

Orthogonality and Hecke operators

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Abstract. In this article we analyze orthogonality relations between old forms and the connection to the theory of Hecke operators.

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1. Introduction and statement of results

Let $\Gamma_1 := SL_2(\mathbf{Z})$ and for $M \in \mathbf{N}$ denote by

$$\Gamma_0(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \mid c \equiv 0 \pmod{M} \right\}$$

and

$$\Gamma_1(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \mid c \equiv 0 \pmod{M}, a, d \equiv 1 \pmod{M} \right\}$$

the corresponding congruence subgroups of Γ_1 of level M . The theory of newforms going back to [1] asserts that the space $S_k(M)$ of cusp forms of even integral weight $k \geq 2$ with respect to $\Gamma_1(M)$ decomposes into the direct sum of the subspace $S_k^{\text{old}}(M)$ of oldforms and the subspace $S_k^{\text{new}}(M)$ of newforms. By definition, $S_k^{\text{old}}(M)$ is the sum of the spaces $S_k(t)|V_d$ where t and d run over all positive integers such that $td|M$ and $t \neq M$, and V_d is the operator replacing a function $f(z)$ on the complex upper half-plane \mathcal{H} by $f(dz)$. Moreover, $S_k^{\text{new}}(M)$ is the orthogonal complement with respect to the Petersson scalar product of $S_k^{\text{old}}(M)$.

By Atkin–Lehner theory, one has a direct sum decomposition

$$S_k^{\text{old}}(M) = \bigoplus_{td|M, t \neq M} S_k^{\text{new}}(t)|V_d, \quad (1)$$

and a quite natural question to ask is to what extent orthogonality may hold for (sums of) pieces of the right-hand side of (1), too.

Here we would like to address this question in a simple case, namely when $M = Np^r$ with $N, r \in \mathbf{N}$ and p is a prime not dividing N . We will consider the sum $S_k(N) \oplus S_k(N)|V_p \subset S_k^{\text{old}}(Np^r)$.

Recall that the Petersson slash operator $|_k$ is defined by

$$(f|_k\gamma)(z) := (ad - bc)^{k/2}(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right)$$

$$\left(f: \mathcal{H} \rightarrow \mathbf{C}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbf{R})\right).$$

For a Dirichlet character $\chi \bmod N$ we denote by

$$S_k(N, \chi) := \left\{ f \in S_k(N); f|_k\gamma = \chi(d) \cdot f, \forall \gamma \in \Gamma_0(N), \gamma = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \right\}$$

the χ -eigenspace. It is a standard result from representation theory of finite groups that one has an orthogonal decomposition

$$S_k(N) = \bigoplus_{\chi} S_k(N, \chi). \quad (2)$$

Recall that the p -th Hecke operator $T_{k,N}(p)$ on $S_k(N, \chi)$ is given by

$$\sum_{n \geq 1} a(n)q^n |T_{k,N}(p) = \sum_{n \geq 1} \left(a(pn) + \chi(p)p^{k-1}a\left(\frac{n}{p}\right) \right) q^n \quad (z \in \mathcal{H}, q = e^{2\pi iz}),$$

with the convention that $a\left(\frac{n}{p}\right) = 0$ if p does not divide n .

Theorem. *Let $V_{k,N,p}$ be the maximal subspace of $S_k(N)$ that is orthogonal to $S_k(N)|V_p$ inside $S_k(Np^r)$. Then*

$$V_{k,N,p} = \ker T_{k,N}(p).$$

So far the statement of the theorem is simply an interesting theoretical statement (at least as we think) and we do not have any immediate application. Its proof only uses simple properties of the Petersson scalar product.

Note that for $k > 2$ one expects that normalized Hecke eigenforms f in $S_k^{\text{new}}(t)$ with trivial character and p -eigenvalue λ_p equal to zero almost all come from complex multiplication. In fact, if $k > 2$ and f does not have complex multiplication and its Fourier coefficients are rational integers, then some simple probability arguments due to Atkin suggest that one should have $\lambda_p \gg_{\epsilon,k,t} p^{(k-3)/2-\epsilon}$ ($\epsilon > 0$), cf. p. 244 of [2]. Even more optimistically, for N fixed and k large with respect to N , it is widely believed that $\dim \ker T_{k,N}(p)$ and hence $\dim V_{k,N,p}$ is very small.

We also recall that one conjectures that $T_{k,1}(p)$ is an isomorphism of $S_k(1)$ (and so $V_{k,1,p} = \{0\}$) for all $k \geq 12$ and all p . This is an obvious generalization of the famous Lehmer conjecture saying that $\tau(p) \neq 0$ for all p , where $\tau(n)$ ($n \in \mathbf{N}$) is the Ramanujan's function, i.e., $\tau(n)$ is the n -th Fourier coefficient of the discriminant function, the unique normalized cusp form of weight 12 on Γ_1 .

2. Proof

If f and g are cusp forms on a group $\gamma\Gamma\gamma^{-1}$, where Γ is a subgroup of finite index in Γ_1 containing -1 and γ is a $(2, 2)$ -matrix with integral coefficients and positive determinant, we normalize the Petersson scalar product by

$$\langle f, g \rangle = \frac{1}{[\Gamma_1: \Gamma']} \int_{\Gamma' \backslash \mathcal{H}} f(z) \overline{g(z)} y^{k-2} dx dy, \quad (z = x + iy \in \mathcal{H})$$

where Γ' is any subgroup of Γ_1 of finite index contained in $\gamma \Gamma \gamma^{-1}$ (the integral is then independent of the choice of Γ'). We simply write $T(p)$ instead of $T_{k,N}(p)$. Since p does not divide N , the operator $T(p)$ is normal and hence $S_k(N)$ has an orthogonal basis $\{f_1, \dots, f_g\}$ of eigenfunctions of $T(p)$. Suppose that

$$f_v | T(p) = \lambda_{v,p} f_v$$

for all $v \in \{1, \dots, g\}$. By the identity (2) it suffices to show our claim for $S_k(N, \chi)$. So we may assume that $f_v \in S_k(N, \chi)$ for all $v \in \{1, \dots, g\}$.

By definition,

$$f | T(p) = f | U_p + \chi(p) p^{k-1} f | V_p,$$

where U_p operates on the Fourier series $\sum_{n \geq 1} a(n) q^n$ by replacing $a(n)$ by $a(pn)$. Then

$$\begin{aligned} \overline{\lambda_{v,p}} \langle f_\mu, f_v \rangle &= \langle f_\mu, f_v | T(p) \rangle \\ &= \langle f_\mu, f_v | U_p \rangle + \overline{\chi(p)} p^{k-1} \langle f_\mu, f_v | V_p \rangle \end{aligned}$$

for all μ and v . If we use the slash operator $|_k$ we can write

$$f_v | U_p = p^{k/2-1} \sum_{\alpha(p)} f_v |_k \begin{pmatrix} 1 & \alpha \\ 0 & p \end{pmatrix}.$$

Therefore, since

$$\langle f, g |_k \gamma \rangle = \langle f |_k \gamma^*, g \rangle, \quad (f, g \in S_k(Np); \gamma \in GL_2^+(\mathbf{R}))$$

where $\gamma^* = \det \gamma \cdot \gamma^{-1}$ as is well-known, we find that

$$\begin{aligned} \langle f_\mu, f_v | U_p \rangle &= p^{k/2-1} \sum_{\alpha(p)} \left\langle f_\mu |_k \begin{pmatrix} p & -\alpha \\ 0 & 1 \end{pmatrix}, f_v \right\rangle \\ &= p^{k/2} \left\langle f_\mu |_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, f_v \right\rangle \\ &= p^k \langle f_\mu | V_p, f_v \rangle. \end{aligned}$$

Thus for all μ and v we obtain

$$\overline{\lambda_{v,p}} \langle f_\mu, f_v \rangle = p^{k-1} (p \langle f_\mu | V_p, f_v \rangle + \overline{\chi(p)} \langle f_\mu, f_v | V_p \rangle). \quad (3)$$

We now have to distinguish two different cases. Let us consider first the case where $v \neq \mu$. Interchanging the roles of μ and v in (3) yields:

$$\overline{\lambda_{\mu,p}} \langle f_v, f_\mu \rangle = p^{k-1} (p \langle f_v | V_p, f_\mu \rangle + \overline{\chi(p)} \langle f_v, f_\mu | V_p \rangle).$$

Since the basis is orthogonal we get

$$0 = p \langle f_v | V_p, f_\mu \rangle + \overline{\chi(p)} \langle f_v, f_\mu | V_p \rangle.$$

Taking complex conjugates on both sides leads to

$$0 = p \overline{\langle f_v | V_p, f_\mu \rangle} + \chi(p) \overline{\langle f_v, f_\mu | V_p \rangle}$$

and hence to

$$0 = \chi(p) \langle f_\mu | V_p, f_v \rangle + p \langle f_\mu, f_v | V_p \rangle.$$

On the other hand because of orthogonality on the left-hand side of (3) we deduce

$$0 = p \langle f_\mu | V_p, f_v \rangle + \overline{\chi(p)} \langle f_\mu, f_v | V_p \rangle.$$

Observing that

$$\det \begin{pmatrix} \chi(p) & p \\ p & \overline{\chi(p)} \end{pmatrix} = 1 - p^2 \neq 0$$

we then obtain

$$\langle f_\mu, f_v | V_p \rangle = 0 \quad (\mu \neq v). \quad (4)$$

Now let $\mu = v$. Since $\langle f_v | V_p, f_v \rangle = \overline{\langle f_v, f_v | V_p \rangle}$, eq. (3) becomes

$$\overline{\lambda_{v,p}} \langle f_v, f_v \rangle = p^{k-1} (p \overline{\langle f_v, f_v | V_p \rangle} + \overline{\chi(p)} \langle f_v, f_v | V_p \rangle). \quad (5)$$

Taking complex conjugates on both sides we get

$$\lambda_{v,p} \langle f_v, f_v \rangle = p^{k-1} (p \langle f_v, f_v | V_p \rangle + \chi(p) \overline{\langle f_v, f_v | V_p \rangle}).$$

Observing $\overline{\lambda_{v,p}} = \overline{\chi(p)} \lambda_{v,p}$, we obtain

$$\overline{\lambda_{v,p}} \langle f_v, f_v \rangle = p^{k-1} (\overline{\chi(p)} p \langle f_v, f_v | V_p \rangle + \overline{\langle f_v, f_v | V_p \rangle}). \quad (6)$$

Thus from eqs (5) and (6) we get a linear equation system. Computing

$$\det \begin{pmatrix} \overline{\chi(p)} & p \\ \chi(p)p & 1 \end{pmatrix} = \overline{\chi(p)} - \chi(p)p^2 \neq 0$$

leads to

$$\langle f_v, f_v | V_p \rangle = 0 \Leftrightarrow \lambda_{v,p} = 0. \quad (7)$$

Our claim that $V_{k,N,p} = \ker T(p)$ now easily follows. Indeed, if $\{f_1, \dots, f_r\}$ ($r \leq g$) is a basis of $\ker T(p)$, then from (4) and (7) we immediately deduce that $\{f_1, \dots, f_r\}$ ($r \leq g$) is a basis of $V_{k,N,p}$, too.

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