ by Chapter VI, Lemma 4.10 of [5]. Let $N = N_1 \times N_2 \times \cdots \times N_s$, where N_i and N_j are nonabelian simple groups, which is isomorphic to each other, and $i, j = 1, 2, \ldots, s$. Then L is direct of some N_i where $N_i \in \{N_1, N_2, \ldots, N_s\}$. Hence if $N_p < L$ or $N_q^x < L$, then we may obtain that $L = N$, a contradiction. Thus $O_p(G) \neq 1$.

(3) $|D| > p$ and $|D : P| > p$. Indeed, if $|D| = p$, then G is a p-nilpotent group by Lemma 2.7, a contradiction. Assume that $|P : D| = p$ and let N be a minimal normal subgroup of G contained in $O_p(G)$, then hypothesis is clearly true for G/N. Hence G/N is a p-nilpotent group. Since the class of all p-nilpotent groups is a saturated formation, we may assume that N is the unique minimal normal subgroup of G contained in $O_p(G)$, we have moreover $O_p(G) \cap \Phi(G) = 1$. Hence there exists a maximal subgroup M of G such that $G = MN$ and $M \cap N = 1$ since $G = O_p(G)M$ and $O_p(G) \cap M$ is normal in G by Lemma 2.4 of [13]. The minimality of N implies that $O_p(G) \cap M = 1$ and therefore $O_p(G) = N$ is a minimal normal subgroup of G.

Let T/N be the normal p'-Hall subgroup of G/N. Then exists a p'-Hall subgroup L of T such that $T = NL = O_p(G)L$ by Schur-Zassenhaus theorem. Now applying Frattini argument we have $G = NN_G(L)$. Let P^* be a Sylow p-subgroup of $N_G(L)$. Without lose of generalization, we may assume that $P^* < P$, hence there exists a subgroup H of P with order $|D|$ such that $P^* \leq H$. Since H is s-semipermutable in G, by the hypothesis, we have that $HL < G$. By Lemma 2.8 of [13], we have HL is normal in G since $|G : HL| = p$. Hence $G = NHL$ and $N \cap HL = 1$, which implies $HL \cong G/N$ is a p-nilpotent group. Thus L is normal in G, which contradicts (1).

(4) Suppose that N is a minimal normal subgroup of G contained in $O_p(G)$. Then G/N is a p-nilpotent group and $N = O_p(G)$.

Let N be a minimal normal subgroup of G contained in $O_p(G)$ and $|N| = p$. Then G/N obviously satisfies the hypothesis of our theorem. Hence G/N is a p-nilpotent group. Applying Lemma 2.8 of [13], we have that G is a p-nilpotent group, which contradicts the choice of G. So we may assume that $|N| > p$. If $|D| > |N|$, then it is easy to see that the hypothesis is still true for G/N, if $|D| = |N|$ and $|P : D| > p$, where $p = 2$. Let K/N be a subgroup of P/N with order 4 and $N < R < K$. Since K is not cyclic, K has a maximal subgroup R^* such that $K = RR^*$. By the hypothesis R and R^* are s-semipermutable in G, and hence K is s-semipermutable in G too. Thus the hypothesis is true for G/N. By the choice of G we know that G/N is a p-nilpotent group. Analogously, one can prove that G/N is a p-nilpotent group if $|D| = |N|$ and $p > 2$. Similar to (3), we can obtain that $O_p(G) = N$ is a minimal normal subgroup of G.

(5) *Final contradiction*. Since $G/O_p(G)$ is a p-nilpotent group, we have that G is solvable. Then G have a normal subgroup M such that $|G : M| = r$, where r is a p'-number or a power of p.

Suppose that r is a p'-number, then we have $P \leq M$. By Lemma 2.1 we know M is a p-nilpotent group. Let T be the Hall p'-subgroup of M, then $T \leq C_G(O_p(G)) \leq O_p(G)$. Hence (1) leads to $T = 1$, which implies that $P = M$ is normal in G. Now by Lemma 2.5 we know that $|D| \geq |N|$. Therefore $|D| = |P|$, which contradicts the choice of P.

If r is a power of p, we may assume that M is normal in G and $|G : M| = p$. Then $P \cap M$ is a Sylow subgroup of M and $P \cap M$ is a maximal subgroup of P. Since $|P : D| > p$, we know that M satisfies the hypothesis. Thus M is a p-nilpotent group. By the same reason as above, we obtain that M is a p-group, and hence G is also a p-group, a contradiction. The proof is complete.

Remark. This theorem generalizes Theorems 3.3 and 3.5 of [12]. By the above theorem, we immediately obtain the following corollary.

COROLLARY 3.2

Let G be a group. For every prime p dividing the order of G, if P has a subgroup D such that $1 < |D| < |P|$ where P is a Sylow p-subgroup of G and every subgroup H of P with order $|D|$ and $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-semipermutable in G, then G is a group with Sylow tower of supersolvable type.

In a similar way as Corollary 3.1, we have the following corollary.

COROLLARY 3.3

Let N be a normal subgroup of a group G such that G/N is a p-nilpotent group, where p be a prime dividing the order of N. Suppose that $(p - 1, |G|) = 1$ and there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with $|H| = |D|$ and $|H| = 2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is s-semipermutable in G, where $P \in \mathrm{Syl}_p(N)$, then G is a p-nilpotent group.

Acknowledgement

The author would like to thank the referee for helpful suggestions. This work is supported by the Scientific Research Foundation of CUIT (No: KYTZ201003).

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