

An elementary approach to gap theorems

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Abstract. Using elementary comparison geometry, we prove: Let (M, g) be a simply-connected complete Riemannian manifold of dimension ≥ 3 . Suppose that the sectional curvature K satisfies $-1 - s(r) \leq K \leq -1$, where r denotes distance to a fixed point in M . If $\lim_{r \rightarrow \infty} e^{2r}s(r) = 0$, then (M, g) has to be isometric to \mathbb{H}^n .

The same proof also yields that if K satisfies $-s(r) \leq K \leq 0$ where $\lim_{r \rightarrow \infty} r^2 s(r) = 0$, then (M, g) is isometric to \mathbb{R}^n , a result due to Greene and Wu.

Our second result is a local one: Let (M, g) be any Riemannian manifold. For $a \in \mathbb{R}$, if $K \leq a$ on a geodesic ball $B_\rho(R)$ in M and $K = a$ on $\partial B_\rho(R)$, then $K = a$ on $B_\rho(R)$.

Keywords. Riemannian manifold; sectional curvature; volume comparison; hyperbolic space.

1. Introduction

The question of when a Riemannian manifold which asymptotically ‘resembles’ \mathbb{R}^n is actually isometric to \mathbb{R}^n is a classical topic in differential geometry. Broadly speaking, attention has been focused on two notions of resemblance. In the first, one makes a weak curvature assumption, such as the nonpositivity of scalar curvature, but one also assumes the existence of a coordinate system (outside a compact set) in which the metric approximates the standard Euclidean metric. The positive mass theorem of Schoen and Yau [4] is the prototype of such a result. In the second class, it is assumed that the sectional curvature has a definite sign and approaches zero at a certain rate. One of the early results in this direction was by Siu and Yau [5]. A by-product of this paper is a completely elementary and short proof of the main result in [5]. A host of theorems was also proved by Greene and Wu in [2]. In particular, they proved:

Theorem 1 [2]. *Let (M, g) be a simply-connected complete Riemannian manifold of dimension ≥ 3 . Suppose that $-s(r) \leq K \leq 0$, where r denotes distance to a fixed point in M .*

If $\lim_{r \rightarrow \infty} r^2 s(r) = 0$ when $\dim M$ is odd or $\int_0^\infty s(r) dr < \infty$ when $\dim M$ is even, then (M, g) is isometric to \mathbb{R}^n .

Results of both kinds have been extended to characterizing hyperbolic manifolds. For instance, Min-Oo [3] proved that a spin n -manifold with scalar curvature $\geq -n(n-1)$ and asymptotic to the hyperbolic metric in a strong sense must be isometric to hyperbolic n -space. In the other direction, Tian and Shi recently proved [6].

Theorem 2 [6]. *Let (M, g) be a simply-connected complete Riemannian manifold of dimension ≥ 3 with $K \leq 0$ and $\text{Ricci} \geq -(n-1)$. If $|K+1| = O(e^{-\alpha r})$ as $r \rightarrow \infty$, for some $\alpha > 2$, then (M, g) is isometric to \mathbb{H}^n .*

In this note, we prove two rigidity results by means of a simple but versatile technique. The first result, Theorem A, is a direct analogue of Theorem 1 above for characterizing hyperbolic space. In fact, the same arguments will also give a quick proof of Theorem 1 (for yet another proof of Theorem 1 involving the Tits metric, see [1]).

Theorem A. *Let (M, g) be a simply-connected complete Riemannian manifold of dimension ≥ 3 . Suppose that $-1 - s(r) \leq K \leq -1$, where r denotes distance to a fixed point in M .*

If $\lim_{r \rightarrow \infty} e^{2r} s(r) = 0$, then (M, g) is isometric to \mathbb{H}^n .

Theorem A complements Theorem 2 in the following sense: While Theorem 2 implies rigidity for the lower bound $K \geq -1$, Theorem A gives rigidity for the upper bound $K \leq -1$. Similarly, in the case of \mathbb{R}^n , some results for $K \geq 0$ were also proved in [2].

The second result, Theorem B, applies to geodesic balls in Riemannian manifolds. It is valid in the presence of positive sectional curvature. If (M, g) is a Riemannian manifold and S any subset of M , we write ' $K = a$ on S ' to mean the following: For any $q \in S$ and any 2-plane $P \subset T_q M$, one has $K(P) = a$, where K is the sectional curvature of g .

Theorem B. *Let (M, g) be a Riemannian manifold of dimension ≥ 3 . Suppose that $K \leq a$ on $B_p(R)$. If $a > 0$, assume that $R \leq \min \left\{ \frac{\pi}{2\sqrt{a}}, \text{inj}(p) \right\}$.*

If $K = a$ on $\partial B_p(R)$, then $K = a$ on $B_p(R)$.

Note that when $a > 0$, we do not need to assume that sectional curvature has a fixed sign in the interior of the geodesic ball. Note also that we are not only demanding that the sectional curvatures achieve their maxima on $\partial B_p(R)$ but also that all curvatures are equal (to a) on $\partial B_p(R)$. Finally, we remark that the above theorem fails to hold if we assume that $K \geq a$ instead of $K \leq a$. An example is given in §3.

The proof of these theorems is based on relative volume comparison for distance spheres. The upper curvature bound implies that the relative volume of distance spheres is increasing and ≥ 1 . On the other hand, this bound also gives lower bounds for the principal curvatures of the distance spheres. Combining this with the lower bound on K one sees that the intrinsic curvature of the distance spheres is bounded below. Another application of volume comparison shows that the relative volume approaches 1 as $r \rightarrow \infty$ (in Theorem A). Hence the relative volume is equal to 1 for all r and one gets the required conclusion.

2. Proofs

We begin by recalling two standard results of comparison geometry. Let (M, g) be a simply-connected Riemannian manifold, not necessarily complete. For $p \in M$ and $r \leq \text{inj}(p)$, let $V(r)$ and $A(r)$ denote the volumes of the ball $B_p(r)$ and the sphere $S_p(r) = \partial B_p(r)$ in M and let $V^a(r)$, $S^a(r)$ denote the corresponding quantities in the simply-connected space-form of curvature a , respectively. Let $\rho(x) := d(p, x)$.

Hessian comparison. Let (M^n, g) , $n \geq 2$, be a Riemannian manifold and $p \in M$. Assume that $K \leq a$ on $B_p(R)$, where $R \in (0, \infty]$ if $a \leq 0$ and $R \leq \min \left\{ \frac{\pi}{2\sqrt{a}}, \text{inj}(p) \right\}$ if $a > 0$.

If λ is any eigenvalue of $\text{Hess}(\rho)$, then $\lambda \geq \lambda_a$, where

$$\lambda_a(r) = \begin{cases} \sqrt{|a|} \coth(\sqrt{|a|}r) & \text{if } a < 0, \\ r^{-1} & \text{if } a = 0, \\ \sqrt{a} \cot(\sqrt{a}r) & \text{if } a > 0. \end{cases}$$

Volume comparison. Let (M^n, g) , $n \geq 2$, be a Riemannian manifold and $p \in M$.

- (i) Let $r \leq R \leq \text{inj}(p)$. If $K \leq a$ on $B_p(R)$, then $\frac{A(r)}{A_a(r)}$ is an increasing function of r .
If $\lim_{r \rightarrow R} \frac{A(r)}{A_a(r)} = 1$, then $K = a$ on $B_p(R)$.
- (ii) If $\text{Ricci} \geq (n-1)a$, then $V(r) \leq V_a(r)$ for any $r > 0$.

The proof begins with the following linear algebra lemma:

Lemma 2.1. Let $S: V \rightarrow V$ be a positive semi-definite symmetric linear operator on an inner-product space V . Let

$$T(X, Y) := \frac{\langle S(X), X \rangle \langle S(Y), Y \rangle - \langle S(X), Y \rangle^2}{\|X\|^2 \|Y\|^2 - \langle X, Y \rangle^2}$$

for linearly independent vectors X and Y . Then

$$\lambda^2 \leq T(X, Y) \leq \mu^2,$$

where λ and μ are the smallest and largest eigenvalues of S , respectively.

Proof. It can be checked that the $T(X, Y)$ depends only on the plane spanned by X and Y , i.e., $T(X, Y) = T(X', Y')$ if X', Y' is another basis for $P = \text{Span}\{X, Y\}$.

We claim that we can find orthogonal vectors $v, w \in P$ such that $\langle S(v), w \rangle = 0$. This will clearly prove the lemma. In fact, let e_1, e_2 be an orthonormal basis for P . We can assume that $\langle S(e_1), e_2 \rangle \neq 0$, otherwise there is nothing to prove. Let $v = e_1 + ce_2, w = e_1 + de_2$, where c, d are to be chosen. We want

$$\langle v, w \rangle = 1 + cd = 0, \quad \langle S(v), w \rangle = a_{22}cd + (c + d)a_{12} + a_{11} = 0,$$

where $a_{ij} = \langle S(e_i), e_j \rangle$. These equations give

$$c^2 + \left(\frac{a_{11} - a_{22}}{a_{12}} \right) c - 1 = 0,$$

which can be solved to give the required v and w . ■

The main ingredient in our proof is the following.

Key Lemma. Let (M^n, g) , $n \geq 3$, be a Riemannian manifold. Suppose that $a - s(r) \leq K \leq a$ on $B_p(R)$, where $R \in (0, \infty]$ if $a \leq 0$ and $R \leq \min \left\{ \frac{\pi}{2\sqrt{a}}, \text{inj}(p) \right\}$ if $a > 0$. Then

$$\frac{A(r)}{A_a(r)} \leq (1 - f_a(r)^2 s(r))^{-\frac{n-1}{2}},$$

for all r such that $1 - f_a(r)^2 s(r) > 0$. Here

$$f_a(r) = \begin{cases} \frac{1}{\sqrt{|a|}} \sinh(\sqrt{|a|}r) & \text{if } a < 0, \\ r & \text{if } a = 0, \\ \frac{1}{\sqrt{a}} \sin(\sqrt{a}r) & \text{if } a > 0. \end{cases}$$

Proof. Let ω_n denote the volume of the unit sphere in \mathbb{R}^n . We then have

$$A^a(r) = f_a(r)^{n-1} \omega_n. \quad (2.1)$$

We first consider the $a < 0$ case. By the Gauss–Codazzi equations for the curvature of the submanifold $S_p(r)$ at a point $q \in S_p(r)$,

$$\tilde{K}(P) = K(P) + \langle S(X), X \rangle \langle S(Y), Y \rangle - \langle S(X), Y \rangle^2, \quad (2.2)$$

for any 2-plane $P \subset T_q S_p(r)$. Here $\{X, Y\}$ is any orthonormal basis of P .

Now, since $\langle S(X), Y \rangle = \text{Hess}(\rho)(X, Y)$, we can apply the Hessian comparison theorem to estimate the eigenvalues of S . Combining these estimates with Lemma 2.1 and using the condition $K \geq a - s(r)$ in (2.2) we get

$$\tilde{K} \geq a - s(r) + |a| \coth(\sqrt{|a|}r)^2.$$

This implies that

$$f_a(r)^2 \tilde{K} \geq k(r) := 1 - |a|^{-1} \sinh(\sqrt{|a|}r)^2 s(r). \quad (2.3)$$

Let us fix an r with $k(r) > 0$. By (2.3), we can apply the Bonnet–Myers theorem to the Riemannian manifold $(N, h) = (S_p(r), f_a(r)^{-2}g)$ to get

$$\text{diam } N \leq \pi k(r)^{-\frac{1}{2}}.$$

From (ii) of the volume comparison theorem, we have: volume of N = volume of $B_p^N(\pi k(r)^{-\frac{1}{2}}) \leq k(r)^{-\frac{n-1}{2}} \omega_n$. Here B^N denotes balls in (N, h) .

Since volume of $N = f_a(r)^{-(n-1)} A(r)$, we have $A(r) \leq f_a(r)^{n-1} k(r)^{-\frac{n-1}{2}} \omega_n$. Combining this with (2.1) gives the required inequality.

The proof goes through without any changes for $a = 0$. When $a > 0$, note that $\lambda \geq \lambda_a \geq 0$ as long as $r \leq \min\{\frac{\pi}{2a}, \text{inj}(p)\}$. Hence Lemma 2.1 can be applied and the rest of the proof goes through. ■

Proof of Theorems A and B. We start with the proof of Theorem A.

Since $K \leq -1$, by the volume comparison theorem, the ratio $F(r) = \frac{A(r)}{A_{-1}(r)} \geq 1$ is a non-decreasing function of r . By the key lemma and the hypothesis that $e^{2r} s(r) \rightarrow 0$ as $r \rightarrow \infty$, we see that $\lim_{r \rightarrow \infty} F(r) \leq 1$. Hence $F(r) = 1$ for all $r > 0$. By the equality part of the volume comparison theorem we obtain $K = -1$ on M .

The proof of Theorem B is similar. In this case the function $F(r) = \frac{A(r)}{A_a(r)} \geq 1$ is increasing for $r \leq R$. Since $K = a$ on $\partial B_p(r)$, we have $K \geq a - s(r)$ where $s(r) \rightarrow 0$ as $r \rightarrow R$. Combining this with the key lemma and arguing as before, we get $K = a$ for $r \leq R$. ■

3. Remarks

The remarks below concern the validity of the theorems under lower bounds on K .

(i) As mentioned earlier, an analogue of Theorem A for the bounds $-1 \leq K \leq -1 + Ce^{-\alpha r} \leq 0$ with $C > 0, \alpha > 2$ is implied by the result in [6].

(ii) Theorem B is no longer true under the bound $K \geq a$. Indeed, consider the metric $g = dr^2 + f(r)^2 g_0$ on the ball $D = \{x: r = \|x\| < \frac{\pi}{2}\}$ in \mathbb{R}^n , where g_0 is the standard round metric on S^{n-1} and

$$f(r) = \begin{cases} \sin(r) & \text{if } r < c - \varepsilon, \\ h(r) & \text{if } c - \varepsilon \leq r \leq c + \varepsilon, \\ -r + \frac{\pi}{2} & \text{if } c + \varepsilon < r < \frac{\pi}{2}. \end{cases}$$

Here h, ε and c are to be chosen. Let $c \in (0, \frac{\pi}{2})$ be the solution to $\sin(r) = -r + \frac{\pi}{2}$. Choose ε so that $[c - \varepsilon, c + \varepsilon] \subset (0, \frac{\pi}{2})$.

Let $h: [c - \varepsilon, c + \varepsilon] \rightarrow (0, \infty)$ be a smooth function with

$$h'' \leq 0 \quad \text{and} \quad -1 \leq h' \leq 1$$

which agrees (up to second order) with $\sin(r)$ at $c - \varepsilon$ and with $-r + \frac{\pi}{2}$ at $c + \varepsilon$. Since the sectional curvatures of the metric $g = dr^2 + f(r)^2 g_0$ lie between the values of

$$-\frac{f''(r)}{f(r)} \quad \text{and} \quad \frac{1 - f'(r)^2}{f(r)^2},$$

we see that g has $K \geq 0$ everywhere and $K = 0$ on $\partial B_0(c + \varepsilon)$. On the other hand, $K = 1$ on $B_0(c - \varepsilon)$. Hence Theorem B fails to hold.

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References

- [1] Ballmann W, Gromov M and Schroeder V, Manifolds of nonpositive curvature, Progress in Mathematics (Boston, MA: Birkhäuser Boston, Inc.) (1985) vol. 61
- [2] Greene R E and Wu H, Gap theorems for noncompact Riemannian manifolds, *Duke Math. J.* **49**(3) (1982) 731–756
- [3] Min-oo M, Scalar curvature rigidity of asymptotically hyperbolic spin manifolds, *Math. Ann.* **285**(4) (1989) 527–539
- [4] Schoen R and Yau S-T, On the proof of the positive mass conjecture in general relativity, *Comm. Math. Phys.* **65**(1) (1979) 45–76
- [5] Siu Y T and Yau S-T, Complete Kähler manifolds with nonpositive curvature of faster than quadratic decay, *Ann. Math. (2)* **105**(2) (1977) 225–264.
- [6] Tian G and Shi Y, Rigidity of asymptotically hyperbolic manifolds, *Comm. Math. Phys.* **259**(3) (2005) 545–559