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On ideals and quotients of $A\mathcal{T}$-algebras

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Abstract. Some results on $A\mathcal{T}$-algebras are given. We study the problem when ideals, quotients and hereditary subalgebras of $A\mathcal{T}$-algebras are $A\mathcal{T}$-algebras or *AT*-algebras, and give a necessary and sufficient condition of a hereditary subalgebra of an $A\mathcal{T}$-algebra being an $A\mathcal{T}$-algebra.

Keywords. $A\mathcal{T}$-algebra; ideal; quotient.

1. Introduction

The classification of amenable C^*-algebras originated from G A Elliott's work on *AT*-algebras. He classified *AT*-algebras of real rank zero [1] and simple *AT*-algebras [2] in 1993 and in 1997, respectively. Since then a number of classification results appeared. Among these results, three important classes of C^*-algebras are classified. In [3], Elliott and Gong classified AH-algebras of real rank zero. Lin [5] gave the classification theorem for unital separable simple nuclear C*-algebras with tracial topological rank zero. Phillips [9] showed the classification theorem for nuclear purely infinite simple C^*-algebras.

$A\mathcal{T}$-algebras were introduced by Lin and Su [7] in 1997. Recall that $A\mathcal{T}$-algebras are inductive limits of finite direct sums of matrix algebras over extensions of circle algebra by compact operators on a separable infinite dimensional Hilbert space. This class contains *AT*-algebras and is larger than *AT* class. Such algebras may not be simple or finite, and also they may not be of real rank zero or of stable rank one. Though Lin and Su classified $A\mathcal{T}$-algebras of real rank zero in [7], for general case, little is known about such algebras till now. For example, are hereditary subalgebras and quotients of $A\mathcal{T}$-algebras still $A\mathcal{T}$-algebras? When a quotient of an $A\mathcal{T}$-algebra is an *AT*-algebra? Whether or not a maximal ideal which is a stable *AT*-algebra is unique in $A\mathcal{T}$-algebra? In this note, we try to answer these questions, and hope to make clear the relation between $A\mathcal{T}$-algebras and extensions of *AT*-algebras in the future.

2. Preliminaries

Let A and B be C^*-algebras. Recall that an extension of A by B is a short exact sequence $0 \to B \xrightarrow{\alpha} E \xrightarrow{\beta} A \to 0$. Denote this extension by (E, α, β).

The extension (E, α, β) is called trivial, if the above sequence splits, i.e. if there is a homomorphism $\gamma\colon A \to E$ such that $\beta \circ \gamma = \mathrm{id}_A$. We called (E, α, β) to be essential, if $\alpha(A)$ is an essential ideal in E.

Let $0 \to B \overset{\alpha}{\to} E \overset{\beta}{\to} A \to 0$ be an extension of A by B. Then there is a unique homomorphism $\sigma \colon E \to M(B)$ such that $\sigma \circ \alpha = \iota$, where $M(B)$ is the multiplier algebra of B, and ι is the inclusion map from B into $M(B)$. It is known that σ is injective if and only if the extension is essential.

The Busby invariant of (E, α, β) is a homomorphism τ from A into the corona algebra $\mathcal{Q}(B) = M(B)/B$ defined by $\tau(a) = \pi(\sigma(b))$ for $a \in A$, where $\pi \colon M(B) \to M(B)/B$ is the quotient map, and $b \in E$ such that $\beta(b) = a$. Therefore the Busby invariant of (E, α, β) is the unique homomorphism such that the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau} & & \\ 0 & \longrightarrow & B & \longrightarrow & M(B) & \xrightarrow{\pi} & M(B)/B & \longrightarrow & 0. \end{array}$$

Hence there is a one-to-one correspondence between extensions of A by B and homomorphisms $\tau \colon A \to \mathcal{Q}(B)$.

If A is unital and the Busby invariant is unital, then (E, α, β) is said to be unital.

Let $0 \to B \overset{\alpha_i}{\to} E_i \overset{\beta_i}{\to} A \to 0$ be two extensions of A by B with Busby invariants τ_i for $i = 1,\ 2$. Then (E_1, α_1, β_1) and (E_2, α_2, β_2) are called strongly equivalent if there exists a unitary $u \in M(B)$ such that $\tau_2(a) = \pi(u)\tau_1(a)\pi(u)^*$ for any $a \in A$.

Let H be a separable infinite dimensional Hilbert space, and let $\mathcal{K}$ be the ideal of compact operators in $B(H)$.

If B is a stable C^*-algebra (i.e. $B \otimes \mathcal{K} \cong B$), then the sum of two extensions τ_1 and τ_2 is defined to be the homomorphism $\tau_1 \oplus \tau_2$, where $\tau_1 \oplus \tau_2 \colon A \to Q(B) \oplus Q(B) \subseteq M_2(Q(B)) \cong Q(B)$.

Let $\mathbb{T}$ be the unit circle in the complex plane and let $C(\mathbb{T})$, which is called circle algebra, denote the set of all continuous functions on $\mathbb{T}$. Sometimes we also call matrix algebras over $C(\mathbb{T})$ circle algebras. We consider unital essential extensions of $C(\mathbb{T})$ by $\mathcal{K}$. By BDF theory, we have that the extension group $\mathrm{Ext}(\mathbb{T})$ is isomorphic to $\mathbb{Z}$. It follows from a well-known theorem of Voiculescu in [11] that extensions with the same index are isomorphic. Suppose that $\mathcal{T}_k$ is the extension of $C(\mathbb{T})$ by $\mathcal{K}$ with index $-k \in \mathbb{Z}$. Then there is a short exact sequence $0 \to \mathcal{K} \to \mathcal{T}_k \to C(\mathbb{T}) \to 0$. Since $\mathcal{T}_k$ is isomorphic to $\mathcal{T}_{-k}$, we only consider the case $k \geq 0$, and call these algebras $\mathcal{T}$-algebras.

One can construct these algebras as below. Let S be the unilateral shift operator on H and let π be the quotient map from $B(H)$ onto the Calkin algebra $\mathcal{Q} = B(H)/\mathcal{K}$. Set $u = \pi(S^k)$ for $k > 0$. Then u is a unitary in $\mathcal{Q}$. Let $\mathrm{sp}_e(S^k)$ denote the essential spectrum of S^k. We have $\mathrm{sp}_e(S^k) = \mathrm{sp}(\pi(S^k)) = \mathbb{T}$. Set $\mathcal{T}_k' = C^*(S^k, \mathcal{K})$. Then we have $0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}_k' \overset{\psi}{\longrightarrow} C(\mathbb{T}) \longrightarrow 0$. It is easy to see that the Busby invariant of the above extension is defined by mapping z to u, where z is the identity function on $\mathbb{T}$. Then $\mathcal{T}_k'$ is a unital essential extension with index $-k$ since z has a lift to S^k in $\mathcal{T}_k'$. Therefore $\mathcal{T}_k' \cong \mathcal{T}_k$.

When $k = 0$, one can choose a unitary $S_0 \in B(H)$ such that $\mathrm{sp}_e(S_0) = \mathbb{T}$. Then $\mathcal{T}_0$ is isomorphic to the C^*-subalgebra $C^*(S_0, \mathcal{K})$ generated by S_0 and $\mathcal{K}$ in $B(H)$.

If A is a C^*-algebra, let $V(A)$ denote the semigroup consisting of Murray–von Neumann equivalence classes of projections in matrix algebras over A. By computation, we have the semigroups of $\mathcal{T}$-algebras (see [7] for details):

$$V(\mathcal{T}_0) = \{(m, n) \colon m \in \mathbb{Z}, n \in \mathbb{Z}^+,\ n = 0,\ m \geq 0\}$$

$$V(\mathcal{T}_1) = \mathbb{Z}^+ \sqcup \mathbb{N}$$

$$V(\mathcal{T}_k) = \mathbb{Z}^+ \sqcup \mathbb{N} \oplus \mathbb{Z}_k.$$

Obviously, $V(\mathcal{T}_0)$ has the cancellation property, so $\mathcal{T}_0$ is finite. In fact, $\mathcal{T}_0$ is an AT-algebra by Theorem 3.5. Moreover, $V(\mathcal{T}_0)$ is not finitely generated. Hence $\mathcal{T}_0$ is quite different from $\mathcal{T}_k$ ($k \geq 1$).

Suppose that E_1 and E_2 are finite direct sums of matrix algebras over $\mathcal{T}$-algebras, that is,

$$E_1 = \bigoplus_{j=1}^{L_1} M_{l_j}(\mathcal{T}_{k_j}), \quad E_2 = \bigoplus_{j=1}^{L_2} M_{m_j}(\mathcal{T}_{r_j}).$$

Then there exist extensions $0 \to I(E_i) \to E_i \to Q(E_i) \to 0$ for $i = 1, 2$, where

$$I(E_1) = \bigoplus_{j=1}^{L_1} M_{l_j}(\mathcal{K}), \qquad I(E_2) = \bigoplus_{j=1}^{L_2} M_{m_j}(\mathcal{K}),$$

$$Q(E_1) = \bigoplus_{j=1}^{L_1} M_{l_j}(C(\mathbb{T})), \quad Q(E_2) = \bigoplus_{j=1}^{L_2} M_{m_j}(C(\mathbb{T})).$$

Let $\pi_i \colon E_i \longrightarrow Q(E_i)$ be the quotient maps. If $\phi \colon E_1 \longrightarrow E_2$ is a homomorphism, then we have $\phi(I(E_1)) \subset I(E_2)$. So there is a commutative diagram in the following:

$$\begin{tikzcd}
0 \arrow[r] & I(E_1) \arrow[r] \arrow[d, "\phi"'] & E_1 \arrow[r] \arrow[d, "\phi"] & Q(E_1) \arrow[r] \arrow[d, "\bar{\phi}"] & 0 \\
0 \arrow[r] & I(E_2) \arrow[r] & E_2 \arrow[r] & Q(E_2) \arrow[r] & 0,
\end{tikzcd}$$

where $\bar{\phi} \colon Q(E_1) \longrightarrow Q(E_2)$ is the induced map of ϕ.

Then $A\mathcal{T}$-algebras are defined to be inductive limits of finite direct sums of matrix algebras over $\mathcal{T}$-algebras. One can see [7] for more details of $A\mathcal{T}$-algebras.

3. The main results

Theorem 3.1.

(1) *Every $A\mathcal{T}$-algebra E contains a stable AF ideal $I(E)$ such that the quotient $Q(E)$ is an AT-algebra.*

(2) *Every quotient of an $A\mathcal{T}$-algebra is an $A\mathcal{T}$-algebra.*

Proof. Let E be an $A\mathcal{T}$-algebra. Suppose $E = \varinjlim(E_n, \phi_{n,n+1})$, where E_n is a finite direct sum of matrix algebras over $\mathcal{T}$-algebras.

(1) Let $I_n = I(E_n)$, $A_n = Q(E_n)$, $I_0 = \varinjlim I(E_n)$ and $A_0 = \varinjlim Q(E_n)$. Then I_0 is a stable AF-ideal of E, and A_0 is an AT-algebra. Moreover, we have a short exact sequence $0 \to I_0 \to E \to A_0 \to 0$.

(2) Suppose that I is a non-zero ideal of E. Then there is an ideal I_n' of E_n for every n such that $I = \varinjlim I_n'$.

Let $E_n = \bigoplus_{i=1}^{k_n} E_{n,i}$, where $E_{n,i}$ is a matrix algebra over $\mathcal{T}$-algebras. Since I_n' is an ideal of E_n, I_n' has the form $I_n' = \bigoplus_{i=1}^{k_n} I_{n,i}'$, where $I_{n,i}'$ is an ideal of $E_{n,i}$. Since $I(E_{n,i})$ is a simple essential ideal, it follows that either $I(E_{n,i}) \subset I_{n,i}'$ or $I_{n,i}' = \{0\}$. Therefore $E_{n,i}/I_{n,i}'$ is equal to $E_{n,i}$ or a quotient $Q(E_{n,i})$. Note that $E_n/I_n' = \bigoplus_{i=1}^{k_n} E_{n,i}/I_{n,i}'$, so we see that E_n/I_n' is a direct sum of matrix algebras over $\mathcal{T}$-algebras and AT-algebras. It follows that the map ϕ_n': $E_n/I_n' \to E/I'$ is an injective homomorphism and $E/I = \varinjlim E_n/I_n'$ since $I = \varinjlim I_n'$. Therefore E/I is an $A\mathcal{T}$-algebra. □

Theorem 3.2. *Let E be an $A\mathcal{T}$-algebra and let B be a hereditary C^*-subalgebra of E. If B is an $A\mathcal{T}$-algebra, then B admits an approximate identity consisting of projections. Conversely, if B admits an approximate identity consisting of projections, then B is an $A\mathcal{T}$-algebra.*

Proof. Suppose that B is an $A\mathcal{T}$-algebra. Let $B = \varinjlim(B_n, \varphi_{n,n+1})$, where B_n is a finite direct sum of matrix algebras over $\mathcal{T}$-algebras and φ_n is the canonical map from B_n into B. Then $\{\varphi_n(1_{B_n})\}$ is an approximate identity for B consisting of projections.

Conversely, let $E = \varinjlim(E_n, \phi_{n,n+1})$ and let B be a hereditary C*-subalgebra of E. Then B is σ-unital. If B admits an approximate identity consisting of projections, then it contains a countable increasing approximate identity consisting of projections by Theorem 6 in [8]. Let $\{e_n\}$ be an approximate identity for B. Since B is hereditary, we have $e_nEe_n \subset B$ and $B = \overline{\bigcup_{n=1}^{\infty} e_nEe_n}$. Note that $e_n \le e_{n+1}$, so it follows that $e_nEe_n \subset e_{n+1}Ee_{n+1}$ and $B = \varinjlim e_nEe_n$.

By Theorem 3.1, E is an extension of an AT-algebra A by a stable AF-algebra I. Let π be the quotient map from E into A. Therefore, for every $n \in \mathbb{N}$, we have a short exact sequence

$$0 \to e_nIe_n \to e_nEe_n \to \pi(e_n)A\pi(e_n) \to 0.$$

Let $I' = \varinjlim e_nIe_n$ and $A' = \varinjlim \pi(e_n)A\pi(e_n)$. Then there is an extension $0 \to I' \to B \to A' \to 0$. Since e_nIe_n is an AF-algebra and $\pi(e_n)A\pi(e_n)$ is a AT-algebra, B is an extension of an AT-algebra by an AF-algebra.

Let $E_n' = \operatorname{im}\phi_n \cong E_n/\ker\phi_n$. Then $E = \varinjlim E_n'$ and E_n' is a direct sum of matrix algebras over $\mathcal{T}$-algebras and AT-algebras.

For any $n \in \mathbb{N}$, since $e_n \in E$, it follows from Lemma 2.7.2 in [4] that there exist a sufficiently large number $k_n \in \mathbb{N}$ and projection $p_{k_n} \in E_{k_n}'$ such that $p_{k_n} \sim e_n$. Then $e_nEe_n \cong p_{k_n}Ep_{k_n}$ and hence $e_nEe_n \cong \overline{\bigcup_{i=1}^{\infty} p_{k_n}E_i'p_{k_n}} = \overline{\bigcup_{i\ge k_n} p_{k_n}E_i'p_{k_n}}$.

Let $E_i' = E_{i,1}' \oplus E_{i,2}' \oplus \cdots \oplus E_{i,k}'$ and $p_{k_n} = q_1 \oplus q_2 \oplus \cdots \oplus q_k \in E_i'$. Then $p_{k_n}E_i'p_{k_n} = \bigoplus_{j=1}^{k} q_jE_{i,j}'q_j$. Note that $q_jE_{i,j}'q_j$ is a unital hereditary C*-subalgebra of $E_{i,j}'$. Hence $q_jE_{i,j}'q_j$ is also an AT-algebra when $E_{i,j}'$ is an AT-algebra,.

If $E_{i,j}' = M_r(\mathcal{T}_l)$, then we have the following extension:

$$0 \to q_jI(E_{i,j}')q_j \to q_jE_{i,j}'q_j \to \pi(q_j)Q(E_{i,j}')\pi(q_j) \to 0.$$

For $q_j \in I(E_{i,j}')(= M_r(\mathcal{K}))$, we have that $q_jE_{i,j}'q_j(= q_jI(E_{i,j}')q_j)$ is isomorphic to a matrix algebra M_d for some positive integer d.

If q_j is not contained in $I(E'_{i,j})$, then $\pi(q_j)Q(E'_{i,j})\pi(q_j)$ is still a circle algebra since it is a corner of $M_r(C(\mathbb{T}))$. Since $I(E'_{i,j})$ is an ideal, $q_jI(E'_{i,j})q_j$ is a hereditary C*-subalgebra of $I(E'_{i,j})$. Note that $q_jI(E'_{i,j})q_j$ is isomorphic to a hereditary C*-subalgebra of $\mathcal{K}$, so it is simple. Since a C^*-subalgebra of $\mathcal{K}$ is isomorphic to a direct sum of a sequence of subalgebras $\mathcal{K}(H_\alpha)$ and $q_jI(E'_{i,j})q_j$ is simple, it follows that $q_jI(E'_{i,j})q_j$ is isomorphic to some $\mathcal{K}(H_\alpha)$. Let $I(E'_{i,j}) \cong \mathcal{K}(H)$. Then $q_j(H)$ is an infinite dimensional subspace of H, and there is a sequence of mutually orthogonal finite projections $\{q'_n\} \subset \mathcal{K}(H)$ such that $q'_n \leq q_j$. Therefore $\{q'_n\} \subset q_jI(E'_{i,j})q_j$, and $q_jI(E'_{i,j})q_j$ is infinite dimensional. So we have $\dim(H_\alpha) = \infty$ and $q_jI(E'_{i,j})q_j \cong \mathcal{K}$. It follows that $q_jE'_{i,j}q_j$ is a corner of a matrix algebra over $\mathcal{T}$-algebra, so e_nEe_n is an $A\mathcal{T}$-algebra. Since an inductive limit of $A\mathcal{T}$-algebras is also an $A\mathcal{T}$-algebra, we conclude that B is an $A\mathcal{T}$-algebra. □

Example 1. By Theorem 3.2, the ideal $C_0(0,1)$ of $C(\mathbb{T})$ is not an $A\mathcal{T}$-algebra. A hereditary C*-subalgebra of $\mathcal{T}$-algebra is not necessarily an $A\mathcal{T}$-algebra. For example, we consider $E = M_m(\mathcal{T}_k)$. Note that $M_m(C_0(0,1))$ is an ideal of $Q(E)$. Let π be the quotient map from E onto $Q(E)$ and let B be the inverse image of $M_m(C_0(0,1))$ under π. Then B is an ideal of E and the sequence $0 \to M_m(\mathcal{K}) \to B \to M_m(C_0(0,1)) \to 0$ is exact. If B is an $A\mathcal{T}$-algebra, then there is an approximate identity $\{p_n\}$ consisting of projections. Therefore $\{\pi(p_n)\}$ is an approximate identity for $M_m(C_0(0,1))$ consisting of projections. This is impossible.

Lemma 3.3. *Let A be a unital finite C^*-algebra. Suppose that A is simple. Then every C^*-subalgebra of A is not isomorphic to $\mathcal{K}$.*

Proof. Let A_0 be a C^*-subalgebra of A which is isomorphic to $\mathcal{K}$. Then the unit 1 of A is not in A_0.

Let p be the minimal projection of A_0 and let $\{p_i\}$ be a sequence of pairwise orthogonal projections of A_0 such that $p_i \sim p$. Since p is full in A, there exist $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in A$ such that

$$x_1px_1^* + x_2px_2^* + \cdots + x_npx_n^* = 1.$$

Let $v = (x_1p, x_2p, \ldots, x_np)$. We have that $vv^* = 1$ and v^*v is a projection. Therefore $v^*v = [px_i^*x_jp]_{n\times n} \leq p \oplus p \oplus \cdots \oplus p$.

Since $p \oplus p \oplus \cdots \oplus p \sim p_1 + p_2 + \cdots + p_n$, there is a projection $q \in A$ such that $1 \sim q \leq p_1 + p_2 + \cdots + p_n < 1$. This is a contradiction since A is finite. Therefore there is no C^*-subalgebra of A which is isomorphic to $\mathcal{K}$. □

Theorem 3.4. *Let E be a unital $A\mathcal{T}$-algebra and let I be a maximal ideal of E. Then E/I is an AT-algebra.*

Proof. Since I is a maximal ideal, E/I is simple. By Theorem 3.1, E/I is an $A\mathcal{T}$-algebra. Suppose that E/I is the inductive limit of $(B_n, \phi_{n,n+1})$, where B_n is finite direct sum of matrix algebras over $\mathcal{T}$-algebras. Let $I' = \varinjlim I(B_n)$. If E/I is infinite, then $I' \neq \{0\}$ (Otherwise, E/I is an *AT*-algebra by Theorem 3.1). Note that $I' \subset E/I$ and E/I is simple, so we have $I' = E/I$. This is a contradiction since E/I is infinite. It follows that E/I is finite if it is simple. By Lemma 3.3, there is no C^*-subalgebra of E/I which is isomorphic to $\mathcal{K}$. Since I' is a stable AF-algebra, we have $I' = \{0\}$. So $E/I \cong Q(E/I)$ is an *AT*-algebra. □

Remark. Let E be a unital $A\mathcal{T}$-algebra. If E admits a stable maximal AT-ideal, then by the above Theorem, E is an extension of an AT-algebra. However, in general, a maximal ideal of an $A\mathcal{T}$-algebra is not necessarily an AT-algebra. For example, we consider $\mathcal{T}_k$. Let I be a maximal ideal of $\mathcal{T}_k$. Then $\mathcal{T}_k/I$ is a simple AT-algebra by Theorem 3.4. Since $\mathcal{K}$ is a minimal ideal of $\mathcal{T}_k$, it follows that $\mathcal{K} \subset I$, and $\mathcal{T}_k/I$ is a quotient of $C(\mathbb{T})$. Therefore $\mathcal{T}_k/I \cong \mathbb{C}$. Since $\mathcal{T}_k/I \cong (\mathcal{T}_k/\mathcal{K})/(I/\mathcal{K})$, $C(\mathbb{T})/(I/\mathcal{K}) \cong \mathbb{C}$ and $I/\mathcal{K} \cong C_0(0, 1)$. So we obtain $I \cong \pi^{-1}(C_0(0, 1))$. It follows that I is not an $A\mathcal{T}$-algebra or an AT-algebra by Example 1.

Theorem 3.5. *If E is an $A\mathcal{T}$-algebra and $\delta_1\colon K_1(Q(E)) \to K_0(I(E))$ is the associated index map, then E is an AT-algebra if and only if $\delta_1 = 0$.*

Proof. Let E be the inductive limit of $\{E_n, \phi_{n,n+1}\}$. Since $I(E)$ and $Q(E)$ are both AT-algebras, it follows from Proposition 4(ii) in [6] that $\delta_1 = 0$ if and only if $\mathrm{sr}(E) = 1$. For any n and i, if $E_{n,i}$ is isomorphic to $M_m(\mathcal{T}_k)$ $(k > 0)$ for some m and k, then we have $\phi_n|_{I(E_{n,i})} = 0$ since $\mathrm{sr}(E) = 1$. Thus $\phi_n(E_n)$ is an AT-algebra. We conclude that E is an AT-algebra by $E = \overline{\bigcup_{n=1}^{\infty} \phi_n(E_n)}$. □

Theorem 3.6. *Let E be an $A\mathcal{T}$-algebra and let I be an ideal of E. Suppose that the partial map vanishes on the ideal $I(M_k(\mathcal{T}_0))$ of $M_k(\mathcal{T}_0)$. Then E/I is an AT-algebra if and only if $I(E) \subset I$.*

Proof. Let $E = \varinjlim(E_n, \phi_{n,n+1})$ and $I_0 = I(E)$. Then $I_0 = \varinjlim I(E_n)$. If $I_0 \not\subseteq I$, then there is $(p \otimes \mathcal{K}) \cap I = \{0\}$ since $\bigcup\{p \otimes \mathcal{K}\colon p \in P(I_0)\}$ is dense in $I_0 \otimes \mathcal{K} \cong I_0$. Note that $(p \otimes \mathcal{K}) \subset I_0$, so we see that there is a positive integer n such that $\phi_n|_{I(E_n)} \neq 0$ and $\phi_n(I(E_n)) \cap (p \otimes \mathcal{K}) \neq \{0\}$. Let $I_n = \phi_n^{-1}(I \cap \phi_n(E_n))$. Then $I(E_n) \not\subseteq I_n$. Otherwise, if $I(E_n) \subset I_n$, then $\phi_n(I(E_n)) \subset \phi_n(I_n) \subset I$.

Since $I(E_n) \not\subseteq I_n$, there is a positive integer i such that $I(E_{n,i}) \cap I_n = \{0\}$. It follows that $\phi_n|_{I(E_{n,i})} \neq 0$. Otherwise, $\phi_n(I(E_{n,i}) = 0$ and hence $I(E_{n,i}) \subset I_n$.

Since $I(E_{n,i})$ is an essential ideal of $E_{n,i}$, ϕ_n is injective on $E_{n,i}$ and $E_{n,i}$ is not isomorphic to matrix algebra over $\mathcal{T}_0$. Notice that I_n is an ideal of E_n, so $I_n \cap E_{n,i}$ is an ideal of $E_{n,i}$. Since $I_n \cap I(E_{n,i}) = \{0\}$, it follows that $I_n \cap E_{n,i} = \{0\}$. Let π_n be the quotient map from E_n onto E_n/I_n. Then $\pi_n(E_{n,i}) = E_{n,i}/(E_{n,i} \cap I_n) \cong E_{n,i}$. Since $E_{n,i}$ is not finite, this is a contradiction for E/I is an AT-algebra. Thus $I(E) \subset I$. □

Remark. The condition the partial map vanishes on the ideal $I(M_k(\mathcal{T}_0))$ of $M_k(\mathcal{T}_0)$ in Theorem 3.6 is necessary. For example, let $E = M_{n_1}(\mathcal{T}_1) \oplus M_{n_2}(\mathcal{T}_0)$. Note that $\mathcal{T}_0$ is an AT-algebra by Theorem 3.5. Then $I(E) = M_{n_1}(\mathcal{K}) \oplus M_{n_2}(\mathcal{K})$, and there is a short exact sequence

$$0 \to M_{n_1}(\mathcal{K}) \oplus M_{n_2}(\mathcal{K}) \to E \to M_{n_1}(C(\mathbb{T})) \oplus M_{n_2}(C(\mathbb{T})) \to 0.$$

Take $I = M_{n_1}(\mathcal{K}) \oplus 0$. Then $E/I \cong M_{n_1}(C(\mathbb{T})) \oplus M_{n_2}(\mathcal{T}_0)$ is an AT-algebra, and $I(E) \not\subseteq I$.

The following lemma must be known to others.

Lemma 3.7. *If A is a stable C^*-algebra and p is a projection in A, then there is a C^*-subalgebra B_0 of A which is isomorphic to $\mathcal{K}$ such that $p \in B_0$.*

Proof. Since $A \otimes \mathcal{K} \cong A$, without loss of generality, we consider $A \otimes \mathcal{K}$. Since $A \otimes \mathcal{K} = \overline{\bigcup_{n=1}^{\infty} M_n(A)}$ and $p \in A \otimes \mathcal{K}$, there exist $n \in \mathbb{N}$ and a projection $q_1 \in M_n(A)$ such that $\|p - q_1\| < 1$. This implies that there is a unitary $u \in \widetilde{A \otimes \mathcal{K}}$ such that $p = \mathrm{Ad}(u)(q_1)$. Since $\bigoplus_{i=1}^{k} M_n(A) \subset M_{kn}(A)$, there are projections $q_1, q_2, \ldots, q_k$ in $M_{kn}(A)$ such that $q_i \sim q_1$ for $1 \leq i \leq k$ and $q_{ij} = 0$ for $i \neq j$. For $k+1$, put $q_{k+1} = (0, 0, \ldots, q_1)$ in $M_{(k+1)n}(A)$, where 0 repeats k times. Then $q_1, q_2, \ldots, q_{k+1}$ are mutually orthogonal and equivalent. Therefore there is a sequence of mutually orthogonal and equivalent projections $\{q_n\}$. Put $p_n = \mathrm{Ad}(u)(q_n)$. Then $\{p_n\}$ is a sequence of mutually orthogonal and equivalent projections in $A \otimes \mathcal{K}$.

Let $v_{1i}v_{1i}^* = p_1$ and $v_{1i}^* v_{1i} = p_{ii}$. Set $v_{ij} = v_{1i}^* v_{1j}$. Then $\{v_{ij}\colon 1 \leq i, j \leq n\}$ is a system of matrix units . Let $A_n = C^*\{v_{ij}\colon 1 \leq i, j \leq n\}$ and $\{e_{ij}\colon 1 \leq i, j \leq n\}$ be the matrix units for M_n. Let $\alpha_n\colon A_n \to M_n$ be the map such that $\alpha_n(v_{ij}) = e_{ij}$, so α_n is an isomorphism. Let $i_n\colon A_n \to A_{n+1}$ be the inclusion map and $j_n\colon M_n \to M_{n+1}$ such that $j_n(a) = a \oplus 0$ for $a \in M_n$.

For any $x \in A_n$, let $x = \sum \lambda_{ij} v_{ij}$. Then

$$\alpha_{n+1} \circ i_n(x) = \sum_{1 \leq ij \leq n} \lambda_{ij} e_{ij} + \sum_{i=n+1 \text{ or } j=n+1} 0 \cdot e_{ij} = j_n(\alpha_n(x)).$$

So we have the following commutative diagram:

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{i_1} & A_2 & \xrightarrow{i_2} & \cdots\cdots & \longrightarrow & \varinjlim A_n \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & & & \\
M_1 & \xrightarrow{j_1} & M_2 & \xrightarrow{j_2} & \cdots\cdots & \longrightarrow & \mathcal{K}
\end{array} .$$

Therefore $\varinjlim A_n \cong \mathcal{K}$. □

Theorem 3.8. *Let* $0 \to B_1 \to E_1 \to A_1 \to 0$ *and* $0 \to B_2 \to E_2 \to A_2 \to 0$ *be two extensions of C^*-algebras and let $\phi\colon E_1 \to E_2$ be a homomorphism. Suppose that B_1 is stable and admits an approximate identity consisting of projections (not necessarily countable), and suppose that A_2 is a unital simple finite algebra. Then $\phi(B_1) \subset B_2$. If we further assume that $E_1 = E_2$ and B_1 is a maximal ideal, then $B_1 = B_2$.*

Proof. Without loss of generality, we may assume that B_i is an ideal of E_i. Let $\pi_i\colon E_i \to A_i$ be the quotient map and $\{p_\alpha\}$ be the approximate identity for B_1 consisting of projections. Since B_1 is stable, for any α, there is a C^*-subalgebra $B_\alpha \cong \mathcal{K}$ such that $p_\alpha \in B_\alpha$ by Lemma 3.7. Note that B_α is simple, so $\pi_2 \circ \phi|_{B_\alpha}$ is injective or $\pi_2 \circ \phi|_{B_\alpha} = 0$. Since A_2 is simple and finite, it follows from Lemma 3.3 that $\pi_2 \circ \phi|_{B_\alpha} = 0$. Therefore $\phi(p_\alpha) \in B_2$.

Since B_2 is an ideal, $\phi(p_\alpha B_1) = \phi(p_\alpha)\phi(B_1) \subset \phi(p_\alpha)E_2 \subset B_2$. For any $b \in B_1$, since $p_\alpha b \to b$, we have $\phi(p_\alpha b) \to \phi(b)$, and then $\phi(b) \in B_2$. Thus $\phi(B_1) \subset B_2$.

Moreover, we assume $E_1 = E_2$ and B_1 is a maximal ideal. Let $\phi\colon E_1 \to E_2$ be an isomorphism. Then $\phi(B_1)$ is a maximal ideal, and therefore $\phi(B_1) = B_2$. By taking $\phi = \mathrm{id}$, we have $B_1 = B_2$. □

COROLLARY 3.9

Let E be a unital $A\mathcal{T}$-algebra. If E has a maximal ideal which is a stable AT-algebra, then such ideal is unique.

Example 2. The requirements for A_2 in Theorem 3.8 are necessary. We now give some examples. By Example 2, we see that an isomorphism between extension algebras of *AT*-algebras by stable *AT*-algebras may not map the ideal into another in the corresponding short exact sequences.

(1) If A_2 is not simple, then the homomorphism ϕ may not preserve the ideal.

For example, take $B_1 = B_2 = \mathcal{K}$, $E_1 = E_2 = \mathcal{K} \oplus \widetilde{\mathcal{K}}$ and $A_1 = A_2 = \widetilde{\mathcal{K}}$. Let ϕ: $E_1 \to E_2$ be a homomorphism such that $\phi|_{\mathcal{K}}$: $\mathcal{K} \to \widetilde{\mathcal{K}}$ is the inclusion map and $\phi|_{\widetilde{\mathcal{K}}}$: $\widetilde{\mathcal{K}} \to \mathcal{K}$ is zero.

(2) If A_2 is non unital, then the conclusion also may not hold.

For example, we set $B_1 = B_2 = \mathcal{K}$, $E_1 = E_2 = \mathcal{K} \oplus \mathcal{K}$ and $A_1 = A_2 = \mathcal{K}$. For any $(a, b) \in E_1$, define $\phi((a, b)) = (b, a)$, and then we have $\phi(B_1) \cap B_2 = \{0\}$.

(3) If A_2 is not finite, then $\phi(B_1) \subset B_2$ may not hold. We consider extensions of Cuntz algebra $\mathcal{O}_2$ by $\mathcal{K}$. Let $0 \to \mathcal{K} \to E \to \mathcal{O}_2 \to 0$ be an essential unital extension, and let $0 \to \mathcal{K} \to E' \xrightarrow{\pi} \mathcal{O}_2 \to 0$ be a trivial essential unital extension. So there exists a homomorphism λ: $\mathcal{O}_2 \to E'$ such that $\pi \circ \lambda = \mathrm{id}_{\mathcal{O}_2}$. Since E is exact, by Kirchberg's exact imbedding theorem [10], there is a unital injective homomorphism η: $E \to \mathcal{O}_2$. Let $\phi = \lambda \circ \eta$. Then ϕ is an injective homomorphism from E into E' and $\phi(\mathcal{K}) \cap \mathcal{K} = \{0\}$.

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