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Relations between bilinear multipliers on $\mathbb{R}^n$, $\mathbb{T}^n$ and $\mathbb{Z}^n$

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Abstract. In this paper we prove the bilinear analogue of de Leeuw's result for periodic bilinear multipliers and some Jodeit type extension results for bilinear multipliers.

Keywords. Bilinear multipliers; Poisson summation formula; sampling theorem; transference methods

1. Introduction

In their much acclaimed work Lacey and Thiele [13, 14] proved the boundedness of the bilinear Hilbert transform and established a longstanding conjecture of AP Calderón. Since then the study of bilinear multiplier operators which commute with simultaneous translations have attracted a great deal of attention. For a comprehensive survey we would like to refer the interested reader to the article of Grafakos and Torres [10].

One of the important themes of study of L^p multipliers is about the relationship between multipliers on the classical Euclidean groups $\mathbb{R}^n$, $\mathbb{T}^n$, $\mathbb{Z}^n$. de Leeuw [8] studied the restrictions of L^p multipliers on $\mathbb{R}^n$ to $\mathbb{T}^n$. These kind of relations between bilinear multiplier operators defined on $\mathbb{R}$ and $\mathbb{T}$ have appeared in the work of Fan and Sato [9], Blasco, Carro and Gillespie [5] and Grafakos [11]. Also, for some other transference results in the bilinear setting, see [2, 3] and [6].

In the same paper de Leeuw observed that periodic multipliers on $\mathbb{R}$ are precisely the ones which are multipliers on $\mathbb{Z}$ and vice versa. In §3 we investigate the bilinear analogue of these results.

The extension question from $\mathbb{T}^n$ to $\mathbb{R}^n$ was not fully explored in de Leeuw's paper. However, in [12], Jodeit addressed some natural extensions. A function on $\mathbb{Z}^n$ is extended to $\mathbb{R}^n$ by forming the sum of integer translates of a suitable function Λ, i.e.

$$\Psi(\xi) = \sum_{k\in\mathbb{Z}^n} \phi(k)\Lambda(\xi - k).$$

(For $n = 1$ if Λ takes the value 1 at zero and has support in $[0, 1)$, then Ψ and ϕ agree at integers.) In [15], Madan and Mohanty addressed the bilinear analogue of this using transference techniques. They have shown that the piecewise constant extension of a bilinear multiplier symbol $\phi(n, m)$ of an operator $\mathcal{P}_\phi$: $L^{p_1}(\mathbb{T}) \times L^{p_2}(\mathbb{T}) \to L^{p_3}(\mathbb{T})$, where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} < 1$, gives a bilinear multiplier on $\mathbb{R}$. In §4, we give other examples of Λ as in Jodeit which complements the results of [15]. Further the results hold for the entire admissible range of exponents p_1, p_2 and p_3.

In §2, we give basic definitions and notation.

2. Preliminaries

Let $\mathcal{S}(\mathbb{R})$ be the space of Schwartz class functions with the usual topology and let $\mathcal{S}'(\mathbb{R})$ be its dual space. We say that a triplet (p_1, p_2, p_3) is Hölder related if $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$, where $p_1, p_2 \geq 1$ and $p_3 \geq \frac{1}{2}$.

For $f, g \in \mathcal{S}(\mathbb{R})$ the bilinear Hilbert transform is given by

$$H(f,g)(x) := p.v. \int_{\mathbb{R}} f(x-t)g(x+t)\frac{\mathrm{d}t}{t}$$

and has the following alternative expression:

$$H(f,g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)(-i)\mathrm{sgn}(\xi-\eta)\mathrm{e}^{2\pi i x(\xi+\eta)}\mathrm{d}\xi\mathrm{d}\eta.$$

In [13, 14], Lacey and Thiele proved the boundedness of the above operator H from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \rightarrow L^{p_3}(\mathbb{R})$ for the Hölder related triplet (p_1, p_2, p_3), where $1 < p_1, p_2 \leq \infty$ and $\frac{2}{3} < p_3 < \infty$.

It is known that for any continuous bilinear operator $\mathcal{C}$: $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R})$, which commutes with simultaneous translations there exists a symbol $\psi_{\mathcal{C}}(\xi, \eta)$ such that for $f, g \in \mathcal{S}(\mathbb{R})$,

$$\mathcal{C}(f,g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)\psi_{\mathcal{C}}(\xi,\eta)\mathrm{e}^{2\pi i x(\xi+\eta)}\mathrm{d}\xi\mathrm{d}\eta$$

In the distributional sense we can write

$$\mathcal{C}(f,g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-u)g(x-v)K_{\mathcal{C}}(u,v)\mathrm{d}u\mathrm{d}v,$$

where $\hat{K_{\mathcal{C}}} = \psi_{\mathcal{C}}$ (in the sense of distributions).

Unlike in the linear case, the boundedness of the symbol $\psi_{\mathcal{C}}$ is not known. In this article we will be dealing with bounded symbols only.

For $\psi \in L^\infty(\mathbb{R}^2)$ and $f, g \in \mathcal{S}(\mathbb{R})$, we write

$$\mathcal{C}_\psi(f,g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)\psi(\xi,\eta)\mathrm{e}^{2\pi i x(\xi+\eta)}\mathrm{d}\xi\mathrm{d}\eta. \tag{1}$$

If for all $f, g \in \mathcal{S}(\mathbb{R})$ the bilinear operator $\mathcal{C}_\psi$ satisfies

$$\|\mathcal{C}_\psi(f,g)\|_{p_3} \leq c\|f\|_{p_1}\|g\|_{p_2}, \tag{2}$$

where c is a constant independent of f, g, then we say that $\mathcal{C}_\psi$ is a bilinear multiplier operator associated with the symbol ψ for the triplet (p_1, p_2, p_3). The set of all bounded bilinear multipliers for the triplet (p_1, p_2, p_3) will be denoted by $M_{p_1,p_2}^{p_3}(\mathbb{R})$. For $p_3 \geq 1$, $M_{p_1,p_2}^{p_3}(\mathbb{R})$ becomes a Banach space under the operator norm, whereas for $p_3 < 1$ it forms a quasi Banach space. We will use the notation $\|\cdot\|$ for the operator norm and for convenience we will not attach any p with it. It will be understood from the context.

Bilinear multiplier operators on $\mathbb{T}$ and $\mathbb{Z}$ can be defined similarly. We say that the operator $\mathcal{P}$ defined by

$$\mathcal{P}(F,G)(x) := \Sigma_n \Sigma_m \hat{F}(n)\hat{G}(m)\phi(n,m)\mathrm{e}^{2\pi i x(n+m)};$$

is bounded from $L^{p_1}(\mathbb{T}) \times L^{p_2}(\mathbb{T}) \to L^{p_3}(\mathbb{T})$ if for some constant c and for all trigonometric polynomials F, G we have

$$\|\mathcal{P}(F,G)\|_{p_3} \leq c\|F\|_{p_1}\|G\|_{p_2}.$$

Similarly on $\mathbb{Z}$ the operator

$$\mathcal{D}(a,b)(l) = \int_{\mathbb{T}}\int_{\mathbb{T}} \hat{a}(\theta)\hat{b}(\rho)\psi(\theta,\rho)\mathrm{e}^{2\pi i l(\theta+\rho)}\mathrm{d}\theta\mathrm{d}\rho$$

is said to be bounded from $l^{p_1}(\mathbb{Z}) \times l^{p_2}(\mathbb{Z}) \to l^{p_3}(\mathbb{Z})$, if for some constant c and for all finite sequences a, b we have

$$\|\mathcal{D}(a,b)\|_{l^{p_3}} \leq c\|a\|_{l^{p_1}}\|b\|_{l^{p_2}}.$$

The space of bounded bilinear multipliers on $\mathbb{T}$ and $\mathbb{Z}$ for the triplet (p_1, p_2, p_3) will be denoted by $M^{p_3}_{p_1,p_2}(\mathbb{T})$ and $M^{p_3}_{p_1,p_2}(\mathbb{Z})$ respectively.

3. Periodic bilinear multipliers

Let $\psi(\xi,\eta) \in M^{p_3}_{p_1,p_2}(\mathbb{R})$ be a periodic function with period one in both variables, i.e. $\psi(\xi,\eta) = \psi(\xi+1,\eta) = \psi(\xi,\eta+1)$. A natural question that arises is whether $\psi(\xi,\eta) \in M^{p_3}_{p_1,p_2}(\mathbb{Z})$. In [4], Blasco proved a partial result in this direction. Conversely, given $\psi \in M^{p_3}_{p_1,p_2}(\mathbb{Z})$ one can ask whether $\psi(\xi,\eta) \in M^{p_3}_{p_1,p_2}(\mathbb{R})$. For linear multipliers, see [12]. We address these questions here for the entire admissible range of exponents. In particular, we show the following.

Theorem 3.1. *Let $\psi \in L^\infty(\mathbb{R}^2)$ be a 1-periodic function in both variables. Then $\psi \in M^{p_3}_{p_1,p_2}(\mathbb{R})$ if and only if $\psi \in M^{p_3}_{p_1,p_2}(\mathbb{Z})$, where the triplet (p_1, p_2, p_3) is Hölder related.*

Proof. First we will prove that if $\psi \in M^{p_3}_{p_1,p_2}(\mathbb{R})$ then $\psi \in M^{p_3}_{p_1,p_2}(\mathbb{Z})$. Let $\Phi \in \mathcal{S}(\mathbb{R})$ be such that $\mathrm{supp}(\Phi) \subseteq [0,1]$, $0 \leq \Phi(x) \leq 1$ and $\Phi(x) = 1, \forall x \in \left[\frac{1}{4}, \frac{3}{4}\right]$. If $\{a_k\}$ and $\{b_l\}$ are two sequences with finitely many non-zero terms, we define two functions f_a, g_b as follows

$$f_a(x) := \sum_k a_k \Phi(x-k)$$

and

$$g_b(x) := \sum_l b_l \Phi(x-l).$$

It is easy to see that $\|f_a\|_{L^{p_1}(\mathbb{R})} \leq \|a\|_{l^{p_1}(\mathbb{Z})}$ and $\|g_b\|_{L^{p_2}(\mathbb{R})} \leq \|b\|_{l^{p_2}(\mathbb{Z})}$. Then

$$
\begin{aligned}
\mathcal{C}_\psi(f_a, g_b)(x) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \psi(\xi, \eta) \hat{f}_a(\xi) \hat{g_b}(\eta) \mathrm{e}^{2\pi i(\xi+\eta)x} \mathrm{d}\xi \mathrm{d}\eta \\
&= \int_0^1 \int_0^1 \psi(\xi, \eta) \mathrm{e}^{2\pi i(\xi+\eta)x} \\
&\quad \times \sum_m \sum_n \hat{f}_a(\xi + m) \hat{g_b}(\eta + n) \mathrm{e}^{2\pi i(m+n)x} \mathrm{d}\xi \mathrm{d}\eta \\
&= \int_0^1 \int_0^1 \psi(\xi, \eta) \mathrm{e}^{2\pi i(\xi+\eta)x} \sum_m \hat{f}_a(\xi + m) \mathrm{e}^{2\pi i m x} \\
&\quad \times \sum_n \hat{g_b}(\eta + n) \mathrm{e}^{2\pi i n x} \mathrm{d}\xi \mathrm{d}\eta \\
&= \int_0^1 \int_0^1 \psi(\xi, \eta) \left(\sum_m f_a(x + m) \mathrm{e}^{-2\pi i \xi m} \right) \\
&\quad \times \left(\sum_n g_b(x + n) \mathrm{e}^{-2\pi i \eta n} \right) \mathrm{d}\xi \mathrm{d}\eta,
\end{aligned}
$$

where we have used the Poisson summation formula in the last step.

For $x \in \left[j + \frac{1}{4}, j + \frac{3}{4}\right] = I_j$, we can write $x = j + (x')$, where (x') is the fractional part of x. Then

$$
\begin{aligned}
\sum_m f_a(x + m) \mathrm{e}^{-2\pi i \xi m} &= \sum_m \sum_k a_k \Phi(j + (x') + m - k) \mathrm{e}^{-2\pi i \xi m} \\
&= \sum_m a_{j+m} \mathrm{e}^{-2\pi i \xi m} \\
&= \sum_m a_m \mathrm{e}^{-2\pi i \xi (m-j)} \\
&= \hat{a}(\xi) \mathrm{e}^{2\pi i \xi j}.
\end{aligned}
$$

Thus, for all $x \in I_j$,

$$
\sum_m f_a(x + m) \mathrm{e}^{-2\pi i \xi m} = \hat{a}(\xi) \mathrm{e}^{2\pi i \xi j}.
$$

Similarly

$$
\sum_n g_b(x + n) \mathrm{e}^{-2\pi i \eta n} = \hat{b}(\eta) \mathrm{e}^{2\pi i \eta j}.
$$

Substituting these for $x \in I_j$, we get

$$
\begin{aligned}
\mathcal{C}_\psi(f_a, g_b)(x) &= \int_0^1 \int_0^1 \psi(\xi, \eta) \hat{a}(\xi) \hat{b}(\eta) \mathrm{e}^{2\pi i(\xi+\eta)j} \mathrm{d}\xi \mathrm{d}\eta \\
&= \mathcal{D}_\psi(a, b)(j),
\end{aligned}
$$

where $\mathcal{D}_\psi$ is the bilinear operator defined on $l^{p_1}(\mathbb{Z}) \times l^{p_2}(\mathbb{Z})$. Now for all $x \in I_j$,

$$\begin{aligned}
|\mathcal{D}_\psi(a,b)(j)|^{p_3} &= |\mathcal{C}_\psi(f_a, g_b)(x)|^{p_3}\chi_{I_j}, \\
&= 2\int_{j+\frac{1}{4}}^{j+\frac{3}{4}} |\mathcal{C}_\psi(f_a, g_b)(x)|^{p_3}\mathrm{d}x.
\end{aligned}$$

Summing over $j \in \mathbb{Z}$,

$$\begin{aligned}
\sum_j |\mathcal{D}_\psi(a,b)(j)|^{p_3} &= 2\sum_j \int_{j+\frac{1}{4}}^{j+\frac{3}{4}} |\mathcal{C}_\psi(f_a, g_b)(x)|^{p_3}\mathrm{d}x \\
&\leq 2\int_{\mathbb{R}} |\mathcal{C}_\psi(f_a, g_b)(x)|^{p_3}\mathrm{d}x \\
&\leq 2\|\mathcal{C}_\psi\|^{p_3}\|f_a\|_{p_1}^{p_3}\|g_b\|_{p_2}^{p_3}.
\end{aligned}$$

Therefore

$$\|\mathcal{D}_\psi(a,b)\|_{p_3} \leq 2^{1/p_3}\ \|\mathcal{C}_\psi\|\ \|a\|_{p_1}\ \|b\|_{p_2}.$$

For the converse, let $\psi \in M_{p_1,p_2}^{p_3}(\mathbb{Z})$. For $f, g \in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned}
\mathcal{C}_\psi(f,g)(x) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)\psi(\xi,\eta)\mathrm{e}^{2\pi i x(\xi+\eta)}\mathrm{d}\xi\mathrm{d}\eta \\
&= \sum_{n,m}\int_0^1\int_0^1 \hat{f}(\xi+n)\hat{g}(\eta+m) \\
&\quad \times \psi(\xi+n,\eta+m)\mathrm{e}^{2\pi i x(\xi+n+\eta+m)}\mathrm{d}\xi\mathrm{d}\eta \\
&= \int_0^1\int_0^1 \sum_n \hat{f}(\xi+n)\mathrm{e}^{2\pi i x(\xi+n)} \\
&\quad \times \sum_m \hat{g}(\eta+m)\mathrm{e}^{2\pi i x(\eta+m)}\psi(\xi,\eta)\mathrm{e}^{2\pi i x(\xi+\eta)}\mathrm{d}\xi\mathrm{d}\eta \\
&= \int_0^1\int_0^1 \left(\sum_n f(x-n)\mathrm{e}^{2\pi i\xi n}\right) \\
&\quad \times \left(\sum_m g(x-m)\mathrm{e}^{2\pi i\eta m}\right)\psi(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta \\
&= \int_0^1\int_0^1 \hat{F}_x(\xi)\hat{G}_x(\eta)\psi(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta,
\end{aligned}$$

where for each $x \in \mathbb{R}$, $F_x(n) = f(x-n)$ and $G_x(n) = g(x-n)$.

Hence,

$$
\begin{aligned}
&\int_{\mathbb{R}} |\mathcal{C}_\psi(f,g)(x)|^{p_3}\mathrm{d}x \\
&\quad = \int_0^1 \sum_l |\mathcal{C}_\psi(f,g)(x+l)|^{p_3}\mathrm{d}x \\
&\quad = \int_0^1 \sum_l \left| \int_0^1 \int_0^1 \hat{F}_x(\xi)\hat{G}_x(\eta)\psi(\xi,\eta)\mathrm{e}^{2\pi i l(\xi+\eta)}\mathrm{d}\xi\mathrm{d}\eta \right|^{p_3} \mathrm{d}x \\
&\quad = \int_0^1 \|\mathcal{D}_\psi(F_x,G_x)\|_{p_3}^{p_3}\mathrm{d}x \\
&\quad \le \int_0^1 \|\mathcal{D}_\psi\|^{p_3}\|F_x\|_{p_1}^{p_3}\|G_x\|_{p_2}^{p_3}\mathrm{d}x \\
&\quad = \int_0^1 \|\mathcal{D}_\psi\|^{p_3} \left(\sum_n |f(x-n)|^{p_1}\right)^{\frac{p_3}{p_1}} \left(\sum_n |g(x-n)|^{p_2}\right)^{\frac{p_3}{p_2}} \mathrm{d}x.
\end{aligned}
$$

Using the Hölder's inequality with the exponents $\frac{p_1}{p_3}, \frac{p_2}{p_3}$, we obtain

$$
\|\mathcal{C}_\psi(f,g)\|_{p_3} \le \|\mathcal{D}_\psi\|\, \|f\|_{p_1}\|g\|_{p_2}. \qquad \blacksquare
$$

We now turn our attention to periodic extensions of compactly supported bilinear multipliers. In what follows J will denote the interval $[-1/2, 1/2)$.

We will prove the following result.

Theorem 3.2. *Let* $\psi \in M_{p_1,p_2}^{p_3}(\mathbb{R})$ *be such that* $\operatorname{supp}(\psi) \subseteq J \times J$. *Consider* $\psi^\sharp$ *the periodic extension of* ψ *given by*

$$
\psi^\sharp(\xi,\eta) = \sum_n \sum_m \psi(\xi-n, \eta-m).
$$

Then $\psi^\sharp \in M_{p_1,p_2}^{p_3}(\mathbb{R})$. *Moreover,* $\|\psi^\sharp\| \le c\|\psi\|$, *where* c *is a constant independent of* ψ.

As a consequence of Theorem 3.1, it would suffice to prove the following.

PROPOSITION 3.3

Let $\psi \in M_{p_1,p_2}^{p_3}(\mathbb{R})$ *be such that* $\operatorname{supp}(\psi) \subseteq J \times J$. *Then* $\psi^\sharp \in M_{p_1,p_2}^{p_3}(\mathbb{Z})$. *Moreover,* $\|\psi^\sharp\| \le c\|\psi\|$, *where* c *is a constant independent of* ψ.

We will need the following two lemmas.

Lemma 3.4. *Let* $\psi \in M_{p_1,p_2}^{p_3}(\mathbb{R})$ *be such that* $\operatorname{supp}(\psi) \subseteq J \times J$. *Then for* $f, g \in \mathcal{S}(\mathbb{R})$, $\operatorname{supp} \widehat{\mathcal{C}_\psi(f,g)} \subset [-2,2]$.

Proof. Let $h \in \mathcal{S}(\mathbb{R})$ be such that supp $\hat{h} \subset [-2, 2]^c$. Then

$$\begin{aligned}\langle \widehat{\mathcal{C}_\psi(f,g)}, h\rangle &= \langle \mathcal{C}_\psi(f,g), \hat{h}\rangle \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)\psi(\xi,\eta)\mathrm{e}^{2\pi i x(\xi+\eta)}\mathrm{d}\xi\mathrm{d}\eta\hat{h}(x)\mathrm{d}x \\ &= \int_J\int_J \hat{f}(\xi)\hat{g}(\eta)\psi(\xi,\eta)\int_{\mathbb{R}} \hat{h}(x)\mathrm{e}^{2\pi i x(\xi+\eta)}\mathrm{d}x\mathrm{d}\xi\mathrm{d}\eta \\ &= \int_J\int_J \hat{f}(\xi)\hat{g}(\eta)\psi(\xi,\eta)h(\xi+\eta)\mathrm{d}\xi\mathrm{d}\eta = 0.\end{aligned}$$

Thus

$$\langle \widehat{\mathcal{C}_\psi(f,g)}, h\rangle = 0.$$

This proves the lemma. ■

For the proof of Proposition 3.3 we will use the following result.

Lemma 3.5 [7]. *Let* $0 < p \le \infty$ *and* g *be a slowly increasing* C^∞ *function such that* $\mathrm{supp}(\hat{g}) \subset [-R, R]$, *then there exists a constant* $C > 0$, *depending on* p *such that*

$$\sum_n |g(n)|^p \le C^p \max(1, R)\int_{\mathbb{R}} |g(x)|^p \mathrm{d}x.$$

This is a well-known sampling lemma.

Now we prove Proposition 3.3.

Proof. Let $\Phi \in \mathcal{S}(\mathbb{R})$. Let $a = \{a_k\}$ and $b = \{b_l\}$ be two sequences with finitely many non-zero terms. We define $f_a(x) := \sum_k a_k\Phi(x-k)$ and $g_b(x) := \sum_l b_l\Phi(x-l)$. It is easy to see that

$$\begin{aligned}\|f_a\|_{L^{p_1}(\mathbb{R})} &\le c\|a\|_{l^{p_1}(\mathbb{Z})}, \\ \|g_b\|_{L^{p_2}(\mathbb{R})} &\le c\|b\|_{l^{p_2}(\mathbb{Z})},\end{aligned}$$

where $c = \sup_{x\in[0,1]}\sum_l |\Phi(x-l)|$.

Also, $\hat{f_a}(\xi) = \hat{\Phi}(\xi)\hat{a}(\xi)$ and $\hat{g_b}(\eta) = \hat{\Phi}(\eta)\hat{b}(\eta)$.

We write the operator

$$\begin{aligned}\mathcal{C}_\psi(f_a, g_b)(x) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f_a}(\xi)\hat{g_b}(\eta)\psi(\xi,\eta)\mathrm{e}^{2\pi i x(\xi+\eta)}\mathrm{d}\xi\mathrm{d}\eta \\ &= \int_J\int_J \hat{\Phi}(\xi)\hat{a}(\xi)\hat{\Phi}(\eta)\hat{b}(\eta)\psi(\xi,\eta)\mathrm{e}^{2\pi i x(\xi+\eta)}\mathrm{d}\xi\mathrm{d}\eta.\end{aligned}$$

Choose Φ such that $\hat{\Phi} \equiv 1$ on J. Then at the integer points we get

$$\mathcal{C}_\psi(f_a, g_b)(n) = \int_J\int_J \hat{a}(\xi)\hat{b}(\eta)\psi(\xi,\eta)\mathrm{e}^{2\pi i n(\xi+\eta)}\mathrm{d}\xi\mathrm{d}\eta = \mathcal{D}_\psi(a,b)(n).$$

Using Lemmas 3.4 and 3.5, we get

$$\begin{aligned}\sum_n |\mathcal{D}_\psi(a,b)(n)|^{p_3} &= \sum_n |\mathcal{C}_\psi(f_a, g_b)(n)|^{p_3}\\ &\le C_{p_3}^{p_3} 2^{p_3} \int_{\mathbb{R}} |\mathcal{C}_\psi(f_a, g_b)(x)|^{p_3} \mathrm{d}x\\ &\le C_{p_3}^{p_3} 2^{p_3} \|\mathcal{C}_\psi\|^{p_3} \|f_a\|_{p_1}^{p_3} \|g_b\|_{p_2}^{p_3}\end{aligned}$$

i.e.

$$\|\mathcal{D}_\psi(a,b)\|_{p_3} \le C'_{p_3} \|\mathcal{C}_\psi\| \; \|a\|_{p_1} \|b\|_{p_2}. \qquad \blacksquare$$

4. Jodeit type extension theorems

In this section we will explore some extensions of bilinear multipliers on $\mathbb{T}$ to bilinear multipliers on $\mathbb{R}$. Essentially our results are analogues of Jodeit type of extensions in the linear case. Our proofs are refinements of Jodeit's original proofs. For the sake of completeness, we include the proofs here. We will need the following lemmas which may be of independent interest.

Lemma 4.1. *Let* $\phi \in M_{p_1,p_2}^{p_3}(\mathbb{T})$.

(i) *If* $p_3 \ge 1$ *and* $a \in l^1(\mathbb{Z}^2)$, *then* $a * \phi \in M_{p_1,p_2}^{p_3}(\mathbb{T})$ *and* $\|a * \phi\| \le \|a\|_1 \|\phi\|$.
(ii) *If* $p_3 < 1$ *and* $a \in l^{p_3}(\mathbb{Z}^2)$, *then* $a * \phi \in M_{p_1,p_2}^{p_3}(\mathbb{T})$ *and* $\|a * \phi\| \le \|a\|_{p_3} \|\phi\|$.

Proof. For $p_3 \ge 1$, this is an immediate consequence of Minkowski's inequality. Assume $p_3 < 1$ and let T' be the operator corresponding to $a * \phi$. For $f \in L^{p_1}(\mathbb{T})$ and $g \in L^{p_2}(\mathbb{T})$,

$$\begin{aligned}&\|T'(f,g)\|_{p_3}^{p_3}\\ &\quad = \int_J \left| \sum_{n,m} a * \phi(n,m) \hat{f}(n) \hat{g}(m) \mathrm{e}^{2\pi i x(n+m)} \right|^{p_3} \mathrm{d}x\\ &\quad = \int_J \left| \sum_{l,k} \sum_{n,m} a(l,k) \phi(n-l, m-k) \hat{f}(n) \hat{g}(m) \mathrm{e}^{2\pi i x(n+m)} \right|^{p_3} \mathrm{d}x\\ &\quad \le \int_J \sum_{l,k} |a(l,k)|^{p_3} \left| \sum_{n,m} \phi(n-l, m-k) \hat{f}(n) \hat{g}(m) \mathrm{e}^{2\pi i x(n+m)} \right|^{p_3} \mathrm{d}x\\ &\quad \le \|T\|^{p_3} \|a\|_{p_3}^{p_3} \|f\|_{p_1}^{p_3} \|g\|_{p_2}^{p_3}.\end{aligned}$$

The first inequality follows from $|\sum_i \alpha_i|^p \le \sum_i |\alpha_i|^p$, $0 < p < 1$ and the second from the assumption that the operator T is bounded. Hence we obtain

$$\|T'(f,g)\|_{p_3} \le \|T\| \|a\|_{p_3} \|f\|_{p_1} \|g\|_{p_2} \qquad \blacksquare$$

Lemma 4.2. *Let $\phi \in M^{p_3}_{p_1,p_2}(\mathbb{T})$. For a positive integer k, we define ϕ_k as follows: $\phi_k(n,m) := \phi(n/k, m/k)$ if k divides both n, m, and $\phi_k(n,m) := 0$ otherwise. Then $\phi_k \in M^{p_3}_{p_1,p_2}(\mathbb{T})$ with norm not exceeding that of ϕ.*

Proof. For $f \in L^\infty(\mathbb{T})$, let $F(x) = \frac{1}{k}\sum_{j=0}^{k-1} f\left(\frac{x+j}{k}\right)$.

$$\begin{aligned}
\hat{F}(n) &= \int_J \frac{1}{k}\sum_{j=0}^{k-1} f\left(\frac{x+j}{k}\right) \mathrm{e}^{-2\pi i x\cdot n}\mathrm{d}x \\
&= \frac{1}{k}\sum_{j=0}^{k-1}\int_J f\left(\frac{x+j}{k}\right) \mathrm{e}^{-2\pi i x\cdot n}\mathrm{d}x \\
&= \int_J f(x)\mathrm{e}^{-2\pi i x\cdot kn}\mathrm{d}x \\
&= \hat{f}(kn).
\end{aligned}$$

Also note that $\|F\|_1 \le \|f\|_1$ and $\|F\|_\infty \le \|f\|_\infty$. Hence

$$\|F\|_p \le \|f\|_p, \quad 1 \le p \le \infty$$

Similarly for $g \in L^\infty(\mathbb{T})$, we define G. Let T_k be the operator corresponding to ϕ_k. Then

$$\begin{aligned}
T_k(f,g)(x) &= \sum_n \sum_m \hat{f}(n)\hat{g}(m)\phi_k(n,m)\mathrm{e}^{2\pi i x(n+m)} \\
&= \sum_n \sum_m \hat{f}(kn)\hat{g}(km)\phi(n,m)\mathrm{e}^{2\pi i x(kn+km)} \\
&= \sum_n \sum_m \hat{F}(n)\hat{G}(m)\phi(n,m)\mathrm{e}^{2\pi i x(kn+km)} \\
&= T(F,G)(kx).
\end{aligned}$$

Hence

$$\begin{aligned}
\|T_k(f,g)\|_{p_3}^{p_3} &= \int_J |T(F,G)(kx)|^{p_3}\mathrm{d}x \\
&= \frac{1}{k}\int_{-k/2}^{k/2} |T(F,G)(x)|^{p_3}\mathrm{d}x \\
&= \int_J |T(F,G)(x)|^{p_3}\mathrm{d}x \\
&\le \|T\|^{p_3}\|F\|_{p_1}^{p_3}\|G\|_{p_2}^{p_3} \\
&\le \|T\|^{p_3}\|f\|_{p_1}^{p_3}\|g\|_{p_2}^{p_3}.
\end{aligned}$$

Thus we obtain that ϕ_k is in $M^{p_3}_{p_1,p_2}(\mathbb{T})$. ■

Our first extension result is the following theorem.

Theorem 4.3. *Let ϕ be in $M^{p_3}_{p_1,p_2}(\mathbb{T})$ and S be a function supported in $\frac{1}{2}J \times \frac{1}{2}J$ such that its periodic extension $S^\sharp$ from $J \times J$ satisfies $\sum_{n,m} |\hat{S^\sharp}(n,m)|^p < \infty$, where $p = \min(1, p_3)$. Then $\psi(\xi,\eta) := \sum_{n,m} \phi(n,m)\hat{S}(\xi-n,\eta-m) \in M^{p_3}_{p_1,p_2}(\mathbb{R})$. Moreover, $\|\psi\| \le 2^{\frac{1}{p}} c\|\phi\|$, where $c = (\sum_{n,m} |\hat{S^\sharp}(n,m)|^p)^{\frac{1}{p}} < \infty$.*

Proof. It is enough to prove that $\psi_r(\xi,\eta) = \sum_{l,k} \phi(l,k)\, r^{|l|+|k|}\hat{S}(\xi-l,\eta-k)$ belongs to $M^{p_3}_{p_1,p_2}(\mathbb{R})$ for $0 < r < 1$ with $\|C_{\psi_r}\| \le c\|\phi\|$. Let K_{ϕ_r} be the kernel corresponding to the bilinear multiplier $\phi(n,m)r^{|n|+|m|}$. Clearly, $\widehat{K_{\phi_r} S} = \psi_r$ is considered as a function on $\mathbb{R}^2$. From Lemma 4.1, $\widehat{K_{\phi_r} S^\#} = \hat{K}_{\phi_r} * \hat{S^\#}$ belongs to $M^{p_3}_{p_1,p_2}(\mathbb{T})$ with norm bounded by $c\|\phi\|$.

Let $f \in \mathcal{S}(\mathbb{R})$ and for each $n \in \mathbb{Z}$, let f_n denote the 1-periodic extension of the function $f(x+n/2)\chi_J(x)$ from J. Then it can be easily verified that

$$\sum_n \|f_n\|^p_{L^p(\mathbb{T})} \le 2\|f\|^p_p.$$

Now for $x \in \frac{1}{2}J$, we have

$$\begin{aligned} C_{\psi_r}(f,g)\left(x+\frac{n}{2}\right) &= \int_{\frac{1}{2}J}\int_{\frac{1}{2}J} f\left(x-t+\frac{n}{2}\right) \\ &\quad \times g\left(x-s+\frac{n}{2}\right)(K_{\phi_r}S)(t,s)\mathrm{d}t\mathrm{d}s \\ &= \int_J\int_J f_n(x-t)g_n(x-s)(K_{\phi_r}S^\#)(t,s)\mathrm{d}t\mathrm{d}s. \end{aligned}$$

Thus,

$$\begin{aligned} \|C_{\psi_r}(f,g)\|^{p_3}_{p_3} &\le \sum_n (c\|\phi\|)^{p_3}\|f_n\|^{p_3}_{p_1}\|g_n\|^{p_3}_{p_2} \\ &\le 2(c\|\phi\|)^{p_3}\|f\|^{p_3}_{p_1}\|g\|^{p_3}_{p_2}. \end{aligned}$$

The last inequality follows as an application of Hölder's inequality. ■

Our next result is the piecewise linear extension of $\phi \in M^{p_3}_{p_1,p_2}(\mathbb{T})$ to $\psi \in M^{p_3}_{p_1,p_2}(\mathbb{R})$.

Theorem 4.4. *Let $\phi \in M^{p_3}_{p_1,p_2}(\mathbb{T})$ and $p_3 > \frac{1}{2}$. If $\Lambda(x_1,x_2) = (1-|x_1|)(1-|x_2|)$ in $[-1,1) \times [-1,1)$ and 0 otherwise. Then the function ψ defined as $\psi(\xi,\eta) := \sum_{n,m}\phi(n,m)\Lambda(\xi-n,\eta-m) \in M^{p_3}_{p_1,p_2}(\mathbb{R})$ and $\|\psi\| \le C_{p_3}\|\phi\|$.*

Proof. Consider $S(x,y) = \frac{\sin^2 4\pi x}{(4\pi x)^2}\frac{\sin^2 4\pi y}{(4\pi y)^2}$. Let $S_{k,l}(x,y)$ be the 1-periodic extension of $\chi_{\frac{k}{2}+\frac{1}{2}J}(x)\chi_{\frac{l}{2}+\frac{1}{2}J}(y)S(x,y)$ from $\left(\frac{k}{2}+J\right)\times\left(\frac{l}{2}+J\right)$. An easy computation using integration by parts gives that for $n, m \neq 0$,

$$|\hat{S}_{k,l}(n,m)| \le \frac{C}{(1+k^2)(1+n^2)(1+l^2)(1+m^2)}.$$

Hence, by Theorem 4.3 we have $\sum_{n,m}\phi(n,m)\hat{S}_{k,l}(\xi-n,\eta-m)$ belongs to $M^{p_3}_{p_1,p_2}(\mathbb{R})$ with bounds not exceeding $\frac{C_{p_3}}{(1+k^2)(1+l^2)}\|\phi\|$.

Thus $\sum_{n,m}\phi(n,m)\hat{S}(\xi-n,\eta-m) = \sum_{n,m}\phi(n,m)\Lambda\left(\frac{\xi-n}{4}, \frac{\eta-m)}{4}\right)$ is in $M^{p_3}_{p_1,p_2}(\mathbb{R})$. By applying Lemma 4.2 we get $\psi \in M^{p_3}_{p_1,p_2}(\mathbb{R})$ with the required bound. ■

As a consequence of this we get the desired piecewise constant extension result.

Theorem 4.5. *Let ϕ be in $M^{p_3}_{p_1,p_2}(\mathbb{T})$, where $p_1, p_2 > 1$. Then $\psi(\xi,\eta) := \sum_{n,m} \phi(n,m)\chi_{J\times J}(\xi - n, \eta - m) \in M^{p_3}_{p_1,p_2}(\mathbb{R})$.*

Proof. Define $\phi_2(n,m) := \phi(n/2, m/2)$ if n, m both are even and $\phi_2(n,m) := 0$ otherwise. By Lemma 4.2, $\phi_2(n,m) \in M^{p_3}_{p_1,p_2}(\mathbb{T})$ and $\|\phi_2\| \leq \|\phi\|$. Consider $\theta_2(n,m) = \phi_2(n,m) + \phi_2(n-1,m) + \phi_2(n,m-1) + \phi_2(n-1,m-1)$. Clearly, $\theta_2 \in M^{p_3}_{p_1,p_2}(\mathbb{T})$ and $\|\theta_2\| \leq 4\|\phi\|$.

Let $\Lambda(\xi,\eta) = (1-|\xi|)(1-|\eta|)$ in $[-1,1)\times[-1,1)$, and 0 elsewhere. Then by Theorem 4.4, $\Theta_2(\xi,\eta) = \sum_{n,m} \theta_2(n,m)\Lambda(\xi - n, \eta - m)$ is in $M^{p_3}_{p_1,p_2}(\mathbb{R})$. Note that $\Theta_2(\xi,\eta) = \phi(n,m)$ if $(\xi,\eta) \in [2n, 2n+1]\times[2m, 2m+1]$.

Let $\tilde{\chi}$ be the 2-periodic extension of a function which is 1 for $0 < x < 1$ and 0 for $-1 < x < 0$. We know that $\tilde{\chi} \in M_p(\mathbb{R})$ for $p > 1$. Hence $\Psi_2(\xi,\eta) = \tilde{\chi}(\xi)\tilde{\chi}(\eta)\Theta_2(\xi,\eta) \in M^{p_3}_{p_1,p_2}(\mathbb{R})$. Now $\Psi_2(\xi,\eta) = \phi(n,m)$ for $2n < \xi < 2n+1$, $2m < \eta < 2m+1$ and $\Psi_2(\xi,\eta) = 0$ otherwise. Since $\psi(\xi,\eta) = \Psi_2(2\xi, 2\eta) + \Psi_2(2\xi+1, 2\eta) + \Psi_2(2\xi, 2\eta+1) + \Psi_2(2\xi+1, 2\eta+1)$, the result follows. ■

Remark. The above theorem does not hold if either of p_1, p_2 is 1. This is very easy to verify. Without loss of generality, we can assume that $p_1 = 1$. Let $\tilde{\phi} \in M_1(\mathbb{T})$ and T be the operator corresponding to the piece-wise constant extension of $\tilde{\phi}$. Put $\phi(n,m) = \tilde{\phi}(n)$. Then by Hölder's inequality $\phi \in M^{p_3}_{1,p_2}(\mathbb{T})$. Suppose the piece-wise constant extension $\psi(\xi,\eta) = \sum_{n,m} \phi(n,m)\chi_{J\times J}(\xi - n, \eta - m) = \sum_n \tilde{\phi}(n)\chi_J(\xi - n)$ is in $M^{p_3}_{1,p_2}(\mathbb{R})$. Then for $f \in L_1(\mathbb{R})$ and $g \in L_{p_2}(\mathbb{R})$, we have $\|C_\psi(f,g)\|_{p_3} = \|T(f).g\|_{p_3} \leq c\|f\|_1\|g\|_{p_2}$. Now notice that $\frac{1}{p_3}$ and $\frac{p_2}{p_3}$ are conjugate indices and $\|f\|_1^{p_3} = \||f|^{p_3}\|_{\frac{1}{p_3}}$. Hence by using duality we get $\|Tf\|_1 \leq c\|f\|_1$. But this is not true for any nonconstant $\tilde{\phi}$.

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