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Integral inequalities for self-reciprocal polynomials

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Abstract. Let $n \geq 1$ be an integer and let $\mathcal{P}_n$ be the class of polynomials P of degree at most n satisfying $z^n P(1/z) = P(z)$ for all $z \in \mathbf{C}$. Moreover, let r be an integer with $1 \leq r \leq n$. Then we have for all $P \in \mathcal{P}_n$:

$$\alpha_n(r) \int_0^{2\pi} |P(\mathrm{e}^{it})|^2 \mathrm{d}t \leq \int_0^{2\pi} |P^{(r)}(\mathrm{e}^{it})|^2 \mathrm{d}t \leq \beta_n(r) \int_0^{2\pi} |P(\mathrm{e}^{it})|^2 \mathrm{d}t$$

with the best possible factors

$$\alpha_n(r) = \begin{cases} \prod_{j=0}^{r-1} \left(\frac{n}{2} - j\right)^2, & \text{if } n \text{ is even,} \\ \frac{1}{2}\left[\prod_{j=0}^{r-1} \left(\frac{n+1}{2} - j\right)^2 + \prod_{j=0}^{r-1} \left(\frac{n-1}{2} - j\right)^2\right], & \text{if } n \text{ is odd,} \end{cases}$$

and

$$\beta_n(r) = \frac{1}{2} \prod_{j=0}^{r-1} (n-j)^2.$$

This refines and extends a result due to Aziz and Zargar (1997).

Keywords. Self-reciprocal polynomials; integral inequalities.

1. Introduction

Let $n \geq 1$ be an integer. A polynomial

$$P(z) = \sum_{k=0}^{n} a_k z^k \quad (a_k \in \mathbf{C};\ k = 0, 1, \ldots, n)$$

is called self-reciprocal of order n if P has a degree of at most n and if

$$z^n P(1/z) = P(z) \quad \text{for all } z \in \mathbf{C},$$

that is, the coefficients satisfy

$$a_k = a_{n-k} \quad \text{for } k = 0, 1, \ldots, n.$$

We denote the class of self-reciprocal polynomials of order n by $\mathcal{P}_n$.

The properties of self-reciprocal polynomials have been studied by several mathematicians. We refer to [1–5] and the references given therein. Here, we are concerned with the following interesting integral inequalities, which were published by Aziz and Zargar [2] in 1997.

PROPOSITION

Let $n \geq 1$ be an integer. For all $P \in \mathcal{P}_n$ we have

$$\frac{n^2}{4}\int_0^{2\pi} |P(e^{it})|^2 dt \leq \int_0^{2\pi} |P'(e^{it})|^2 dt \leq \frac{n^2}{2}\int_0^{2\pi} |P(e^{it})|^2 dt. \tag{1.1}$$

The sign of equality holds on the right-hand side if $P(z) = c(z^n + 1)$ $(c \in \mathbf{C})$ and on the left-hand side if $P(z) = cz^{n/2}$ $(c \in \mathbf{C})$, where n is even.

In view of these results it is natural to ask:

(i) Is it possible to replace in (1.1) the factor $n^2/4$ by a larger term, if n is odd?
(ii) What are the best possible bounds for the ratio $\int_0^{2\pi} |P^{(r)}(e^{it})|^2 dt / \int_0^{2\pi} |P(e^{it})|^2 dt$, where $r > 1$?

It is our aim to answer both questions. In §2 we present a lemma, which we need to establish our main result, given in Section 3.

2. Lemma

We define for $z \in \mathbf{C}$ and $r \in \mathbf{N}$:

$$p_r(z) = \prod_{j=0}^{r-1} (z - j)^2.$$

Throughout this paper, we maintain this notation. The following elementary lemma provides some inequalities for p_r.

Lemma. Let k, r, N, a and b be integers.

(i) *If $0 \leq k \leq N-1$ and $r \geq 1$, then*

$$2p_r(N) \leq p_r(k) + p_r(2N - k). \tag{2.1}$$

(ii) *If $0 \leq k \leq N$ and $r \geq 1$, then*

$$p_r(N) + p_r(N+1) \leq p_r(k) + p_r(2N + 1 - k). \tag{2.2}$$

(iii) *If $a \geq 0$, $b \geq 0$ and $r \geq 1$, then*

$$p_r(a) + p_r(b) \leq p_r(a + b). \tag{2.3}$$

Proof.

(i) If $N \leq r - 1$, then we have $p_r(N) = 0$. Hence, let $N \geq r$. We consider two cases.

Case 1. $r \leq k$.

Thus, $1 \leq r \leq k \leq N - 1$. Inequality (2.1) is equivalent to

$$2 \leq \prod_{j=0}^{r-1} (1 - x_j)^2 + \prod_{j=0}^{r-1} (1 + x_j)^2, \tag{2.4}$$

where $x_j = (N-k)/(N-j) \in (0,1)$ for $j = 0, 1, \ldots, r-1$. Let Q_r be the expression on the right-hand side of (2.4). We have

$$Q_1 = 2x_0^2 + 2 \geq 2.$$

And, if $Q_r \geq 2$, then

$$\begin{aligned} Q_{r+1} &= (1-x_r)^2 Q_r + 4x_r \prod_{j=0}^{r-1} (1+x_j)^2 \\ &\geq 2(1-x_r)^2 + 4x_r = 2 + 2x_r^2 \geq 2. \end{aligned}$$

By induction, we conclude that (2.4) holds for all $r \geq 1$.

Case 2. $k \leq r-1$.
Then we have $0 \leq k \leq r-1 \leq N-1$. This gives $p_r(k) = 0 < p_r(N)$, so that we obtain

$$\begin{aligned} \frac{p_r(k) + p_r(2N-k)}{p_r(N)} &= \prod_{j=0}^{r-1} \left(1 + \frac{N-k}{N-j}\right)^2 \\ &\geq \left(1 + \frac{N-k}{N-(r-1)}\right)^2 > \left(1 + \frac{1}{2}\right)^2 > 2. \end{aligned}$$

(ii) If $k = N$, then equality holds in (2.2). Let $k \leq N-1$. If $N+2 \leq r$, then $p_r(N) = p_r(N+1) = 0$. Next, let $r \leq N+1$. Applying (2.1) gives

$$\begin{aligned} &p_r(k) + p_r(2N+1-k) - p_r(N) - p_r(N+1) \\ &\quad \geq p_r(N) - p_r(N+1) + p_r(2N+1-k) - p_r(2N-k) \\ &\quad = r(4N+2-2k-r)p_{r-1}(2N-k) - r(2N+2-r)p_{r-1}(N). \end{aligned}$$

Since

$$4N+2-2k-r \geq 2N+2-r > 0 \quad \text{and} \quad p_{r-1}(2N-k) \geq p_{r-1}(N) \geq 0,$$

we conclude that (2.2) is valid.

(iii) Let $0 \leq a \leq b$. If $b \leq r-1$, then $p_r(a) = p_r(b) = 0$. We assume that $b \geq r$.

Case 1. $0 \leq a \leq r-1$.
Since

$$0 < \frac{b-j}{a+b-j} \leq 1 \quad \text{for} \quad j = 0, 1, \ldots, r-1,$$

we obtain

$$\frac{p_r(a) + p_r(b)}{p_r(a+b)} = \prod_{j=0}^{r-1} \left(\frac{b-j}{a+b-j}\right)^2 \leq 1.$$

Case 2. $r \le a$.

We have

$$0 < \frac{a-j}{a+b-j} \le \frac{b-j}{a+b-j} < 1 \quad \text{for} \quad j = 0, 1, \ldots, r-1.$$

Hence,

$$\frac{p_r(a) + p_r(b)}{p_r(a+b)} = \prod_{j=0}^{r-1} \left(\frac{a-j}{a+b-j}\right)^2 + \prod_{j=0}^{r-1} \left(\frac{b-j}{a+b-j}\right)^2$$
$$\le \left(\frac{a}{a+b}\right)^2 + \left(\frac{b}{a+b}\right)^2 < 1.$$

This proves (2.3). □

3. Main result

We are now able to answer the two questions posed in §1. The following theorem refines and extends the Proposition.

Theorem. *Let n, r be integers with $1 \le r \le n$. For all $P \in \mathcal{P}_n$ we have*

$$\alpha_n(r) \int_0^{2\pi} |P(\mathrm{e}^{it})|^2 \mathrm{d}t \le \int_0^{2\pi} |P^{(r)}(\mathrm{e}^{it})|^2 \mathrm{d}t \le \beta_n(r) \int_0^{2\pi} |P(\mathrm{e}^{it})|^2 \mathrm{d}t \quad (3.1)$$

with the best possible factors

$$\alpha_n(r) = \begin{cases} \prod_{j=0}^{r-1} \left(\frac{n}{2} - j\right)^2, & \text{if } n \text{ is even}, \\ \frac{1}{2}\left[\prod_{j=0}^{r-1} \left(\frac{n+1}{2} - j\right)^2 + \prod_{j=0}^{r-1} \left(\frac{n-1}{2} - j\right)^2\right], & \text{if } n \text{ is odd}, \end{cases} \quad (3.2)$$

and

$$\beta_n(r) = \frac{1}{2} \prod_{j=0}^{r-1} (n-j)^2. \quad (3.3)$$

Proof. Let $P(z) = \sum_{k=0}^{n} a_k z^k$ with $a_k = a_{n-k}$ for $k = 0, 1, \ldots, n$. Differentiation gives

$$P^{(r)}(z) = \sum_{k=r}^{n} a_k\, k(k-1) \cdots (k-r+1) z^{k-r}$$
$$= \sum_{k=0}^{n-r} a_{k+r} (k+r)(k+r-1) \cdots (k+1) z^k,$$

so that Parseval's identity leads to

$$\frac{1}{2\pi}\int_0^{2\pi} |P(\mathrm{e}^{it})|^2 \mathrm{d}t = \sum_{k=0}^{n} |a_k|^2 \tag{3.4}$$

and

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |P^{(r)}(\mathrm{e}^{it})|^2 \mathrm{d}t &= \sum_{k=0}^{n-r} |a_{k+r}|^2 \prod_{j=0}^{r-1}(k+r-j)^2 \\ &= \sum_{k=0}^{n} |a_k|^2 \prod_{j=0}^{r-1}(k-j)^2. \end{aligned}\tag{3.5}$$

First, we prove the left-hand side of (3.1) with $\alpha_n(r)$ as given in (3.2). Let

$$S_n(r) = \frac{1}{2\pi}\left(\int_0^{2\pi} |P^{(r)}(\mathrm{e}^{it})|^2 \mathrm{d}t - \alpha_n(r)\int_0^{2\pi} |P(\mathrm{e}^{it})|^2 \mathrm{d}t\right).$$

Case 1. $n = 2N$ ($N \in \mathbf{N}$).

Applying (3.4) and (3.5) yields

$$\begin{aligned} S_n(r) &= \sum_{k=0}^{n}[p_r(k) - p_r(n/2)]|a_k|^2 \\ &= \sum_{k=0}^{N-1}[p_r(k) - p_r(n/2)]|a_k|^2 + \sum_{k=N+1}^{2N}[p_r(k) - p_r(n/2)]|a_k|^2 \\ &= \sum_{k=0}^{N-1}[p_r(k) - p_r(N)]|a_k|^2 + \sum_{k=0}^{N-1}[p_r(2N-k) - p_r(N)]|a_{2N-k}|^2 \\ &= \sum_{k=0}^{N-1}[p_r(k) + p_r(2N-k) - 2p_r(N)]|a_k|^2. \end{aligned}$$

From part (i) of the lemma, we conclude that $S_n(r) \geq 0$.

Case 2. $n = 2N+1$ ($0 \leq N \in \mathbf{Z}$).

We obtain

$$\begin{aligned} S_n(r) &= \sum_{k=0}^{n}[p_r(k) - \alpha_n(r)]|a_k|^2 \\ &= \sum_{k=0}^{N}[p_r(k) - \alpha_{2N+1}(r)]|a_k|^2 + \sum_{k=N+1}^{2N+1}[p_r(k) - \alpha_{2N+1}(r)]|a_k|^2 \\ &= \sum_{k=0}^{N}[p_r(k) - \alpha_{2N+1}(r)]|a_k|^2 \\ &\quad + \sum_{k=0}^{N}[p_r(2N+1-k) - \alpha_{2N+1}(r)]|a_{2N+1-k}|^2 \end{aligned}$$

$$= \sum_{k=0}^{N} [p_r(k) + p_r(2N+1-k) - 2\alpha_{2N+1}(r)]|a_k|^2$$

$$= \sum_{k=0}^{N} [p_r(k) + p_r(2N+1-k) - p_r(N) - p_r(N+1)]|a_k|^2.$$

Using part (ii) of the lemma implies $S_n(r) \geq 0$.

Next, we prove the second inequality in (3.1) with $\beta_n(r)$ as given in (3.3). We define

$$T_n(r) = \frac{1}{2\pi}\left(\beta_n(r)\int_0^{2\pi} |P(\mathrm{e}^{it})|^2 \mathrm{d}t - \int_0^{2\pi} |P^{(r)}(\mathrm{e}^{it})|^2 \mathrm{d}t\right).$$

Applying (3.4) and (3.5) we get

$$2T_n(r) = \sum_{k=0}^{n} [p_r(n) - 2p_r(k)]|a_k|^2. \tag{3.6}$$

Also, we have

$$2T_n(r) = \sum_{k=0}^{n} [p_r(n) - 2p_r(n-k)]|a_{n-k}|^2. \tag{3.7}$$

From (3.6) and (3.7) we obtain

$$2T_n(r) = \sum_{k=0}^{n} [p_r(n) - p_r(k) - p_r(n-k)]|a_k|^2,$$

so that part (iii) of the lemma yields $T_n(r) \geq 0$.

It remains to show that the factors given in (3.2) and (3.3) are the best possible. Let

$$U(z) = z^{n/2} \quad (n \text{ even}), \quad V(z) = z^{(n-1)/2}(z+1) \quad (n \text{ odd}), \quad W(z) = z^n + 1.$$

Then we have $U, V, W \in \mathcal{P}_n$. A short calculation yields

$$\int_0^{2\pi} |U(\mathrm{e}^{it})|^2 \mathrm{d}t = 2\pi, \quad \int_0^{2\pi} |U^{(r)}(\mathrm{e}^{it})|^2 \mathrm{d}t = 2\pi \prod_{j=0}^{r-1}\left(\frac{n}{2} - j\right)^2, \tag{3.8}$$

$$\int_0^{2\pi} |V(\mathrm{e}^{it})|^2 \mathrm{d}t = 4\pi, \tag{3.9}$$

$$\int_0^{2\pi} |V^{(r)}(\mathrm{e}^{it})|^2 \mathrm{d}t = 2\pi\left[\prod_{j=0}^{r-1}\left(\frac{n+1}{2} - j\right)^2 + \prod_{j=0}^{r-1}\left(\frac{n-1}{2} - j\right)^2\right], \tag{3.10}$$

and

$$\int_0^{2\pi} |W(\mathrm{e}^{it})|^2 \mathrm{d}t = 4\pi, \quad \int_0^{2\pi} |W^{(r)}(\mathrm{e}^{it})|^2 \mathrm{d}t = 2\pi \prod_{j=0}^{r-1}(n-j)^2. \tag{3.11}$$

From (3.8)–(3.11) we conclude that the factors $\alpha_n(r)$ and $\beta_n(r)$ are the best possible. □

Remark. The theorem reveals that the left-hand side of (1.1) can be improved, if n is odd: $n^2/4$ can be replaced by the best possible factor $(n^2+1)/4$.

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