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A note on two Camina's theorems on conjugacy class sizes

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Abstract. Let G be a finite group. We mainly investigate how certain arithmetical conditions on conjugacy class sizes of some elements of biprimary order of G influence the structure of G. Some known results are generalized.

Keywords. Conjugacy class sizes; p-nilpotent groups; solvable groups.

1. Introduction

It is well-known in finite group theory that there is a strong relation between properties such as solvability or p-nilpotency of a group and the sizes of its conjugacy classes. There exist several other known results studying the structure of a group under some arithmetical conditions on its conjugacy class sizes. For example, Camina [2] proved the following theorem: Let G be a group such that p^a is the highest power of the prime p which divides the index of an element of G. Assume that there is a p-element in G whose index is precisely p^a. Then G has a normal p-complement. Later, Camina and Camina [3] proved the following theorem: Let G be an A-group which has an element of index 2^a, where 2^a is the maximal power of 2 which divides the index of any element of G. Then G is solvable. In this note, we vary the former two results by replacing conditions for all conjugacy classes by conditions referring to only some conjugacy classes. Our main result is the following:

Theorem A. *Let G be a group which has a 2-element of index 2^a where 2^a is the maximal power of 2 which divides the index of any $\{2, q\}$-element of G, where $q \neq 2$ is an arbitrary prime dividing the order of G. Then G is 2-nilpotent. In particular, G is solvable.*

All groups considered in this note are finite. If G is a group, then x^G denotes the conjugacy class containing x, $|x^G|$ the size of x^G (following [1], we call $\mathrm{Ind}_G(x) = |x^G| = |G\colon C_G(x)|$, the index of x in G). The rest of our notation and terminology are standard. The reader may refer to [6].

2. Basic definitions and preliminary results

In this section, we give some lemmas which are useful for our main results.

Lemma 2.1 (*Lemma* 6 *of* [1]). *$O_p(G)$ contains every element in G whose order and index are powers of p.*

Lemma 2.2 (*Chap.* 5, *Theorem* 3.4 *of* [4]). *Let $A \times B$ be a group of automorphisms of the p-group P with A a p'-group and B a p-group. If A acts trivially on $C_P(B)$, then $A = 1$.*

Lemma 2.3 (*Theorem* 5 *of* [5]). *Let G be a finite group and p a prime divisor of $|G|$. Then there is in G no p'-element of prime power order whose index is divisible by p if and only if $G = P \times H$, where P is a Sylow p-subgroup of G.*

3. The proof of Theorem A

The proof of Theorem A. By the hypothesis we suppose that x is a 2-element of G such that $[G: C_G(x)] = 2^a$. By Lemma 2.1, it is easy to know that the normal closure of x will be a 2-group, say H. Let $Z = C_G(H)$. Now $[G: C_G(x)] = 2^a$, and so if $y \in C_G(x)$ and y has prime power order prime to 2, $[C_G(x): C_G(xy)]$ is prime to 2. For otherwise 2^{a+1} would divide the index of xy, contrary to the hypothesis. However, $C_G(xy) = C_G(x) \cap C_G(y)$, as x and y has coprime order and $[x, y] = 1$. As $[C_G(x): C_G(xy)]$ is prime to 2, we can assume that $C_G(xy)$ contains a Sylow 2-subgroup of $C_G(x)$, say P. Obviously $H \cap C_G(x) \leq H$. Since H is normal in G, we have that $H \cap C_G(x) \leq O_2(C_G(x)) \leq P$. Thus $H \cap C_G(x) \leq H \cap P$. As $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(y)$, we get that $P \leq C_G(y)$, it follows that $H \cap C_G(x) \leq H \cap C_G(y)$ or $C_H(x) \leq C_H(y)$. We can now use Lemma 2.2 to deduce that $C_H(y) = H$.

So H centralizes every element of prime power order prime to 2 in $C_G(x)$. We can conclude that H centralizes every element of order prime to 2 in $C_G(x)$. In fact, for any element of order prime to 2 in $C_G(x)$, we can write $z = z_1 z_2 \ldots z_s$, where z_i is a power of a prime distinct from to 2 and the z_i commute pairwise, and $z_i \in C_G(x)$. By the above paragraph it follows that H centralizes every element of order prime to 2 in $C_G(x)$.

As $[G: C_G(x)] = 2^a$, we can deduce that $[G: Z]$ is a power of 2. Now let w be any $2'$-element of prime power order in Z. By the previous argument, $[C_G(x): C_G(w) \cap C_G(x)]$ is prime to 2, but, as Z is a normal subgroup of $C_G(x)$, we have that $[Z: C_Z(w)]$ is prime to 2. Thus every $2'$-element of prime power order in Z has index in Z prime to 2 and so, by Lemma 2.3 we have that $Z = K \times P_1$, where K has order prime to 2 and P_1 is the Sylow 2-subgroup of Z. As $[G: Z]$ is a power of 2, K is a normal 2-complement of G. Our proof is now complete.

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