Proc. Indian Acad. Sci. (Math. Sci.) Vol. 119, No. 3, June 2009, pp. 345–350.

On infinitesimal conformal transformations of the tangent bundles with the synectic lift of a Riemannian metric

AYDIN GEZER

Department of Mathematics, Faculty of Science and Arts, Ataturk University, 25240, Erzurum, Turkey
E-mail: gezeraydin@hotmail.com; agezer@atauni.edu.tr

MS received 26 December 2007; revised 31 July 2008

Abstract. The purpose of the present article is to investigate some relations between the Lie algebra of the infinitesimal fibre-preserving conformal transformations of the tangent bundle of a Riemannian manifold with respect to the synectic lift of the metric tensor and the Lie algebra of infinitesimal projective transformations of the Riemannian manifold itself.

Keywords. Fibre-preserving vector fields; infinitesimal projective transformations; Riemannian metric; synectic lift.

1. Introduction

Let M be an n-dimensional manifold with a Riemannian metric g and V be a vector field on M. Let us consider the local one-parameter group $\{\phi_t\}$of local transformations of M generated by V. The vector field V is called an infinitesimal projective transformation if each ϕ_t is a local projective transformation of M. As is well-known, V is an infinitesimal projective transformation if and only if there exist a covariant vector field ξ on M with the components ξ_i such that $L_V\Gamma^h_{ji} = \delta^h_j\xi_i + \delta^h_i\xi_j$, where L_V denotes the Lie derivation with respect to V and Γ^h_{ji} the components of the Riemannian connection of M.

Let $T(M)$ be the tangent bundle over M and Φ be a transformation of $T(M)$. If the transformation Φ preserves the fibres, it is called a fibre-preserving transformation. Consider a vector field $\tilde{X}$ on $T(M)$ and the local one-parameter group $\{\Phi_t\}$ of local transformations of $T(M)$ generated by $\tilde{X}$. The vector field $\tilde{X}$ is called an infinitesimal fibre-preserving transformation if each Φ_t is a local fibre-preserving transformation of $T(M)$. An infinitesimal fibre-preserving transformation $\tilde{X}$ on $T(M)$ is called an infinitesimal fibre-preserving conformal transformation if each Φ_t is a local fibre-preserving conformal transformation of $T(M)$. Let $\tilde{g}$ be a Riemannian or a pseudo-Riemannian metric on $T(M)$. It is well-known that $\tilde{X}$ is an infinitesimal conformal transformation of $T(M)$ if and only if there exist a scalar function Ω on $T(M)$ such that $L_{\tilde{X}}\tilde{g} = 2\Omega\tilde{g}$, where $L_{\tilde{X}}$ denotes the Lie derivation with respect to $\tilde{X}$.

The purpose of the present paper is to prove the following theorem:

Theorem. *Let M be an n-dimensional Riemannian manifold, and let $T(M)$ be its tangent bundle with the synectic lift of a Riemannian metric. Every infinitesimal fibre-preserving conformal transformation $\tilde{X}$ on $T(M)$ naturally induces an infinitesimal projective transformation V on M. Moreover the correspondence $\tilde{X} \to V$ gives a homomorphism of the*

Lie algebra of infinitesimal fibre-preserving conformal transformations on $T(M)$ onto the Lie algebra of infinitesimal projective transformations on M.

2. Preliminaries

In this section, we shall summarize all the basic definitions and results on $T(M)$ that are needed later. Most of them are well-known and details can be found in [8]. Indices $a, b, c, i, j, \ldots$ have range in $\{1, \ldots, n\}$ while indices $\alpha, \lambda, \mu, \ldots$ have range in $\{1, \ldots, n; n+1, \ldots, 2n\}$. We put $\bar{i} = n + i$. Summation over repeated indices is always implied.

Coordinate systems in M are denoted by (U, x^h), where U is the coordinate neighborhood and x^h the coordinate functions. Components in (U, x^h) of geometric objects on M will be referred to simply as components. We denote partial differentiation $\partial/\partial x^h$ by ∂_h.

Let (M, g) be a Riemannian manifold, ∇ the Riemannian connection of g and Γ^a_{ji} the coefficients of ∇, i.e., $\nabla_{\partial_j}\partial_i = \Gamma^a_{ji}\partial_a$ with respect to the natural frame $\{\partial_h\}$. The curvature tensor R of ∇ has components R^h_{kji}.

With the Riemannian connection ∇ given on M, we can introduce on each induced coordinate neighborhood $\pi^{-1}(U)$ of $T(M)$ a frame field which is very useful in our computation. It is called the adapted frame on $\pi^{-1}(U)$ and consists of the following $2n$ linearly independent vector fields $\{E_\lambda\} =\} = \{E_i, E_{\bar{i}}\}$ on $\pi^{-1}(U)$:

$$E_i = \partial_i - y^b\Gamma^a_{bi}\partial_{\bar{a}}, \quad E_{\bar{i}} = \partial_{\bar{i}},$$

where $\{x^h, y^h\}$ is the induced coordinates of $T(M)$. $\{dx^h, \delta y^h\}$ is the dual frame of $\{E_i, E_{\bar{i}}\}$, where $\delta y^h = dy^h + y^b\Gamma^h_{ba}dx^a$. By the straightforward calculation, we have the following:

Lemma 2.1. *The Lie brackets of the adapted frame of $T(M)$ satisfy the following identities*:

$$\begin{cases} [E_j, E_i] = y^b R^a_{ijb} E_{\bar{a}}, \\ [E_j, E_{\bar{i}}] = \Gamma^a_{ji} E_{\bar{a}}, \\ [E_{\bar{j}}, E_{\bar{i}}] = 0, \end{cases} \tag{2.1}$$

where R^a_{ijb} denotes the components of the curvature tensor of M [1,8].

Let $\tilde{X}$ be a vector field on $T(M)$ with components $(v^h, v^{\bar{h}})$ with respect to the adapted frame $\{E_h, E_{\bar{h}}\}$. The vector field $\tilde{X}$ is a fibre-preserving vector field on $T(M)$ if and only if v^h depend only on the variables (x^h). Therefore, every fibre-preserving vector field $\tilde{X}$ on $T(M)$ induces a vector field $V = v^h\frac{\partial}{\partial x^h}$ on M.

Let $L_{\tilde{X}}$ be the Lie derivation with respect to $\tilde{X}$. Then we have the following lemma.

Lemma 2.2 (*see* [6]). *The Lie derivations of the adapted frame and its dual basis are given as follows*:

(1) $L_{\tilde{X}}E_h = -\partial_h v^a E_a + \{y^b v^c R^{\ a}_{hcb} - v^{\bar{b}}\Gamma^a_{bh} - E_h(v^{\bar{a}})\}E_{\bar{a}}$.
(2) $L_{\tilde{X}}E_{\bar{h}} = \{v^b\Gamma^a_{bh} - E_{\bar{h}}(v^{\bar{a}})\}E_{\bar{a}}$.
(3) $L_{\tilde{X}}dx^h = \partial_m v^h dx^m$.
(4) $L_{\tilde{X}}\delta y^h = -\{y^b v^c R^{\ h}_{mcb} - v^{\bar{b}}\Gamma^h_{bm} - E_m(v^{\bar{h}})\}dx^m - \{v^b\Gamma^h_{bm} - E_{\bar{m}}(v^{\bar{h}})\}\delta y^m$.

Let g be a Riemannian metric with components g_{ji}, then we see that

$$\tilde{g} = a_{ji} dx^j dx^i + 2g_{ji} dx^j \delta y^i \tag{2.2}$$

is non-singular and can be regarded as pseudo-Riemannian metric on $T(M)$, where $a = (a_{ji})$ is a symmetric tensor field of the type (0, 2) on M. The metric (2.2) has components

$$\tilde{g} = (\tilde{g}_{\beta\gamma}) = \begin{pmatrix} a_{ji} & g_{ji} \\ g_{ji} & 0 \end{pmatrix}$$

with respect to the adapted frame on $T(M)$, that is, it coincides with $\tilde{g} = {}^{\mathrm{C}}g + {}^{\mathrm{V}}a$, where ${}^{\mathrm{C}}g$ and ${}^{\mathrm{V}}a$ denote the complete and vertical lifts of g and a to $T(M)$, respectively. The metric $\tilde{g}$, which is called a synectic lift of the Riemannian metric g, was introduced by Talantova and Shirokov [4] to study the differential geometry of tangent bundles of Riemannian manifolds (for details, also see [5]). Their paper is concerned with the geometry of the space of n dual variables. The concept of a dual number is the analogue of a complex number $x + jy$, with $j^2 = 0$. Since the set of dual numbers is represented geometrically by R^2, the set of n dual variables is represented by $R^{2n} = R^n \times R^n = TR^n$. They showed that the space TR^n with a certain metric represents a space of n dual variables with purely dual constant curvature. This special metric on TR^n is related projectively to the complete lift of the standard metric on R^n. Also, Shirokov [3] investigated infinitesimal holomorphically projective transformations on $T(M)$ with respect to the synectic lift of the metric tensor. Afterwards, Pavlov [2] studied the tangent bundles with a metric $\lambda{}^{\mathrm{C}}g + {}^{\mathrm{V}}a$ and also proved that the substitution of the metric ${}^{\mathrm{C}}g \to \lambda{}^{\mathrm{C}}g + {}^{\mathrm{V}}a$ is a necessary and sufficient condition on preserving the ‘angles’ between holomorphic planes.

Remark. In the case of $a = g$, the synectic lift of the Riemannian metric g to $T(M)$ coincides with the lift metric I+II on $T(M)$, where $a = (a_{ji})$ is a symmetric tensor field of the type (0, 2) on M and $g = (g_{ji})$ is a Riemannian metric on M. In [7], Yamauchi proved that every infinitesimal fibre-preserving conformal transformation $\tilde{X}$ on $T(M)$ with the lift metric I + II naturally induces an infinitesimal projective transformation V on M and also showed that the correspondence $\tilde{X} \to V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving conformal transformations on $T(M)$ onto the Lie algebra of infinitesimal projective transformations on M.

Lemma 2.3. *The Lie derivative of $\tilde{g}$ with respect to the fibre-preserving vector field $\tilde{X}$ is given as follows*:

$$\begin{aligned} L_{\tilde{X}}\tilde{g} = {} & \{L_V a_{ji} - 2g_{jm}(y^b v^c R_{icb}^{\ m} - v^{\bar{b}}\Gamma_{bi}^m - E_i(v^{\bar{m}}))\}dx^j dx^i \\ & + 2\{L_V g_{ji} - g_{jm}(\nabla_i v^m - E_{\bar{i}}(v^{\bar{m}}))\}dx^j \delta y^i, \end{aligned}$$

where $L_V g_{ji}$ and $L_V a_{ji}$ denote the components of the Lie derivative $L_V g$ and $L_V a$, and also $\nabla_i v^m$ denote the components of the covariant derivative of V.

3. Infinitesimal conformal transformations of the tangent bundles with the synectic lift of a Riemannian metric

Let $T(M)$ be the tangent bundle over M with the synectic lift of the metric on Riemannian manifold M, and let $\tilde{X}$ be an infinitesimal fibre-preserving conformal transformation on

$T(M)$ such that $L_{\tilde{X}}\tilde{g} = 2\Omega\tilde{g}$. From Lemma 2.3, we have

$$\begin{aligned}
&\{L_V a_{ji} - 2g_{jm}(y^b v^c R_{icb}^{\ m} - v^{\bar{b}}\Gamma_{bi}^m - E_i(v^{\bar{m}}))\}dx^j dx^i \\
&\quad + 2\{L_V g_{ji} - g_{jm}(\nabla_i v^m - E_{\bar{i}}(v^{\bar{m}})\}dx^j \delta y^i \\
&= 2\Omega a_{ji} dx^j dx^i + 4\Omega g_{ji} dx^j \delta y^i
\end{aligned}$$

from which we get

$$L_V g_{ji} - 2\Omega g_{ji} = g_{jm}(\nabla_i v^m - E_{\bar{i}}(v^{\bar{m}}) \tag{3.1}$$

and

$$\begin{aligned}
L_V a_{ji} - 2\Omega a_{ji} &= g_{jm}(y^b v^c R_{icb}^{\ m} - v^{\bar{b}}\Gamma_{bi}^m - E_i(v^{\bar{m}})) \\
&\quad + g_{mi}(y^b v^c R_{jcb}^{\ m} - v^{\bar{b}}\Gamma_{bj}^m - E_j(v^{\bar{m}})).
\end{aligned} \tag{3.2}$$

PROPOSITION 3.1

The scalar function Ω on $T(M)$ depends only on the variables (x^h) with respect to the induced coordinates (x^h, y^h).

Proof. Applying $E_{\bar{k}}$ to both sides of eq. (3.1), we have

$$2E_{\bar{k}}(\Omega)g_{ji} = g_{jm}E_{\bar{k}}E_{\bar{i}}(v^{\bar{m}})$$

from which we get

$$E_{\bar{k}}(\Omega)g_{ji} = E_{\bar{i}}(\Omega)g_{jk}.$$

It follows that

$$(n-1)E_{\bar{k}}(\Omega) = 0.$$

This shows that the scalar function Ω on $T(M)$ depends only on the variables (x^h) with respect to the induced coordinates (x^h, y^h), thus we can regard Ω as a function on M and in the following we write ρ instead of Ω.

From (3.1) and Proposition 3.1, $E_{\bar{i}}(v^{\bar{m}})$ depend only on the variables (x^h), thus we can put

$$v^{\bar{h}} = y^a A_a^h + B^h, \tag{3.3}$$

where A_a^h and B^h are certain functions which depend only on the variable (x^h). Furthermore we can show that A_a^h and B^h are the components of a (1,1) tensor field and a contravariant vector field on M, respectively.

Substituting (3.3) into (3.1) and (3.2), we have

$$L_V g_{ji} - 2\rho g_{ji} - g_{jm}\nabla_i v^m + g_{jm}A_i^m = 0, \tag{3.4}$$

$$L_V a_{ji} - 2\rho a_{ji} + g_{jm}\nabla_i B^m + g_{im}\nabla_j B^m = 0 \tag{3.5}$$

and

$$v^a(R_{aikj} + R_{ajki}) + g_{jm}\nabla_i A_k^m + g_{mi}\nabla_j A_k^m = 0, \tag{3.6}$$

where $\nabla_i B^m$ and $\nabla_i A_k^m$ denote the components of the covariant derivative of the vector field $B = (B^h)$ and the (1,1) tensor field $A = (A_i^h)$ on M.

PROPOSITION 3.2

The vector field V with components (v^h) is an infinitesimal projective transformation on M.

Proof. Applying the covariant derivative ∇_k to both sides of eq. (3.4), we obtain

$$\begin{aligned} g_{jm}\nabla_k A_i^m &= \nabla_k(2\rho g_{ji} + g_{jm}\nabla_i v^m - L_V g_{ji}) \\ &= 2\rho_k g_{ji} + g_{jm}\nabla_k\nabla_i v^m - \nabla_k(L_V g_{ji}). \end{aligned}$$

From this formula: $L_V\Gamma_{ki}^m = \nabla_k\nabla_i v^m + R_{aki}^m\ v^a$, we can write

$$\begin{aligned} g_{jm}\nabla_k A_i^m &= 2\rho_k g_{ji} + g_{jm}(L_V\Gamma_{ki}^m - R_{aki}^{\ \ m}\ v^a) \\ &\quad - (L_V\nabla_k g_{ji} + L_V\Gamma_{kj}^a g_{ai} + L_V\Gamma_{ki}^a g_{ja}) \end{aligned}$$

from which, we get

$$g_{jm}\nabla_k A_i^m = 2\rho_k g_{ji} - R_{akij}v^a - L_V\Gamma_{kj}^a g_{ai}. \tag{3.7}$$

Substituting (3.7) into (3.6), we have

$$L_V\Gamma_{ji}^h = \delta_j^h\rho_i + \delta_i^h\rho_j, \tag{3.8}$$

where $\rho_i = \nabla_i\rho$. Hence, V is an infinitesimal projective transformation on M.

Now we consider the converse problem, that is, let M admit an infinitesimal projective transformation $V = v^h\frac{\partial}{\partial x^h}$. Then we have the following proposition:

PROPOSITION 3.3

The vector field $\tilde{X}$ on $T(M)$ defined by

$$\tilde{X} = v^h E_h + y^a(A_a^{\ h} + B^h)E_{\bar{h}}$$

is an infinitesimal fibre-preserving conformal transformation on $T(M)$, where $A_i^{\ h} = g^{ha}A_{ai}$, $A_{ji} = 2\rho g_{ji} + \nabla_i v_j - L_V g_{ji}$ and $g_{ji}B^j = B_i$, $\nabla_j B_i = \rho a_{ji} - \frac{1}{2}L_V a_{ji}$.

Proof. By Lemma 2.3, it follows that

$$\begin{aligned} L_{\tilde{X}}\tilde{g} &= L_{\tilde{X}}(a_{ji}dx^j dx^i + 2g_{ji}dx^j\delta y^i) \\ &= \tilde{X}(a_{ji})dx^j dx^i + a_{ji}(L_{\tilde{X}}dx^j)dx^i + a_{ji}dx^j(L_{\tilde{X}}dx^i) \\ &\quad + 2\tilde{X}(g_{ji})dx^j\delta y^i + 2g_{ji}(L_{\tilde{X}}dx^j)\delta y^i + 2g_{ji}dx^j(L_{\tilde{X}}\delta y^i) \\ &= [L_V a_{ji} + g_{jm}\nabla_i B^m + g_{mi}\nabla_j B^m + y^k(v^a(R_{aikj} + R_{ajki}) \\ &\quad + g_{jm}\nabla_i A_k^m + g_{mi}\nabla_j A_k^m)]dx^j dx^i + (L_V g_{ji} - \nabla_i v_j + A_{ji})dx^j\delta y^i. \end{aligned}$$

On the other hand, by means of (3.7) and (3.8)

$$\begin{aligned} \nabla_j A_{ia} &= g_{im}\nabla_j\nabla_i v^m + 2\rho_j g_{ia} - (L_V\Gamma_{j\ i}^{\ m})g_{ma} - (L_V\Gamma_{j\ a}^{\ m})g_{im} \\ &= -v^b R_{bjai} + 2\rho_j g_{ia} - (\delta_j^m\rho_i + \delta_i^m\rho_j)g_{ma} \\ &= -v^b R_{bjai} + \rho_j g_{ia} - \rho_i g_{ja}, \end{aligned}$$

from which we obtain

$$L_{\tilde{X}}\tilde{g} = 2\rho a_{ji}dx^j dx^i + 4\rho g_{ji}dx^i \delta y^i = 2\rho\tilde{g}.$$

Hence, $\tilde{X}$ is an infinitesimal fibre-preserving conformal transformation on $T(M)$.

Proof of Theorem. Let M be an n-dimensional Riemannian manifold, and let $T(M)$ be its tangent bundle with the synectic lift of a Riemannian metric. Summing up Propositions 3.1–3.3, it is clear that the correspondence $\tilde{X} \to V$ gives a homomorphism of the Lie algebra of infinitesimal fibre-preserving conformal transformations on $T(M)$ onto the Lie algebra of infinitesimal projective transformations on M.

References

[1] Hasegawa I and Yamauchi K, Infinitesimal conformal transformations on tangent bundles with the lift metric I + II, *Scientiae Mathematicae Japonicae* **57(1)** (2003) 129–137, e7, 437–445

[2] Pavlov E, Conformal-holomorphic metrics, *Tensor, New Ser.* **51(1)** (1992) 26–32

[3] Shirokov A P, On holomorphically projective transformations in a tangent bundle, *Trudy Geom. Sem. Kazan. Univ.* **11** (1979) 111–114

[4] Talantova N V and Shirokov A P, A remark on a certain metric in the tangent bundle, *Izv. Vyss. Uchebn. Zaved. Math.* **6(157)** (1975) 143–146

[5] Vishnevskii V V, Shirokov A P and Shurygin V V, Spaces over algebras (Kazan University Press) (1985)

[6] Yamauchi K, On infinitesimal conformal transformations of the tangent bundles over Riemannian manifolds, *Ann Rep. Asahikawa. Med. Coll.* **15** (1994) 1–10

[7] Yamauchi K, On infinitesimal conformal transformations of the tangent bundles with the metric I + II over Riemannian manifolds, *Ann Rep. Asahikawa. Med. Coll.* **17** (1996) 1–7

[8] Yano K and Ishihara S, Tangent and cotangent bundles (New York: Marcel Dekker, Inc.) (1973)