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The Poincaré series of a local Gorenstein ring of multiplicity up to 10 is rational

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Abstract. Let R be a local, Gorenstein ring with algebraically closed residue field k of characteristic 0 and let $P_R(z) := \sum_{p=0}^{\infty} \dim_k(\mathrm{Tor}_p^R(k,k))z^p$ be its Poincaré series. We compute P_R when R belongs to a particular class defined in the Introduction, proving its rationality. As a by-product we prove the rationality of P_R for all local, Gorenstein rings of multiplicity at most 10.

Keywords. Gorenstein; Artinian; Poincaré series.

1. Introduction and notation

Let R be a local, Noetherian ring with maximal ideal $\mathfrak{N}$ and residue field $k := R/\mathfrak{N}$.

The Poincaré (or Betti) series of R, namely,

$$P_R(z) := \sum_{p=0}^{\infty} \dim_k(\mathrm{Tor}_p^R(k,k))z^p,$$

has a particular interest in commutative algebra. It is the generating function of the Tor-algebra which is a commutative Hopf algebra. Its dual Hopf algebra is isomorphic to the Yoneda Ext-algebra of R (see [G-L] for an account on these and other general properties of P_A). Moreover the relationships between P_R and some particular quotients of R have been deeply inspected (see for e.g. [Ta], [A-L], [G-L], [RR]).

Since Serre conjectured the rationality of P_R (see e.g. [Se]) and proved it when R is regular, many authors spent their efforts to prove the conjecture. However Anick [An] gave an example of a local ring R having trascendental P_R (see also [Bo] for an example of local, Gorenstein ring with the same property).

Without any intention of completeness we recall that the conjecture is, in any case, true for several classes of rings R: complete intersection ring (see [Ta]), rings R such that $\mathrm{emdim}(R) - 2 \leq \mathrm{depth}(R)$ (see [Sc]), Gorenstein rings with $\mathrm{emdim}(R) - 4 \leq \mathrm{depth}(R)$ (see [Wi] and [J-K-M]), Koszul algebras (see [Mo]).

Let $J \subseteq R$ be a parametric ideal which is a reduction of $\mathfrak{N}$ (see §14 of [Ma]). We can thus consider the local, Artinian ring $A := R/J$ and its maximal ideal $\mathfrak{M} := \mathfrak{N}/J$. With the above notations, following Sally (see [Sa1]), we say that a ring R is stretched if $\mathrm{length}(\mathfrak{M}^2/\mathfrak{M}^3) = 1$. In [Sa2] the rationality of P_R is proved when R is a stretched Cohen-Macaulay ring. In [E-V2] such a notion has been generalized introducing almost stretched rings, i.e. rings such that $\mathrm{length}(\mathfrak{M}^2/\mathfrak{M}^3) = 2$. Imitating the proof in [Sa2], the authors prove the rationality of P_R also for almost stretched Gorenstein rings.

In this short note, following the argument of [Sa2] (and of [E-V2]), we deal with the next case, namely the case of Gorenstein rings R satisfying, with the above notations, $\mathrm{length}(\mathfrak{M}^2/\mathfrak{M}^3) = 3$ and $\mathfrak{M}^4 = 0$. In particular we prove the following.

Theorem A. *Let R be a local, Gorenstein ring satisfying, with the above notations, $\mathrm{length}(\mathfrak{M}^2/\mathfrak{M}^3) = 3$ and $\mathfrak{M}^4 = 0$. Assume that the residue field is algebraically closed of characteristic different from 2, 3. Then P_R is rational.* ■

As a by-product, using all the above-mentioned known results about the rationality of P_R and taking into account that passing to R/J does not change the multiplicity (see [Ma], Theorems 14.13 and 17.11), we are finally able to prove the following.

Theorem B. *Let R be a local, Gorenstein ring with multiplicity at most 10. Assume that the residue field is algebraically closed of characteristic 0. Then P_R is rational.* ■

Notation. For the following definitions and results, we refer to [Ma].

A ring is a Noetherian ring. Let R be any local ring with maximal ideal $\mathfrak{N}$ and residue field $k := R/\mathfrak{N}$. We will denote by $\mathrm{char}(k)$ the characteristic of k.

The Samuel function of R, $\chi_R(t) := \mathrm{length}(R/\mathfrak{N}^{t+1})$, coincides for large t with a polynomial $p_R(t)$ of degree $\dim(R)$. The multiplicity of R is

$$\mathrm{mult}(R) := \lim_{t\to+\infty} \frac{\dim(R)!}{t^{\dim(R)}}\chi_R(t).$$

The embedding dimension of R is $\mathrm{emdim}(R) := \chi_R(1) - \chi_R(0) = \mathrm{length}(\mathfrak{N}/\mathfrak{N}^2)$.

For a ring R we denote by $\mathrm{depth}(R)$ the maximum length of a regular sequence in $\mathfrak{N}$. Recall that R is Cohen-Macaulay if $\dim(R) = \mathrm{depth}(R)$. If, in addition, the injective dimension of R is finite, then R is called Gorenstein.

Let A be an Artinian ring with maximal ideal $\mathfrak{M}$. If $\mathfrak{M}^e \neq 0$ and $\mathfrak{M}^{e+1} = 0$, following [Re], we define the level of A as e and denote it by $\mathrm{lev}(A)$ (some other authors prefer to use *maximum Socle degree* instead of level). In this case the Samuel function coincides with $\mathrm{length}(A)$ in the range $t = e, \dots, +\infty$, hence $\mathrm{mult}(A) = \mathrm{length}(A)$.

The Hilbert function of A, $H_A(t) := \mathrm{length}(\mathfrak{M}^t/\mathfrak{M}^{t+1})$, is the first difference function of χ_A. It is non-zero only in the range $t = 0, \dots, e$. We will thus write simply $H_A = (H_A(0), \dots, H_A(e))$. Moreover $\mathrm{length}(A) = \sum_{t=0}^{e} H_A(t)$.

The ring A is Cohen-Macaulay and it is Gorenstein if and only if its Socle $\mathrm{Soc}(A) := 0:_A \mathfrak{M}$ has dimension 1 over k. In particular $H_A(0) = H_A(e) = 1$.

2. Reduction to the Artinian case

Let R be a local ring with maximal ideal $\mathfrak{N}$ and let $r \in \mathfrak{N}$ be a regular element. We recall that in [Ta], it is proved

$$P_R(z) = \begin{cases} (1+z)P_{R/(r)}(z), & r \in \mathfrak{N}\setminus\mathfrak{N}^2,\\ (1-z^2)P_{R/(r)}(z), & r \in \mathfrak{N}^2. \end{cases}$$

In particular assume now that R is Cohen-Macaulay of positive Krull dimension $h := \dim(R)$. Then there is a parameter ideal $J \subseteq R$ which is a reduction of $\mathfrak{N}$ and such that $A := R/J$ is a local, Artinian, Cohen-Macaulay ring with maximal ideal $\mathfrak{M} := \mathfrak{N}/J$ and $\dim_k(A) = \mathrm{mult}(A) = \mathrm{mult}(R)$ (see Theorems 14.14, 14.13 and 17.11 of [Ma]).

Since A is Artinian, it follows that it is complete. Thus $A = S/I$, where S is a regular local ring and $I \subseteq S$ (see Theorem 29.4 of [Ma]). We can assume that $\mathrm{emdim}(A) = \mathrm{emdim}(S) = \dim(S)$ (i.e. I is contained in the square of the maximal ideal of S). Moreover it is well-known that, if R is Gorenstein, then the same is true for A.

Then, in order to prove the rationality of the Poincaré series of the rings R we considered in Theorem B, it is necessary and sufficient to prove the same property for local, Artinian, Gorenstein rings A with $\mathrm{length}(A) \le 10$.

In this case we have the following possibilities. Either A is a complete intersection or $\mathrm{emdim}(A) \le 3$ and A is not a complete intersection or $\mathrm{emdim}(A) = 4$ or, finally, H_A is one of the following:

$$(1, n, 1, \dots, 1), \quad (1, n, 2, 1, \dots, 1), \quad (1, 5, 3, 1),$$

where $n \ge 5$. We examine all the above cases in this section but the last one will be the object of the next section. To this purpose we recall some classical results in the following two theorems.

Theorem 2.1. *Let* $\mathrm{char}(k) \neq 2$ *and let A be a local, Artinian, Gorenstein ring with* $\mathrm{emdim}(A) \le 4$. *Then P_A is rational.*

Proof. If A is a complete intersection, see e.g. [Se] and [Ta]. If A is Gorenstein (but not a complete intersection) then the statement has been proved in the remaining cases in Theorem 9 of [Wi], when $n = 3$, and in Theorem A of [J-K-M] when $n = 4$. ■

If $H_A = (1, n, 1, \dots, 1)$, then A is called stretched (see [Sa1]). In [Sa2] the following theorem is proved.

Theorem 2.2. *Let* $\mathrm{char}(k) \neq 2$ *and let A be a local, Artinian, Gorenstein stretched ring. Then P_A is rational.* ■

Finally, if $H_A = (1, n, 2, 1, \dots, 1)$, then A is called almost stretched (see [E-V1]). In [E-V2] it is proved as follows.

Theorem 2.3. *Let* $\mathrm{char}(k) = 0$ *and let A be a local, Artinian, Gorenstein almost stretched ring. Then P_A is rational.* ■

Thus, in this section, we have checked Theorem B for each local, Gorenstein ring R of multiplicity at most 10, except when the quotient $A := R/J$ of R has Hilbert function $H_A = (1, 5, 3, 1)$.

Remark 2.4. In all the above cases the rationality of P_A, hence of P_R, is proved by computing it explicitly. More precisely, setting $n := \mathrm{emdim}(A)$, we have that P_A is

$$\frac{1}{(1-z)^n}$$

if A is a complete intersection,

$$\frac{(1+z)^3}{1-\varepsilon z^2-\varepsilon z^3+z^5},$$

where ε denotes the minimal number of generators of I, if $n=3$ and A is Gorenstein but not a complete intersection,

$$\frac{(1+z)^4}{f_A(z)},$$

if $n=4$ and A is Gorenstein but not a complete intersection, where $f_A(t)$ is one of the following according the structure of the $\mathrm{Tor}_*^S(A,k)$,

$$1-\varepsilon z^2-(2\varepsilon-2)z^3-\varepsilon z^4+z^6,$$

$$1-\varepsilon z^2-(2\varepsilon-5)z^3-(\varepsilon-6)z^4+2z^5-z^6-z^7,$$

$$1-\varepsilon z^2-(2\varepsilon-2-p)z^3-(\varepsilon-1-2p)z^4+(p+1)z^5-z^7$$

for some $p=1,\dots,\varepsilon$ and ε as for $n=3$, and finally

$$\frac{1}{1-nz+z^2}$$

if A is either a stretched or almost stretched Gorenstein ring with $n\geq 2$.

3. A class of local, Artinian, Gorenstein rings

As explained in the previous section we are reduced to study the case of rings R such that the quotient $A:=R/J$ satisfies $H_A=(1,5,3,1)$. So it is necessary to inspect such rings more carefully. In the present section we will deal with local, Artinian, Gorenstein rings A satisfying the more general condition $H_A=(1,n,3,1)$: if $n=2$, then A, being Gorenstein, is a complete intersection, thus $\mathrm{length}(\mathfrak{M}^3/\mathfrak{M}^4)\geq 2$ necessarily. Hence the Gorenstein condition forces $n\geq 3$.

When k is algebraically closed and $\mathrm{char}(k)\neq 2,3$, in §4 of [C-N] we gave a complete classification of such rings A assuming the restrictive extra hypothesis that A is also a k-algebra. We will recall the method and we will show that such an hypothesis is actually unnecessary.

Let $\mathfrak{M}=(a_1,a_2,a_3,\dots,a_n)$. If $a^2\in\mathfrak{M}^3$ for each $a\in\mathfrak{M}$, then $2a_1a_2=(a_1+a_2)^2-a_1^2-a_2^2\in\mathfrak{M}^3$. Thus we can assume a_1^2 is an element of a minimal set of generators of $\mathfrak{M}^2$. Repeating the above argument for $A/(a_1^2)$ we can finally assume that $\mathfrak{M}^2=(a_1^2,a_2^2,a_3^2)$.

Let $L:=(a_1,a_2,a_3)$ and let $V\subseteq\mathfrak{M}/\mathfrak{M}^2$ be the corresponding subspace. Thus we have three relations in I of the form

$$\alpha_1a_1^2+\alpha_2a_2^2+\alpha_3a_3^2+2\bar\alpha_1a_2a_3+2\bar\alpha_2a_1a_3+2\bar\alpha_3a_1a_2\in\mathfrak{M}^3, \tag{3.1}$$

where $\alpha_i,\bar\alpha_j\in A$, $i,j=1,2,3$, which induce analogous linearly independent relations in V. Hence we have a net $\mathcal{N}$ of conics in the projective space $\mathbb{P}(V)$. Let Δ be the discriminant curve of $\mathcal{N}$ in $\mathbb{P}(V)$. Then Δ is a plane cubic and the classification of $\mathcal{N}$ depends on the structure of Δ as explained in [Wa] (notice that the classification described in [Wa] in the complex case, holds as well if the base field k is an algebraically closed field with $\mathrm{char}(k) \neq 2, 3$).

In what follows we will describe how to modify the results proved in §4 of [C-N] for k-algebras, in order to extend them to the case of rings. We will examine only one case, namely the case of integral discriminant curve Δ (see §4.2 of [C-N]), the other ones being similar.

In this case we obtain that relations (3.1) thus become $a_1a_2+a_3^2, a_1a_3, a_2^2-6pa_3^2+qa_1^2 \in \mathfrak{M}^3$, where $p, q \in A$. We notice that if one of them is in $\mathfrak{M}$ then the corresponding monomial is in $\mathfrak{M}^3$, thus it can be assumed to be 0 in what follows. Hence, from now on we will assume that p, q are either invertible or 0.

We have

$$a_1^2a_2 = a_1^2a_3 = a_1a_2a_3 = a_1a_3^2 = a_2^2a_3 = a_3^3 = 0,$$
$$a_2a_3^2 = -a_1a_2^2 = qa_1^3, \quad a_2^3 = -6pqa_1^3,$$

thus $\mathfrak{M}^2 = (a_1^2, a_3^2, a_2a_3)$ and $\mathfrak{M}^3 = (a_1^3)$. Relation (3.1) thus become

$$a_1a_2 = -a_3^2 + \beta_{1,2}a_1^3, \quad a_1a_3 = \beta_{1,3}a_1^3, \quad a_2^2 = \alpha_{2,2}^1 a_1^2 + \alpha_{2,2}^3 a_3^2 + \beta_{2,2}a_1^3,$$

where $\alpha_{i,j}^h, \beta_{i,j} \in A$ are either invertible or 0, $\alpha_{2,2}^1 = -q$, $\alpha_{2,2}^3 = 6p$.

In general, we have relations of the form

$$a_ia_j = \alpha_{i,j}^1 a_1^2 + \alpha_{i,j}^2 a_2a_3 + \alpha_{i,j}^3 a_3^2 + \beta_{i,j}a_1^3, \quad i \geq 1,\ j \geq 4,$$

where the $\alpha_{i,j}^h, \beta_{i,j} \in A$ are as above and $\alpha_{i,j}^h = \alpha_{j,i}^h$, $\beta_{i,j} = \beta_{j,i}$.

Via $(a_2, a_3) \mapsto (a_2 + \beta_{1,2}a_1^2, a_3 + \beta_{1,3}a_1^2)$, we can assume $\beta_{1,2} = \beta_{1,3} = 0$.

If $\alpha_{2,2}^1 = 0$ then $a_2a_3 \in \mathrm{Soc}(A)\setminus \mathfrak{M}^3$, a contradiction. In particular $\alpha := \beta_{2,2}a_1 - \alpha_{2,2}^1 \in A\setminus \mathfrak{M}$, whence

$$a_1a_2 = -a_3^2, \quad a_1a_3 = 0, \quad a_2^2 = -\beta a_1^2 + \alpha a_3^2, \tag{3.2}$$

where β is invertible and $\alpha := \alpha_{2,2}^3$.

If $n = 3$ we have finished. Thus we will assume $n \geq 4$ from now on. Via $a_j \mapsto a_j + \alpha_{1,j}^1 a_1 + \beta_{1,j}a_1^2 + \alpha_{2,j}^2 a_3 + \beta_{2,j}a_3^2 + \alpha_{3,j}^2 a_2 + \beta_{3,j}a_2a_3$, we can assume $\alpha_{1,j}^1 = \beta_{1,j} = \alpha_{2,j}^2 = \beta_{2,j} = \alpha_{3,j}^2 = \beta_{3,j} = 0$, $j \geq 4$.

Since $a_1a_3 = 0$, it follows $\alpha_{1,j}^2 a_2a_3^2 = (a_1a_j)a_3 = a_1a_3a_j = (a_3a_j)a_1 = \alpha_{3,j}^1 a_1^3$. Thus $\alpha_{1,j}^2 = \alpha_{3,j}^1 = 0$, $j \geq 4$. Since $a_1a_2a_j = -a_3^2a_j = (a_3a_j)a_3 = 0$, we have $\alpha_{1,j}^3 a_2a_3^2 = (a_1a_j)a_2 = a_1a_2a_j = (a_2a_j)a_1 = \alpha_{2,j}^1 a_1^3$, thus $\alpha_{1,j}^3 = \alpha_{2,j}^1 = 0$, $j \geq 4$. Moreover $0 = (a_2a_j)a_3 = a_2a_3a_j = (a_3a_j)a_2 = \alpha_{3,j}^3 a_2a_3^2$, thus $\alpha_{3,j}^3 = 0$, $j \geq 4$. Finally $0 = a_2^2a_j = (a_2a_j)a_2 = \alpha_{2,j}^3 a_2a_3^2$, thus $\alpha_{3,j}^3 = 0$, $j \geq 4$. We conclude that $a_1a_j = a_2a_j = a_3a_j = 0$, $j \geq 4$.

It follows that $0 = (a_1a_i)a_j = (a_ia_j)a_1 = \alpha_{i,j}^1 a_1^3$, $0 = (a_2a_i)a_j = (a_ia_j)a_2 = \alpha_{i,j}^3 a_2a_3^2$, $0 = (a_3a_i)a_j = (a_ia_j)a_3 = \alpha_{i,j}^2 a_2a_3^2$, thus $a_ia_j = \beta_{i,j}a_1^3$, $i, j \geq 4$.

Consider the symmetric matrix $B := (\beta_{i,j})_{i,j\geq 4}$ and its class $\bar{B}$ modulo $\mathfrak{M}$. If $\det(B) \in \mathfrak{M}$, there would exist a non-zero $y := {}^t(y_4, \ldots, y_n) \in k^{\oplus n-3}$ such that $\bar{B}y = 0$. Let $\alpha_i \in A$

whose residue class in k is y_i, $i = 4, \dots, n$. By construction the element $a := \sum_{i=4}^n \alpha_i a_i$ has non-zero class in $\mathfrak{M} / \mathfrak{M}^2$, thus $a \notin \mathfrak{M}^3$.

On one hand, $a_1 a_j = a_2 a_j = a_3 a_j = 0$ (see above). On the other hand,

$$aa_j = \sum_{i=4}^n \alpha_i a_i a_j = \left(\sum_{i=4}^n \alpha_i \beta_{i,j}\right) a_1^3$$

if $j \geq 4$. Due to the choice of $\alpha_4, \dots, \alpha_n$ we have $\sum_{i=4}^n \alpha_i \beta_{i,j} \in \mathfrak{M}$. Thus $aa_j = 0$, $j \geq 4$, hence $a \in \mathrm{Soc}(A) \setminus \mathfrak{M}^3$, a contradiction, thus B is invertible. We can thus make a linear change on $a_4, \dots, a_n$ in such a way that

$$a_i a_j = \delta_{i,j} u_i a_1^3, \quad i, j \geq 4, \tag{3.3}$$

where $u_i \in A$ is invertible.

The field k is algebraically closed and the ring A is Artinian, thus complete. Due to the following immediate consequence of Hensel's lemma (see Theorem 8.3 of [Ma]), it follows that via a suitable homothety of $(a_1, \dots, a_n)$ we can assume $u_4 = \dots = u_n = 1$.

Lemma 3.4. *Let A be an Artinian ring with algebraically closed residue field. If $u \in A$ is invertible, for each positive integer $n \geq 1$ there exists $v \in A$ such that $v^n = u$* ■

Recall that we are assuming that $A = S/I$, where S is a regular local ring and $I \subseteq S$ is a suitable ideal. Combining equalities (3.2) and (3.3) above, we have found a minimal set of generators $\{x_1, \dots, x_n\}$ of the maximal ideal of S such that

$$(x_1x_2 + x_3^2, x_1x_3, x_2^2 + x_1^2 - \alpha x_3^2, x_ix_j, x_h^2 - x_1^3)_{\substack{1 \leq i < j \leq n,\ 4 \leq j \\ 4 \leq h \leq n}} \subseteq I,$$

where $\alpha \in A$.

Theorem 3.5. *Let k, A, S and I be as above. Then there is a minimal set of generators $\{x_1, \dots, x_n\}$ of the maximal ideal of S such that I is one of the following*:

$$I_1 := (x_1x_2 + x_3^2, x_1x_3, x_1^2 + x_2^2 - \alpha x_3^2, x_ix_j, x_h^2 - x_1^3)_{\substack{1 \leq i < j \leq n,\ 4 \leq j, \\ 4 \leq h \leq n}} \quad \alpha \in A,$$

$$I_{3-p} := (x_1^2, x_2^2, x_3^2 + 2px_1x_2, x_ix_j, x_h^2 - x_1x_2x_3)_{\substack{1 \leq i < j \leq n,\ 4 \leq j, \\ 4 \leq h \leq n}} \quad p = 0, 1,$$

$$I_4 := (x_2^3 - x_1^3, x_3^3 - x_1^3, x_ix_j, x_h^2 - x_1^3)_{\substack{1 \leq i < j \leq n, \\ 4 \leq h \leq n}},$$

$$I_5 := (x_1^2, x_1x_2, x_2x_3, x_2^3 - x_3^3, x_1x_3^2 - x_3^3, x_ix_j, x_h^2 - x_3^3)_{\substack{1 \leq i < j \leq n,\ 4 \leq j, \\ 4 \leq h \leq n}},$$

$$I_6 := (x_1^2, x_1x_2, 2x_1x_3 + x_2^2, x_3^3, x_2x_3^2, x_ix_j, x_h^2 - x_1x_3^2)_{\substack{1 \leq i < j \leq n,\ 4 \leq j. \\ 4 \leq h \leq n}}$$

Proof. When Δ is integral we have checked that $I_1 \subseteq I$, thus we have an epimorphism $\varphi \colon S/I_1 \twoheadrightarrow S/I$, hence $\mathrm{length}(S/I_1) \geq \mathrm{length}(S/I)$.

We claim that $\mathrm{length}(S/I_1) \leq \mathrm{length}(S/I)$. To this purpose we denote by $\mathcal{P}$ the maximal ideal of S/I_1 and we define the algebra

$$\mathrm{gr}(S/I_1) = \bigoplus_{i=0}^{\infty} \mathcal{P}^i / \mathcal{P}^{i+1} = k \oplus \mathcal{P}/\mathcal{P}^2 \oplus \mathcal{P}^2/\mathcal{P}^3 \oplus \mathcal{P}^3.$$

A set of generators of $\mathrm{gr}(S/I_1)$ as k-vector space is given by the monomials of degree up to 3 in the classes $\bar{x}_1, \dots, \bar{x}_n$ of $x_1, \dots, x_n$ respectively. The following relations modulo I_1 hold true in S:

$$\begin{aligned} x_1x_2 &= -x_3^2, \\ x_1x_3 &= 0, \\ x_2^2 &= -x_1^2 + \alpha x_3^2, \\ x_ix_j &= 0, \quad 1 \le i < j \le n,\ 4 \le j \\ x_h^2 &= x_1^3, \quad 4 \le h \le n. \end{aligned}$$

It follows that we have

$$\begin{aligned} \bar{x}_1\bar{x}_2 &= -\bar{x}_3^2, \\ \bar{x}_1\bar{x}_3 &= 0, \\ \bar{x}_2^2 &= -\bar{x}_1^2 + \alpha x_3^2, \\ \bar{x}_i\bar{x}_j &= 0, \quad 1 \le i < j \le n,\ 4 \le j \\ \bar{x}_h^2 &= 0, \quad 4 \le h \le n, \end{aligned}$$

in $\mathrm{gr}(S/I_1)$. We conclude that a set of generators of $\mathrm{gr}(S/I_1)$ as k-vector space is actually given by

$$1, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \dots, \bar{x}_n, \bar{x}_1^2, \bar{x}_2\bar{x}_3, \bar{x}_3^2, \bar{x}_1^3,$$

which are exactly $n+5$. Since $\mathrm{length}(\mathrm{gr}(S/I_1)) = \mathrm{length}(S/I_1)$ we deduce $\mathrm{length}(S/I_1) \le \mathrm{length}(S/I)$. We conclude that the equality $\mathrm{length}(S/I_1) = \mathrm{length}(S/I)$ holds, whence φ turns out to be an isomorphism i.e. $I_1 = I$.

Thus we have proved the above theorem when Δ is integral. When Δ is not integral the argument is similar: one has to imitate the methods of §4.3 to 4.7 of [C-N]. ∎

4. The proof of Theorem A

In this section we will prove the rationality of P_A for the rings defined in the previous section, thus completing the proof of Theorem A. To this purpose we will imitate again the proof given in [Sa2] (or [E-V2]). We will assume that the residue field of A is algebraically closed with $\mathrm{char}(k) \ne 2, 3$.

We have to examine the six ideals described in Theorem 3.5. Since the computations are similar, we will consider again only one case, e.g. the last ideal. In this case

$$I_6 := (x_1^2, x_1x_2, 2x_1x_3 + x_2^2, x_3^3, x_2x_3^2, x_ix_j, x_h^2 - x_1x_3^2)_{\substack{1 \le i < j \le n,\ 4 \le j \\ 4 \le h \le n}}.$$

Notice that in our case the Socle of A, $\mathrm{Soc}(A)$, is generated by the class of $x_1x_3^2$. We recall that

$$P_A(z) = \frac{P_{A/\mathrm{Soc}(A)}(z)}{1 + z^2 P_{A/\mathrm{Soc}(A)}(z)}, \tag{4.1}$$

(see [RR] or [A-L]), hence it suffices to prove the rationality of $P_{A/\mathrm{Soc}(A)}(z)$. We have

$$\begin{aligned} A/\mathrm{Soc}(A) &\cong A/(x_1x_3^2) \\ &\cong S/(x_1^2, x_1x_2, 2x_1x_3+x_2^2, x_3^3, x_2x_3^2, x_ix_j, x_h^2, x_1x_3^2)_{\substack{1\le i<j\le n,\ 4\le j \\ 4\le h\le n}}. \end{aligned}$$

Since the classes of $x_4, \dots, x_n$ are trivially in $\mathrm{Soc}(A/(x_1x_3^2))$, hence it follows from Proposition 3.4.4 of [G-L] that

$$P_{A/\mathrm{Soc}(A)}(z) = \frac{P_{S_0/J}(z)}{1-(n-3)zP_{S_0/J}(z)}, \tag{4.2}$$

where $S_0 := S/(x_4, \dots, x_n)$ and $J := (x_1^2, x_1x_2, 2x_1x_3 + x_2^2, x_3^3, x_2x_3^2, x_1x_3^2)$, hence it suffices to prove the rationality of $P_{S_0/J}(z)$. Consider now the ideal $H := (x_1^2, x_1x_2, 2x_1x_3+x_2^2, x_3^3, x_2x_3^2) \subseteq S_0$. The quotient S_0/H is a local, Artinian, Gorenstein ring since H coincides with the ideal I_6 defined in Theorem 3.5 when $n = 3$ and its Socle is generated by the class of $x_1x_3^2$. Thus again by [RR] or [A-L] we obtain that

$$P_{S_0/H}(z) = \frac{P_{S_0/J}(z)}{1+z^2P_{S_0/J}(z)},$$

hence

$$P_{S_0/J}(z) = \frac{P_{S_0/H}(z)}{1-z^2P_{S_0/H}(z)}. \tag{4.3}$$

Thus it suffices to prove the rationality of $P_{S_0/H}(z)$.

We finally conclude with the proof of Theorem A in the following.

Theorem 4.4. *Let k be algebraically closed with* $\mathrm{char}(k) \neq 2, 3$ *and let A be a local, Artinian, Gorenstein ring with Hilbert function $H_A = (1, n, 3, 1)$. Then P_A is rational.* ■

Proof. Since $\mathrm{emdim}(S_0/H) = 3$, the statement follows by combining formulas (4.1), (4.2), (4.3) and Theorem 2.2 in the case $A \cong S/I_1$. The same argument gives the rationality of the Poincaré series also for $A \cong S/I_t$ with $t = 1, \dots, 5$. ■

As explained in the previous section, an immediate consequence of the above theorem is the proof of Theorem B when k is algebraically closed with $\mathrm{char}(k) = 0$.

Remark 4.5. Again P_A, hence P_R, can be explicitely computed via the method described above. Indeed

$$P_A(z) = \begin{cases} \frac{1}{1-nz+3z^2-z^3}, & \text{if } t \le 3, \\ \frac{(1+z)^3}{1-(n-3)z-\varepsilon z^2-\varepsilon z^3+z^5}, & \text{if } t \ge 4, \end{cases}$$

where $\varepsilon = \binom{n}{2} + 1$. Notice that, also in this case, ε is the minimal number of generators of I.

Remark 4.6. We list here some comments to our proof and about some possible improvements.

In order to prove Theorem B we assumed $\mathrm{char}(k) = 0$. This is necessary only for proving the rationality of P_A when A is almost stretched. In order to have rings of multiplicity at most 10 the only interesting cases for the Hilbert function of A are only $(1, 5, 2, 1)$, $(1, 5, 2, 1, 1)$ and $(1, 6, 2, 1)$, because the other ones are covered by the results in [Ta], [Wi], [J-K-M]. Imitating the method described in §3 of [C-N], modified as explained in §3 above, one can easily extend the result of rationality of P_A also to the case $\mathrm{char}(k) \neq 2, 3$ (and k not necessarily algebraically closed).

We now have a few words about the other condition on k, namely algebraic closure. We have not checked all the details but we feel that, via a careful analysis of the classification of nets of conics given in [Wa] which is at the base of the proof of Theorem 3.5, such a condition can be discarded.

If this is true, Theorem A obviously holds under the general hypothesis that the characteristic of k is neither 2 nor 3, for each ring R for which there exists a parameter ideal $J \subseteq R$ satisfying $\mathrm{length}(\mathfrak{M}^2 / \mathfrak{M}^3) = 3$, where, as usual $\mathfrak{M} := \mathfrak{N} / J$ and $\mathfrak{N}$ is the maximal ideal of R.

Finally we notice that, in order to extend our result also to all Gorenstein rings of multiplicity 11, we need to study the Poincaré series of local Artinian, Gorenstein rings A satisfying either $H(A) = (1, 5, 3, 1, 1)$ or $H(A) = (1, 5, 4, 1)$. This second case seems to be quite difficult since, in our philosophy, it is related to the classification up to projectivities of linear systems of projective dimension 3 of quadrics in $\mathbb{P}^3_k$.

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