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New characterizations of fusion frames (frames of subspaces)

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Abstract. In this article, we give new characterizations of fusion frames, on the properties of their synthesis operators, on the behavior of fusion frames under bounded operators with closed range, and on erasures of subspaces of fusion frames. Furthermore we show that every fusion frame is the image of an orthonormal fusion basis under a bounded surjective operator.

Keywords. Frame; fusion frame (frames of subspaces); exact fusion frame; Bessel fusion sequence; orthonormal fusion basis; pseudo-inverses.

1. Introduction

A frame is a redundant set of vectors in a Hilbert space $\mathcal{H}$ with the property that provide usually non-unique representations of vectors in terms of the frame elements, and they have been applied in wavelet and frequency analysis theories, filter bank theory, signal and image processing. For more details about the theory and applications of frames, we refer the reader to [7], [9], [11], [14] and [15]. The fusion frames (frames of subspaces) which were introduced by Casazza and Kutyniok in [3] and Fornasier in [12] are a natural generalization of frame theory and related to the construction of global frames from local frames in Hilbert spaces. We extend some of the known results of frames to fusion frames.

The paper is organized as follows. In §2, we briefly recall the definitions and basic properties, and several characterizations of fusion frames. In §3, we study the relationship between operators and fusion frames. In §4, we study the erasure of subspaces of a fusion frame.

Let $\mathcal{H}$ be a separable Hilbert space and let I, J, J_i be countable (or finite) index sets. If W is a closed subspace of $\mathcal{H}$, we denote the orthogonal projection of $\mathcal{H}$ onto W by π_W.

A sequence $\mathcal{F} = \{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f\ ,\ f_i\rangle|^2 \leq B\|f\|^2 \quad \text{for all } f \in \mathcal{H}. \tag{1}$$

The numbers A, B are called lower and upper frame bounds, respectively.

2. The characterizations of fusion frames

In this section we get several characterizations of fusion frames. We refer to [1, 3, 5, 6, 12] for the basic theory of fusion frames and their applications.

DEFINITION 2.1

Let $\mathcal{W} = \{W_i\}_{i\in I}$ be a sequence of closed subspaces in $\mathcal{H}$, and let $\mathcal{V} = \{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. We say that $\mathcal{W}_v = \{(W_i, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$, if there exist constants $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{i\in I} v_i^2 \|\pi_{W_i}(f)\|^2 \leq D\|f\|^2 \qquad \text{for all } f \in \mathcal{H}. \tag{2}$$

The numbers C, D are called the fusion frame bounds. The family $\mathcal{W}_v$ is called a C-tight fusion frame if $C = D$, it is a Parseval fusion frame if $C = D = 1$, and v-uniform if $v = v_i = v_j$ for all $i, j \in I$. If the right-handed inequality of (2) holds, then we say that $\mathcal{W}_v$ is a Bessel fusion sequence with Bessel fusion bound D. Moreover we say that $\mathcal{W} = \{W_i\}_{i\in I}$ is an orthonormal fusion basis for $\mathcal{H}$ if $\mathcal{H} = \bigoplus_{i\in I} W_i$.

For each Bessel fusion sequence $\mathcal{W}_v = \{(W_i, v_i)\}_{i\in I}$ of $\mathcal{H}$, we define the representation space associated with $\mathcal{W}_v$ by

$$\ell^2(\mathcal{H}, I) = \left\{ \{f_k\}_{k\in I} | f_k \in \mathcal{H} \quad \text{and} \quad \sum_{k\in I} \|f_k\|^2 < \infty \right\} \tag{3}$$

with inner product given by

$$\langle \{f_k\}_{k\in I} \ , \ \{g_k\}_{k\in I} \rangle = \sum_{i\in I} \langle f_i, g_i \rangle. \tag{4}$$

Let $\mathfrak{E} = \{e_j\}_{j\in J}$ be an orthonormal basis for $\mathcal{H}$. Define $e_{ij} = \{\delta_{ik} e_j\}_{k\in I}$ for all $i \in I, j \in J$ where δ_{ik} is the Kronecker delta. Then $\mathcal{E}_{\mathfrak{E}} = \{e_{ij}\}_{i\in I, j\in J}$ is an orthonormal basis for $\ell^2(\mathcal{H}, I)$. The sequence $\mathcal{E}_{\mathfrak{E}}$ is called the associated orthonormal basis to $\mathfrak{E}$ in $\ell^2(\mathcal{H}, I)$.

DEFINITION 2.2

Let $\mathcal{W}_v$ be a Bessel fusion sequence for $\mathcal{H}$. The *synthesis* operator $T_{\mathcal{W}_v} \colon \ell^2(\mathcal{H}, I) \longrightarrow \mathcal{H}$ is defined by

$$T_{\mathcal{W}_v}(\{f_i\}_{i\in I}) = \sum_{i\in I} v_i \pi_{W_i}(f_i) \quad \forall \{f_i\}_{i\in I} \in \ell^2(\mathcal{H}, I). \tag{5}$$

The adjoint operator $T^*_{\mathcal{W}_v} \colon H \longrightarrow \ell^2(H, I)$ given by $T^*_{\mathcal{W}_v}(f) = \{v_i \pi_{W_i}(f)\}_{i\in I}$ is called the *analysis* operator.

We can also characterize Bessel fusion sequences in terms of their synthesis operators as in frame theory.

Theorem 2.3. *A sequence $\mathcal{W}_v$ is a Bessel fusion sequence with Bessel fusion bound D for $\mathcal{H}$ if and only if the synthesis operator $T_{\mathcal{W}_v}$ is a well-defined bounded operator from $\ell^2(\mathcal{H}, I)$ into $\mathcal{H}$ and $\|T_{\mathcal{W}_v}\| \leq \sqrt{D}$.*

Proof. This claim follows immediately from the fact that for each $J \subseteq I$ with $|J| < \infty$ and each $\{f_i\}_{i\in I} \in \ell^2(\mathcal{H}, I)$ we have

$$\begin{aligned}
\left\| \sum_{i\in J} v_i \pi_{W_i}(f_i) \right\|^2 &= \sup_{\|g\|=1} \left| \left\langle g\ ,\ \sum_{i\in J} v_i \pi_{W_i}(f_i) \right\rangle \right|^2 \\
&= \sup_{\|g\|=1} \left| \sum_{i\in J} v_i \langle \pi_{W_i}(g)\ ,\ f_i \rangle \right|^2 \\
&\le \sup_{\|g\|=1} \left(\sum_{i\in J} v_i \|\pi_{W_i}(g)\|\ \|f_i\| \right)^2 \\
&\le \sup_{\|g\|=1} \sum_{i\in J} {v_i}^2 \|\pi_{W_i}(g)\|^2 \sum_{i\in J} \|f_i\|^2 \le D \sum_{i\in J} \|f_i\|^2.
\end{aligned}$$

The opposite implication is obvious. ∎

If $\mathcal{W}_\upsilon = \{(W_i, v_i)\}_{i\in\mathbb{N}}$ is a Bessel fusion sequence in $\mathcal{H}$, then the operator $T^*_{\mathcal{W}_\upsilon} T_{\mathcal{W}_\upsilon} \colon \ell^2(\mathcal{H}) \longrightarrow \ell^2(\mathcal{H})$ given by $T^*_{\mathcal{W}_\upsilon} T_{\mathcal{W}_\upsilon}(\{f_i\}_{i\in\mathbb{N}}) = \{\sum_{i\in\mathbb{N}} v_k v_i \pi_{W_k}\pi_{W_i}(f_i)\}_{k\in\mathbb{N}}$ is a bounded operator. If $\mathfrak{E} = \{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$, then the *Gram* matrix associated with $\mathcal{W}_\upsilon$ with respect to $\mathcal{E}_{\mathfrak{E}}$ is defined by

$$\{v_i v_m < \pi_{W_i}(e_j), \pi_{W_m}(e_n) >\}_{i,j,m,n\in\mathbb{N}}.$$

Theorem 2.4. *The following conditions are equivalent*:

(i) *$\mathcal{W}_\upsilon$ is a Bessel fusion sequence with Bessel fusion bound D.*
(ii) *The Gram matrix associated with $\mathcal{W}_\upsilon$ with respect to $\mathcal{E}_{\mathfrak{E}}$ defines a bounded operator on $\ell^2(\mathcal{H})$, with norm at most D.*

Proof. The implication (i) ⇒ (ii) follows from Theorem 2.3. To prove (ii) ⇒ (i) suppose that $\{f_i\}_{i\in\mathbb{N}} \in \ell^2(\mathcal{H})$, then for every $n, m \in \mathbb{N}, n > m$, we have

$$\begin{aligned}
\left\| \sum_{k=1}^{n} v_k \pi_{W_k}(f_k) - \sum_{k=1}^{m} v_k \pi_{W_k}(f_k) \right\|^4 &= \left\| \sum_{k=n+1}^{m} v_k \pi_{W_k}(f_k) \right\|^4 \\
&= \left| \left\langle \sum_{k=n+1}^{m} v_k \pi_{W_k}(f_k), \sum_{i=n+1}^{m} v_i \pi_{W_i}(f_i) \right\rangle \right|^2 \\
&= \left| \sum_{k=n+1}^{m} \left\langle f_k, \sum_{i=n+1}^{m} v_k v_i \pi_{W_k}\pi_{W_i}(f_i) \right\rangle \right|^2 \\
&\le \left(\sum_{k=n+1}^{m} \|f_k\|^2 \right) \left(\sum_{k=n+1}^{m} \left\| \sum_{i=n+1}^{m} v_k v_i \pi_{W_k}\pi_{W_i}(f_i) \right\|^2 \right) \\
&\le D^2 \left(\sum_{k=n+1}^{m} \|f_k\|^2 \right)^2.
\end{aligned}$$

By Theorem 2.3, $\mathcal{W}_\upsilon$ is a Bessel fusion sequence with Bessel fusion bound D. ∎

Let $\mathcal{H}, \mathcal{K}$ be two Hilbert spaces and $B(\mathcal{H}, \mathcal{K})$ the set of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$. The range and the null space of $T \in B(\mathcal{H}, \mathcal{K})$ are denoted by R_T and N_T, respectively. Suppose that the operator $T \in B(\mathcal{H}, \mathcal{K})$ has a closed range. Then there exists a unique bounded operator $T^{\dagger}\colon \mathcal{K} \longrightarrow \mathcal{H}$ satisfying:

$$TT^{\dagger}T = T, \quad T^{\dagger}TT^{\dagger} = T^{\dagger}, \quad (T^{\dagger}T)^* = T^{\dagger}T, \quad (TT^{\dagger})^* = TT^{\dagger}. \tag{6}$$

The operator $T^{\dagger}$ is called the pseudo-inverse operator of T. If T is a bounded invertible operator, then $T^{\dagger} = T^{-1}$.

DEFINITION 2.5

The reduced minimum modulus $\gamma(T)$ of an operator $T \in B(\mathcal{H}, \mathcal{K})$ is defined by

$$\gamma(T) = \inf\{\|T(f)\|\colon \|f\| = 1, f \in N_T^{\perp}\}. \tag{7}$$

It is well-known that $\gamma(T) = \gamma(T^*) = \gamma(T^*T)^{\frac{1}{2}}$. It was proved in [10] that an operator T has closed range if and only if $\gamma(T) > 0$, and in Hilbert spaces $\gamma(T) = \|T^{\dagger}\|^{-1}$.

Let $\mathcal{W}_\upsilon$ be a fusion frame for $\mathcal{H}$ with fusion frame bounds C and D. The fusion frame operator $S_{\mathcal{W}_\upsilon}$ for $\mathcal{W}_\upsilon$ is defined by

$$S_{\mathcal{W}_\upsilon}\colon \mathcal{H} \longrightarrow \mathcal{H} \qquad S_{\mathcal{W}_\upsilon}(f) = T_{\mathcal{W}_\upsilon} T^*_{\mathcal{W}_\upsilon}(f) = \sum_{i \in I} \upsilon_i^2 \pi_{W_i}(f), \tag{8}$$

which is a positive, self-adjoint, invertible operator on $\mathcal{H}$ with $C \cdot \mathrm{Id}_{\mathcal{H}} \leq S_{\mathcal{W}_\mathcal{V}} \leq D \cdot \mathrm{Id}_{\mathcal{H}}$. This provides for all $f \in \mathcal{H}$ the reconstruction formula as follows.

$$\begin{aligned}\sum_{i \in I} \upsilon_i^2 S^{-1}_{\mathcal{W}_\upsilon} \pi_{W_i}(f) &= S^{-1}_{\mathcal{W}_\upsilon} S_{\mathcal{W}_\mathcal{V}}(f) = f \\ &= S_{\mathcal{W}_\mathcal{V}} S^{-1}_{\mathcal{W}_\upsilon}(f) = \sum_{i \in I} \upsilon_i^2 \pi_{W_i} S^{-1}_{\mathcal{W}_\upsilon}(f).\end{aligned} \tag{9}$$

We now give a characterization of fusion frames in terms of the associated synthesis and analysis operators. A similar characterization of fusion frames appeared in [3].

Theorem 2.6. *Let $\mathcal{W} = \{W_i\}_{i \in I}$ be a family of closed subspaces in $\mathcal{H}$, and let $\mathcal{V} = \{\upsilon_i\}_{i \in I}$ be a family of weights. Then the following conditions are equivalent*:

(i) *$\mathcal{W}_\upsilon$ is a fusion frame for $\mathcal{H}$.*
(ii) *The synthesis operator $T_{\mathcal{W}_\upsilon}$ is a bounded linear operator from $\ell^2(\mathcal{H}, I)$ onto $\mathcal{H}$.*
(iii) *The analysis operator $T^*_{\mathcal{W}_\upsilon}$ is injective with closed range.*

Proof. This claim holds in an analogous way as in frame theory. ■

COROLLARY 2.7

The optimal fusion frame bounds for $\mathcal{W}_\upsilon$ are

$$C = \|T^{\dagger}_{\mathcal{W}_\upsilon}\|^{-2} = \gamma(T_{\mathcal{W}_\upsilon})^2 \quad \textit{and} \quad D = \|S_{\mathcal{W}_\upsilon}\| = \|T_{\mathcal{W}_\upsilon}\|^2.$$

Proof. By using Theorem 2.6 we have

$$C = \inf_{\|f\|=1} \sum_{i\in I} {v_i}^2 \|\pi_{W_i}(f)\|^2 = \inf_{\|f\|=1} \|T^*_{\mathcal{W}_v}(f)\|^2 = \gamma(T^*_{\mathcal{W}_v})^2$$
$$= \gamma(T_{\mathcal{W}_v})^2 = \|T^{\dagger}_{\mathcal{W}_v}\|^{-2},$$
$$D = \sup_{\|f\|=1} \sum_{i\in I} {v_i}^2 \|\pi_{W_i}(f)\|^2 = \sup_{\|f\|=1} |\langle S_{\mathcal{W}_v}(f),\ f\rangle|$$
$$= \|S_{\mathcal{W}_v}\| = \|T_{\mathcal{W}_v}\|^2. \qquad \blacksquare$$

The definition shows that if $\mathcal{W}_v$ is a fusion frame for $\mathcal{H}$, then $\mathcal{W} = \{W_i\}_{i\in I}$ is complete in $\mathcal{H}$, that is $\overline{\text{span}}\{W_i\}_{i\in I} = \mathcal{H}$. We say that $\mathcal{W}_v$ is a fusion frame sequence if it is a fusion frame for $\overline{\text{span}}\{W_i\}_{i\in I}$. Theorem 2.6 leads to a statement about fusion frame sequences.

COROLLARY 2.8

A sequence $\mathcal{W}_v$ is a fusion frame sequence if and only if

$$T_{\mathcal{W}_v}\colon \ell^2(\mathcal{H}, I) \longrightarrow \mathcal{H}, \quad T_{\mathcal{W}_v}(\{f_i\}_{i\in I}) = \sum_{i\in I} v_i \pi_{W_i}(f_i),$$

is a well-defined bounded operator with a closed range.

Proof. This follows immediately from Theorem 2.6. $\blacksquare$

Let W and Z be two closed subspaces of $\mathcal{H}$. The angle from W to Z is defined as the unique number $\theta(W, Z) \in \left[0, \frac{\pi}{2}\right]$ for which

$$\cos(\theta(W, Z)) = \inf_{f\in W, \|f\|=1} \|\pi_Z(f)\|.$$

Theorem 2.9. *Let $\mathcal{W}_v = \{(W_i, v_i)\}_{i\in I}$ and $\mathcal{Z}_\lambda = \{(Z_i, \lambda_i)\}_{i\in I}$ be two fusion frame sequences for $\mathcal{H}$, and let $W = \overline{\text{span}}\{W_i\}_{i\in I}$, $Z = \overline{\text{span}}\{Z_i\}_{i\in I}$. The following conditions are equivalent*:

(i) $\mathcal{H} = W \bigoplus Z^{\perp}$.
(ii) $\mathcal{H} = Z \bigoplus W^{\perp}$.
(iii) *The operator $T^*_{\mathcal{Z}_\lambda} T_{\mathcal{W}_v}$ is a bounded, invertible operator from $R_{T^*_{\mathcal{W}_v}}$ onto $R_{T^*_{\mathcal{Z}_\lambda}}$.*
(iv) $\cos(\theta(W, Z)) > 0$ *and* $\cos(\theta(Z, W)) > 0$.

Proof.
(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv). This follows from Theorem 2.3 of [17].
(i) $\Rightarrow$ (iii). By Corollary 2.8, $T^*_{\mathcal{Z}_\lambda} T_{\mathcal{W}_v}$ is a bounded operator. Therefore it suffices to show that $T^*_{\mathcal{Z}_\lambda} T_{\mathcal{W}_v}$ is bijective. Let $\{f_i\}_{i\in I} \in R_{T^*_{\mathcal{W}_v}}$ and let $h = T_{\mathcal{W}_v}(\{f_i\}_{i\in I})$. If $T^*_{\mathcal{Z}_\lambda} T_{\mathcal{W}_v}(\{f_i\}_{i\in I}) = 0$, then $h \in Z^{\perp}$, thus $h = 0$ and it follows that $\{f_i\}_{i\in I} \in N_{T_{\mathcal{W}_v}}$. Since $N_{T_{\mathcal{W}_v}} = R^{\perp}_{T^*_{\mathcal{W}_v}}$ hence $\{f_i\}_{i\in I} \in R_{T^*_{\mathcal{W}_v}} \cap R^{\perp}_{T^*_{\mathcal{W}_v}} = \{0\}$, which implies that $T^*_{\mathcal{Z}_\lambda} T_{\mathcal{W}_v}$ is injective. Now let $\{f_i\}_{i\in I} \in R_{T^*_{\mathcal{Z}_v}}$. Then there is some $f \in W$ such that $T^*_{\mathcal{Z}_\lambda}(f) = \{f_i\}_{i\in I}$

and by (9) we have

$$\begin{aligned}\{f_i\}_{i\in I} &= T^*_{\mathcal{Z}_\lambda}(f) = T^*_{\mathcal{Z}_\lambda}\left(\sum_{i\in I} \upsilon_i^2 \pi_{W_i} S^{-1}_{\mathcal{W}_\upsilon}(f)\right)\\ &= T^*_{\mathcal{Z}_\lambda} T_{\mathcal{W}_\upsilon}(\{v_i \pi_{W_i} S^{-1}_{\mathcal{W}_\upsilon}(f)\}_{i\in I}).\end{aligned}$$

It follows that $T^*_{\mathcal{Z}_\lambda} T_{\mathcal{W}_\upsilon}$ is surjective.

(iii) $\Rightarrow$ (i). Suppose that $h \in \mathcal{H}$. Then $h = g_1 + g_2$ where $g_1 \in Z$ and $g_2 \in Z^\perp$. Put $T^*_{\mathcal{Z}_\lambda}(g_1) = \{f_i\}_{i\in I}$ and $u = T_{\mathcal{W}_\upsilon}(T^*_{\mathcal{Z}_\lambda} T_{\mathcal{W}_\upsilon})^{-1}(\{f_i\}_{i\in I})$. Then $u \in W$ and we have

$$T^*_{\mathcal{Z}_\lambda}(g_1 - u) = T^*_{\mathcal{Z}_\lambda}(g_1) - T^*_{\mathcal{Z}_\lambda}(u) = \{f_i\}_{i\in I} - \{f_i\}_{i\in I} = 0,$$

which implies that $g_1 - u \in Z^\perp$. Thus $h = u + (g_1 - u + g_2) \in W + Z^\perp$. Let $f \in W \cap Z^\perp$, since $T^*_{\mathcal{W}_\upsilon} S^{-1}_{\mathcal{W}_\upsilon}(f) = \{v_i \pi_{W_i} S^{-1}_{\mathcal{W}_\upsilon}(f)\}_{i\in I}$, hence $\{v_i \pi_{W_i} S^{-1}_{\mathcal{W}_\upsilon}(f)\}_{i\in I} \in R_{T^*_{\mathcal{W}_\upsilon}}$. Thus by (9) we compute

$$T^*_{\mathcal{Z}_\lambda} T_{\mathcal{W}_\upsilon}(\{v_i \pi_{W_i} S^{-1}_{\mathcal{W}_\upsilon}(f)\}_{i\in I}) = T^*_{\mathcal{Z}_\lambda}\left(\sum_{i\in I} \upsilon_i^2 \pi_{W_i} S^{-1}_{\mathcal{W}_\upsilon}(f)\right) = T^*_{\mathcal{Z}_\lambda}(f) = 0.$$

This show that $f = 0$, and so $\mathcal{H} = W \bigoplus Z^\perp$. ∎

In the next theorem we give a characterization of fusion frames which keeps the information about the fusion frame bounds.

Theorem 2.10. *A sequence $\mathcal{W}_\upsilon$ is a fusion frame for $\mathcal{H}$ with bounds C, D if and only if the following conditions are satisfied.*

(i) *$\mathcal{W} = \{W_i\}_{i\in I}$ is complete in $\mathcal{H}$.*

(ii) *The synthesis operator $T_{\mathcal{W}_\upsilon}$ is well-defined on $\ell^2(\mathcal{H}, I)$ and for every $\{f_i\}_{i\in I} \in N^\perp_{T_{\mathcal{W}_\upsilon}}$,*

$$C \sum_{i\in I} \|f_i\|^2 \le \|T_{\mathcal{W}_\upsilon}(\{f_i\}_{i\in I})\|^2 \le D \sum_{i\in I} \|f_i\|^2. \tag{10}$$

Proof. Let $\mathcal{W}_\upsilon$ be a fusion frame with bounds C and D. Then by Lemma 3.4 of [3], $\mathcal{W}$ is complete in $\mathcal{H}$. Theorem 2.3 shows that the right-handed inequality (10) holds, and Theorem 2.6 implies that $N^\perp_{T_{\mathcal{W}_\upsilon}} = R_{T^*_{\mathcal{W}_\upsilon}}$, that is

$$N^\perp_{T_{\mathcal{W}_\upsilon}} = \{\{v_i \pi_{W_i}(f)\}_{i\in I} : \quad f \in \mathcal{H}\}.$$

For every $f \in \mathcal{H}$ we also have

$$\begin{aligned}\left(\sum_{i\in I} {v_i}^2 \|\pi_{W_i}(f)\|^2\right)^2 &= |\langle S_{\mathcal{W}_\upsilon}(f)\ ,\ f\rangle|^2 \le \|S_{\mathcal{W}_\upsilon}(f)\|^2 \|f\|^2\\ &\le \frac{1}{C}\|S_{\mathcal{W}_\upsilon}(f)\|^2 \sum_{i\in I} {v_i}^2 \|\pi_{W_i}(f)\|^2,\end{aligned}$$

hence

$$C\sum_{i\in I} {v_i}^2\|\pi_{W_i}(f)\|^2 \leq \|S_{\mathcal{W}_v}(f)\|^2 = \|T_{\mathcal{W}_v}(\{v_i\pi_{W_i}(f)\}_{i\in I})\|^2.$$

To prove the converse implication, assume that the conditions (i) and (ii) are satisfied. Then by the right-handed inequality (10) and Theorem 2.3, $\mathcal{W}_v$ satisfies the upper fusion frame condition with bound D. For lower fusion frame condition, we prove that $T_{\mathcal{W}_v}$ is onto. First we show that $R_{T_{\mathcal{W}_v}}$ is closed. Suppose that $\{g_n\}_{n=1}^{\infty}$ is a sequence in $R_{T_{\mathcal{W}_v}}$. Then we can find a sequence $\{f_n\}_{n=1}^{\infty}$ in $N^{\perp}_{T_{\mathcal{W}_v}}$ such that $g_n = T_{\mathcal{W}_v}(f_n)$ for all $n \in \mathbb{N}$. Now if $\{g_n\}_{n=1}^{\infty}$ converges to some $g \in \mathcal{H}$, then (10) implies that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Therefore $\{f_n\}_{n=1}^{\infty}$ converges to some $f \in \ell^2(\mathcal{H}, I)$ which by continuity of $T_{\mathcal{W}_v}$ we have $T_{\mathcal{W}_v}(f) = g$. Thus $R_{T_{\mathcal{W}_v}}$ is closed. By Theorem 2.6, $\mathcal{W}_v$ is a fusion frame for $R_{T_{\mathcal{W}_v}}$ and hence by $R_{T_{\mathcal{W}_v}} = \overline{\operatorname{span}}\{W_i\}_{i\in I}$, it follows that $\mathcal{H} = R_{T_{\mathcal{W}_v}}$. By (6) we know that the operators $T^{\dagger}_{\mathcal{W}_v}T_{\mathcal{W}_v}$ and $T_{\mathcal{W}_v}T^{\dagger}_{\mathcal{W}_v}$ are the orthogonal projections onto $N^{\perp}_{T_{\mathcal{W}_v}}$ and $R_{T_{\mathcal{W}_v}}$ respectively. Thus for each $\{f_i\}_{i\in I} \in \ell^2(\mathcal{H}, I)$ we have

$$C\|T^{\dagger}_{\mathcal{W}_v}T_{\mathcal{W}_v}(\{f_i\}_{i\in I})\|^2 \leq \|T_{\mathcal{W}_v}T^{\dagger}_{\mathcal{W}_v}T_{\mathcal{W}_v}(\{f_i\}_{i\in I})\|^2 = \|T_{\mathcal{W}_v}(\{f_i\}_{i\in I})\|^2.$$

We also have $N_{T^{\dagger}_{\mathcal{W}_v}} = R^{\perp}_{T_{\mathcal{W}_v}} = \{0\}$, so $\|T^{\dagger}_{\mathcal{W}_v}\|^2 \leq \frac{1}{C}$. From Lemma 2.4 of [8], we obtain $\|(T^*_{\mathcal{W}_v})^{\dagger}\| = \|(T^{\dagger}_{\mathcal{W}_v})^*\| = \|T^{\dagger}_{\mathcal{W}_v}\|$ hence $\|(T^*_{\mathcal{W}_v})^{\dagger}\|^2 \leq \frac{1}{C}$. Again by (6) the operator $(T^*_{\mathcal{W}_v})^{\dagger}T^*_{\mathcal{W}_v}$ is the orthogonal projection onto $N^{\perp}_{T^*_{\mathcal{W}_v}} = R_{T_{\mathcal{W}_v}} = \mathcal{H}$, hence for all $f \in \mathcal{H}$ we compute

$$\begin{aligned}\|f\|^2 &= \|(T^*_{\mathcal{W}_v})^{\dagger}T^*_{\mathcal{W}_v}(f)\|^2 \leq \|(T^*_{\mathcal{W}_v})^{\dagger}\|^2\|T^*_{\mathcal{W}_v}(f)\|^2\\ &\leq \frac{1}{C}\|T^*_{\mathcal{W}_v}(f)\|^2 = \frac{1}{C}\sum_{i\in I} v_i^2\|\pi_{W_i}(f)\|^2.\end{aligned}$$

This show that the lower fusion frame condition satisfies. ■

In the following theorems we give two more characterizations of fusion frames in terms of orthonormal fusion bases. First we show that each fusion frame is the image of the orthonormal fusion basis under the synthesis operator (which is a bounded surjective operator).

Theorem 2.11. *Let $\mathcal{W}_v$ be a fusion frame for $\mathcal{H}$. Then there is an orthonormal fusion basis $\mathcal{N} = \{N_i\}_{i\in I}$ for $\ell^2(\mathcal{H}, I)$ such that $T_{\mathcal{W}_v}(N_i) = W_i$ for every $i \in I$.*

Proof. Let $\mathfrak{E} = \{e_j\}_{j\in J}$ be an orthonormal basis for $\mathcal{H}$, then $\mathcal{F}_i = \{\pi_{W_i}(e_j)\}_{j\in J}$ is a Parseval frame for W_i and hence $W_i = \overline{\operatorname{span}}\{\pi_{W_i}(e_j)\}_{j\in J}$. Let $\mathcal{E}_{\mathfrak{E}} = \{e_{ij}\}_{i\in I, j\in J}$ be the associated orthonormal basis to $\mathfrak{E} = \{e_j\}_{j\in J}$ of $\ell^2(\mathcal{H}, I)$, and let $N_i = \overline{\operatorname{span}}\{e_{ij}\}_{j\in J}$. Then $\mathcal{N} = \{N_i\}_{i\in I}$ is an orthonormal fusion basis for $\ell^2(\mathcal{H}, I)$. Now if $f \in N_i$, then we can write $f = \sum_{j\in J}\langle f\ ,\ e_{ij}\rangle e_{ij}$, thus

$$T_{\mathcal{W}_v}(f) = \sum_{j\in J}\langle f\ ,\ e_{ij}\rangle T_{\mathcal{W}_v}(e_{ij}) = \sum_{j\in J} v_i\langle f\ ,\ e_{ij}\rangle \pi_{W_i}(e_j),$$

and this shows that $T_{\mathcal{W}_v}(f) \in W_i$. Finally if $g \in W_i$, then we have

$$\begin{aligned} g &= \sum_{j\in J} \langle g\ ,\ \pi_{W_i}(e_j)\rangle \pi_{W_i}(e_j) = \sum_{j\in J} \frac{1}{v_i}\langle g\ ,\ e_j\rangle T_{\mathcal{W}_v}(e_{ij}) \\ &= T_{\mathcal{W}_v}\left(\sum_{j\in J} \frac{1}{v_i}\langle g\ ,\ e_j\rangle e_{ij}\right). \end{aligned}$$

Thus $g \in T_{\mathcal{W}_v}(N_i)$. Altogether we have $T_{\mathcal{W}_v}(N_i) = W_i$. ■

Theorem 2.12. *Let $\mathcal{W}_v$ be a fusion frame for $\mathcal{H}$, and let $\mathfrak{E}_i = \{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i for all $i \in I$. Then there exists an orthonormal fusion basis $\mathcal{N} = \{N_i\}_{i\in I}$ for $\mathcal{H}$ and a bounded, surjective operator $U\colon \mathcal{H} \longrightarrow \mathcal{H}$ such that $U(N_i) = W_i$.*

Proof. According to Theorem 3.2 of [3], $\mathfrak{E}_v = \{v_i e_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$. Let $\mathcal{U} = \{u_{ij}\}_{i\in I, j\in J_i}$ be an arbitrary orthonormal basis for $\mathcal{H}$. By Theorem 5.5.5 of [7], there is a bounded, surjective operator $U\colon \mathcal{H} \longrightarrow \mathcal{H}$ such that $U(u_{ij}) = v_i e_{ij}$ for all $i \in I$, $j \in J_i$. Write $N_i = \overline{\operatorname{span}}\{u_{ij}\}_{j\in J_i}$, then $\mathcal{N} = \{N_i\}_{i\in I}$ is an orthonormal fusion basis for $\mathcal{H}$ and $U(N_i) = W_i$. ■

3. Fusion frames and operators

In this section we study the relationship between operators and fusion frames for a Hilbert space $\mathcal{H}$. We first consider the behavior of fusion frames under a bounded linear operator with closed range.

Theorem 3.1. *Let $\mathcal{W}_v$ be a fusion frame for $\mathcal{H}$ with fusion frame bounds C and D and let $U\colon \mathcal{H} \longrightarrow \mathcal{K}$ be a bounded operator with closed range such that $U^*UW_i \subseteq W_i$ for all $i \in I$. Then $\mathcal{U}\mathcal{W}_v = \{(\overline{UW_i},\ v_i)\}_{i\in I}$ is a fusion frame sequence with fusion frame bounds $C\|U^\dagger\|^{-2}\|U\|^{-2}$ and $D\|U^\dagger\|^2\|U\|^2$ respectively.*

Proof. Since $U^*UW_i \subseteq W_i$, hence $\pi_{\overline{UW_i}}U = U\pi_{W_i}$. By (6) the operator $UU^\dagger$ is the orthogonal projection onto R_U. Therefore for every $g \in R_U$ we have

$$\begin{aligned} \sum_{i\in I} v_i^2\|\pi_{\overline{UW_i}}(g)\|^2 &= \sum_{i\in I} v_i^2\|\pi_{\overline{UW_i}}UU^\dagger(g)\|^2 \\ &= \sum_{i\in I} v_i^2\|U\pi_{W_i}U^\dagger(g)\|^2 \\ &\le \|U\|^2\sum_{i\in I} v_i^2\|\pi_{W_i}U^\dagger(g)\|^2 \\ &\le D\|U\|^2\|U^\dagger\|^2\|g\|^2. \end{aligned}$$

For the lower fusion frame condition, suppose that $g \in R_U$. Then (6) implies $g = (UU^\dagger)^*(g) = (U^\dagger)^*U^*(g)$. Moreover by Lemma 2.3 of [13], we have $\pi_{W_i}U^* = \pi_{W_i}U^*\pi_{\overline{UW_i}}$. Therefore we obtain

$$\begin{aligned} \|g\|^2 &= \|(U^\dagger)^*U^*(g)\|^2 \\ &\le \|U^\dagger\|^2\|U^*(g)\|^2 \end{aligned}$$

$$\leq \frac{\|U^\dagger\|^2}{C}\sum_{i\in I} v_i^2\|\pi_{W_i}U^*(g)\|^2$$
$$= \frac{\|U^\dagger\|^2}{C}\sum_{i\in I} v_i^2\|\pi_{W_i}U^*\pi_{\overline{UW_i}}(g)\|^2$$
$$\leq \frac{\|U\|^2\|U^\dagger\|^2}{C}\sum_{i\in I} v_i^2\|\pi_{\overline{UW_i}}(g)\|^2. \qquad \blacksquare$$

COROLLARY 3.2

Let $\mathcal{W}_v$ be a fusion frame for $\mathcal{H}$ with fusion frame bounds C and D and let P denote the orthogonal projection onto a closed subspace V such that $PW_i \subseteq W_i$ for all $i \in I$. Then $\mathcal{P}\mathcal{W}_v = \{(PW_i, v_i)\}_{i\in I}$ is a fusion frame for V with fusion frame bounds C, D.

Proof. This follows immediately from Theorem 3.1 ■

Under the same assumptions as Theorem 3.1, if $S_{\mathcal{W}_v}$ and $S_{\mathcal{U}\mathcal{W}_v}$ be fusion frame operators associated with $\mathcal{W}_v$ and $\mathcal{U}\mathcal{W}_v$, respectively, then for every $f \in \mathcal{H}$ we have

$$US_{\mathcal{W}_v}(f) = \sum_{i\in I} v_i^2 U\pi_{W_i}(f) = \sum_{i\in I} v_i^2\pi_{\overline{UW_i}}U(f) = S_{\mathcal{U}\mathcal{W}_v}U(f). \tag{11}$$

COROLLARY 3.3

With the same assumptions as in Theorem 3.1, *if $\mathcal{W}_v$ be a Parseval fusion frame for $\mathcal{H}$, then $\mathcal{U}\mathcal{W}_v$ is a Parseval fusion frame sequence in $\mathcal{K}$.*

Proof. This follows immediately from (11) and Proposition 3.22 of [3]. ■

In the following result we restrict ourselves to finite fusion frames. Notice that this result does not hold for an infinite fusion frame (see, Example 7.1 of [16]).

COROLLARY 3.4

Let $\mathcal{W}_v$ be a fusion frame for $\mathcal{H}$ with fusion frame bounds C, D. If $U \in B(\mathcal{H},\mathcal{K})$ is a bounded, surjective operator, then for every $g \in \mathcal{K}$ we have

$$C\|U^\dagger\|^{-2}\|U\|^{-2}\|g\|^2 \leq \sum_{i\in I} v_i^2\|\pi_{\overline{UW_i}}(g)\|^2.$$

Hence, if $|I| < \infty$, then $\mathcal{U}\mathcal{W}_v = \{(\overline{UW_i}, v_i)\}_{i\in I}$ is a fusion frame for $\mathcal{K}$.

Proof. This follows from the arguments in Theorem 3.1. ■

COROLLARY 3.5

Let $\mathcal{W}_v$ be a fusion frame for $\mathcal{H}$ with fusion frame bounds C, D. If $U \in B(\mathcal{K},\mathcal{H})$ is a bounded, surjective operator, then $\mathcal{U}^*\mathcal{W}_v = \{(U^*W_i, v_i)\}_{i\in I}$ is a fusion frame sequence for $\mathcal{K}$ with fusion frame bounds $C\|U^\dagger\|^{-2}\|U\|^{-2}$ and $D\|U^\dagger\|^2\|U\|^2$ respectively.

Proof. This follows immediately from Theorem 2.4 of [13]. ■

4. Erasures of subspaces

Our purpose in this section is to study conditions for removing an element from a fusion frame to again obtain a fusion frame. We say that $\mathcal{W}_v$ is an exact fusion frame, if it ceases to be a fusion frame whenever anyone of its element is removed.

Now we state an useful result of fusion frames that is proved in [2].

Theorem 4.1. *Let $\mathcal{W}_v$ be a fusion frame for $\mathcal{H}$, and let $i \in I$ and $T_i \in B(\mathcal{H}, W_i)$. If $f \in \mathcal{H}$ and $f = \sum_{i\in I} v_i^2 T_i(f)$, then we have*

(i) $\sum_{i\in I} v_i^2 \|T_i(f)\|^2 = \sum_{i\in I} v_i^2 \|\pi_{W_i} S_{\mathcal{W}_v}^{-1}(f) - T_i(f)\|^2 + \sum_{i\in I} v_i^2 \|\pi_{W_i} S_{\mathcal{W}_v}^{-1}(f)\|^2$.

(ii) $\sum_{i\in I} v_i^2 \|\pi_{W_i} S_{\mathcal{W}_v}^{-1}(f) - \pi_{W_i}(f)\|^2 + \sum_{i\in I} v_i^2 \|\pi_{W_i} S_{\mathcal{W}_v}^{-1}(f) - T_i(f)\|^2 = \sum_{i\in I} v_i^2 \|T_i(f) - \pi_{W_i}(f)\|^2$.

Proof. See Theorem 2.2 of [2]. ■

COROLLARY 4.2

Let $\mathcal{W}_v$ be a fusion frame for $\mathcal{H}$, and let $T_{\mathcal{W}_v}$ be an associated synthesis operator. Then the pseudo-inverse operator $T_{\mathcal{W}_v}^{\dagger}: \mathcal{H} \longrightarrow \ell^2(\mathcal{H}, I)$ is given by

$$T_{\mathcal{W}_v}^{\dagger}(f) = \{v_i \pi_{W_i} S_{\mathcal{W}_v}^{-1}(f)\}_{i\in I} \quad \forall f \in \mathcal{H}. \tag{12}$$

Proof. Let $f \in \mathcal{H}$. Then by Theorem 2.1 of [8], the equation $T_{\mathcal{W}_v}(\{f_i\}_{i\in I}) = f$ has exactly one solution with minimal norm. This solution is $T_{\mathcal{W}_v}^{\dagger}(f)$. The result now follows by combining (9) and Theorem 4.1(i). ■

If we remove an element from a fusion frame, we obtain either another fusion frame or an incomplete set. The following theorem shows this result with extra information about the bounds of fusion frame.

Theorem 4.3. *Let $\mathcal{W}_v$ be a fusion frame for $\mathcal{H}$ with fusion frame bounds C, D and let $j \in I$ and $\mathcal{W}_v^{\mathcal{J}} = \{(W_i, v_i)\}_{i\in I, i\neq j}$. Then,*

(i) *if there is some $g \in W_j - \{0\}$ such that $\pi_{W_j} S_{\mathcal{W}_v}^{-1}(g) = \frac{1}{v_j^2} g$, then $\mathcal{W}^{\mathcal{J}} = \{W_i\}_{i\in I, i\neq j}$ is an incomplete set in $\mathcal{H}$;*

(ii) *if $\mathrm{Id}_{\mathcal{H}} - v_j^2 \pi_{W_j} S_{\mathcal{W}_v}^{-1}$ is a bounded, invertible operator on $\mathcal{H}$, then $\mathcal{W}_v^{\mathcal{J}}$ is a fusion frame with fusion frame bounds $\frac{C^2}{C + v_j^2 \|(\mathrm{Id}_{\mathcal{H}} - v_j^2 \pi_{W_j} S_{\mathcal{W}_v}^{-1})^{-1}\|^2}$ and D.*

Proof. (i) Define $T_i: \mathcal{H} \longrightarrow W_i$ by $T_i = v_i^{-2} \delta_{ij} \pi_{W_i}$ for all $i \in I$, where δ_{ij} is the Kronecker delta. Then we have $\sum_{i\in I} v_i^2 T_i(g) = \sum_{i\in I} \delta_{ij} \pi_{W_i}(g) = \pi_{W_j}(g) = g$. Now by Theorem 4.1(i) we compute

$$\sum_{i\in I} v_i^2 \|v_i^{-2} \delta_{ij} \pi_{W_i}(g)\|^2 = \sum_{i\in I} v_i^2 \|\pi_{W_i} S_{\mathcal{W}_v}^{-1}(g) - v_i^{-2} \delta_{ij} \pi_{W_i}(g)\|^2 + \sum_{i\in I} v_i^2 \|\pi_{W_i} S_{\mathcal{W}_v}^{-1}(g)\|^2.$$

Consequently,

$$\frac{1}{v_j^2}\|g\|^2 = \frac{1}{v_j^2}\|g\|^2 + 2\sum_{\substack{i\in I\\ i\neq j}} v_i^2 \|\pi_{W_i} S_{\mathcal{W}_v}^{-1}(g)\|^2.$$

Hence $S_{\mathcal{W}_v}^{-1}(g) \in (\overline{\text{span}}\{W_i\}_{i\in I, i\neq j})^{\perp}$, since $S_{\mathcal{W}_v}^{-1}(g) \neq 0$, and it follows that $\mathcal{W}^{\mathcal{J}}$ is an incomplete set in $\mathcal{H}$.

(ii) By (9) for every $f \in \mathcal{H}$, we have $f = \sum_{i\in I} v_i^2 S_{\mathcal{W}_v}^{-1} \pi_{W_i}(f)$, hence

$$\pi_{W_j}(f) = \sum_{i\in I} v_i^2 \pi_{W_j} S_{\mathcal{W}_v}^{-1} \pi_{W_i}(f).$$

Consequently

$$(I\mathrm{d}_H - v_j^2 \pi_{W_j} S_{\mathcal{W}_v}^{-1})\pi_{W_j}(f) = \sum_{\substack{i\in I\\ i\neq j}} v_i^2 \pi_{W_j} S_{\mathcal{W}_v}^{-1} \pi_{W_i}(f)\ .$$

Now by using the Schwarz inequality we compute

$$\begin{aligned}
\|(\mathrm{Id}_H - v_j^2 \pi_{W_j} S_{\mathcal{W}_v}^{-1})\pi_{W_j}(f)\|^2 &= \left\| \sum_{\substack{i\in I\\ i\neq j}} v_i^2 \pi_{W_j} S_{\mathcal{W}_v}^{-1} \pi_{W_i}(f) \right\|^2 \\
&= \sup_{\|g\|=1} \left| \left\langle g\ ,\ \sum_{\substack{i\in I\\ i\neq j}} v_i^2 \pi_{W_j} S_{\mathcal{W}_v}^{-1} \pi_{W_i}(f) \right\rangle \right|^2 \\
&= \sup_{\|g\|=1} \left| \sum_{\substack{i\in I\\ i\neq j}} v_i^2 \langle \pi_{W_i} S_{\mathcal{W}_v}^{-1} \pi_{W_j}(g)\ ,\ \pi_{W_i}(f)\rangle \right|^2 \\
&\leq \sup_{\|g\|=1} \left(\sum_{\substack{i\in I\\ i\neq j}} v_i^2 \|\pi_{W_i} S_{\mathcal{W}_v}^{-1} \pi_{W_j}(g)\| \|\pi_{W_i}(f)\| \right)^2 \\
&\leq \sup_{\|g\|=1} \left(\sum_{\substack{i\in I\\ i\neq j}} v_i^2 \|\pi_{W_i} S_{\mathcal{W}_v}^{-1} \pi_{W_j}(g)\|^2 \right) \left(\sum_{\substack{i\in I\\ i\neq j}} v_i^2 \|\pi_{W_i}(f)\|^2 \right) \\
&\leq \frac{1}{C} \left(\sum_{\substack{i\in I\\ i\neq j}} v_i^2 \|\pi_{W_i}(f)\|^2 \right),
\end{aligned}$$

which implies that

$$C\|f\|^2 \leq \sum_{\substack{i\in I\\ i\neq j}} v_i^2 \|\pi_{W_i}(f)\|^2 + v_j^2 \|\pi_{W_j}(f)\|^2$$

$$\leq \sum_{\substack{i\in I\\ i\neq j}} v_i^2\|\pi_{W_i}(f)\|^2 + v_j^2\|(\mathrm{Id}_H - v_j^2\pi_{W_j}S_{\mathcal{W}_v}^{-1})^{-1}\|^2$$

$$\times\frac{1}{C}\sum_{\substack{i\in I\\ i\neq j}} v_i^2\|\pi_{W_i}(f)\|^2$$

$$=\left(1+\frac{v_j^2}{C}\|(\mathrm{Id}_H - v_j^2\pi_{W_j}S_{\mathcal{W}_v}^{-1})^{-1}\|^2\right)\sum_{\substack{i\in I\\ i\neq j}} v_i^2\|\pi_{W_i}(f)\|^2.$$

Hence $\mathcal{W}_v^{\mathcal{J}}$ satisfies the lower fusion frame condition with lower bound as required. Clearly $\mathcal{W}_v^{\mathcal{J}}$ also holds the upper fusion frame condition. ∎

COROLLARY 4.4

Suppose that $\mathcal{W}_v$ is a fusion frame for $\mathcal{H}$, and let $j \in I$. If $\|S_{\mathcal{W}_v}^{-1}\| < \frac{1}{v_j^2}$, then $\mathcal{W}_v^{\mathcal{J}}$ is a fusion frame for $\mathcal{H}$, with same fusion frame bounds in Theorem 4.3(ii).

Proof. This claim follows immediately from the fact that we have

$$\|v_j^2\pi_{W_j}S_{\mathcal{W}_v}^{-1}\| \leq v_j^2\|S_{\mathcal{W}_v}^{-1}\| < 1.$$

∎

The following corollary is proved in Corollary 3.3(iii) of [4]. We give another proof of this corollary with extra information about the bounds.

COROLLARY 4.5

Let $\mathcal{W}_v$ be a fusion frame for $\mathcal{H}$ with fusion frame bounds C, D and let $j \in I$. If $v_j^2 < C$, then $\mathcal{W}^{\mathcal{J}}$ is a fusion frame with same fusion frame bounds in Theorem 4.3(ii).

Proof. The result follows from Corollary 4.4 and the following fact that

$$\|S_{\mathcal{W}_v}^{-1}\| \leq \frac{1}{C} < \frac{1}{v_j^2}.$$

∎

Remark 4.6. In Corollary 4.5 the inequality is strict. To see this, let $\mathfrak{E} = \{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$, and define $W_i = \mathrm{span}\{e_i\}$ for all $i \in I$. Then $\mathcal{W}_1 = \{(W_i, 1)\}_{i\in I}$ is an orthonormal fusion basis for $\mathcal{H}$, and so $\mathcal{W}_1$ is an exact Parseval fusion frame. Indeed for each $j \in I$ we have $\|S_{\mathcal{W}_1}^{-1}\| = \frac{1}{v_j^2} = 1$.

COROLLARY 4.7

Suppose that $\mathcal{W}_v$ is a fusion frame for $\mathcal{H}$, and let $j \in I$. If $W_j = \mathcal{H}$ and $\|S_{\mathcal{W}_v}^{-1}\| \neq \frac{1}{v_j^2}$, then $\mathcal{W}_v^{\mathcal{J}}$ is a fusion frame for $\mathcal{H}$, with same fusion frame bounds as in Theorem 4.3(ii).

Proof. Since $W_j = \mathcal{H}$, we have $\pi_{W_j} = I\mathrm{d}_{\mathcal{H}}$. Define

$$T_i\colon H \longrightarrow W_i \qquad T_i(f) = v_i^{-2}\delta_{ij}\pi_{W_i}(f) \quad \forall f \in \mathcal{H}.$$

Then by Theorem 4.1(i) we also have

$$\begin{aligned}\langle S_{\mathcal{W}_v}^{-1}(f), f\rangle &= \sum_{i\in I} v_i^2 \|\pi_{W_i} S_{\mathcal{W}_v}^{-1}(f)\|^2 \\ &\le \sum_{i\in I} v_i^2 \|T_i(f)\|^2 = \frac{1}{v_j^2}\|f\|^2,\end{aligned}$$

which implies that $\|S_{\mathcal{W}_v}^{-1}\| \le \frac{1}{v_j^2}$. This shows that the proof holds. ■

COROLLARY 4.8

Let $\mathcal{W}_v$ be an exact fusion frame for $\mathcal{H}$. Then for every $i \in I$ we have $\|S_{\mathcal{W}_v}^{-1}\| \ge \frac{1}{v_i^2}$.

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