

Uniqueness of solutions to Schrödinger equations on complex semi-simple Lie groups

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Abstract. In this note we study the time-dependent Schrödinger equation on complex semi-simple Lie groups. We show that if the initial data is a bi-invariant function that has sufficient decay and the solution has sufficient decay at another fixed value of time, then the solution has to be identically zero for all time. We also derive Strichartz and decay estimates for the Schrödinger equation. Our methods also extend to the wave equation. On the Heisenberg group we show that the failure to obtain a parametrix for our Schrödinger equation is related to the fact that geodesics project to circles on the contact plane at the identity.

Keywords. Schrödinger equation; uniqueness; Strichartz estimates; complex Lie groups; Heisenberg group.

1. Introduction

Let G denote a Lie group. We are concerned here with the initial value problem for the time-dependent Schrödinger equation

$$-iu_t = \Delta u, \quad u|_{t=0} = f(x), \quad x \in G. \quad (1.1)$$

Δ will denote the Laplace–Beltrami operator with respect to an appropriate G invariant metric. Our main aim is to obtain conditions on f which will guarantee that the solution $u(x, t)$, vanishes identically for all $t > 0$. The results that we have may all be viewed as consequences of a well-known theorem of Hardy on the Euclidean Fourier transform, which is a form of the uncertainty principle. The Hardy theorem has been extended to the Lie group setting by various authors (see [2, 9]). However, we do not need any Lie group version of the Hardy theorem. The original Euclidean version will suffice in our applications to Lie groups. We plan to extend our results to the real semi-simple Lie groups and to obtain versions on the Heisenberg group later. On nilpotent groups there is a serious difficulty since the geodesics project to circles on the contact plane and the existence of such closed loops creates difficulties in writing down a parametrix for the solution operator to (1.1) (see §3). Our results may be viewed as a statement that a concentrated wave packet at initial time will spread out and if it still remains concentrated, then it must be trivial. Lastly, in §4 we derive decay estimates and Strichartz estimates for bi-invariant solutions to the Schrödinger equation on complex semi-simple groups. The methods extend to the wave equation on complex Lie groups.

2. The uniqueness theorem

We recall the theorem of Hardy. Let the Fourier transform be defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} f(x) \, dx.$$

Theorem 1 [4]. *Let $f(x)$ satisfy $|f(x)| \leq Ce^{-a|x|^2}$. Furthermore, assume $|\hat{f}(\xi)| \leq Be^{-b|\xi|^2}$. If $4ab > 1$, then $f \equiv 0$.*

We shall now derive from this theorem a uniqueness theorem for solutions to (1.1) when $G = \mathbb{R}^n$.

Theorem 2. *Let us consider the initial value problem for $u(x, t)$,*

$$-iu_t = \Delta u, \quad u|_{t=0} = f(x).$$

Assume that $|f(x)| \leq Ae^{-a|x|^2}$, and $|u(x, t_0)| \leq Be^{-b|x|^2}$. If $16abt_0^2 > 1$, then $u(x, t) \equiv 0$, for all $t \geq 0$.

Proof. We recall that using the fundamental solution of (1.1), we may write

$$u(x, t) = \frac{c_n}{t^{n/2}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} f(y) \, dy.$$

We may re-write the last identity as

$$u(x, t) = \frac{c_n e^{i\frac{|x|^2}{4t}}}{t^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle \frac{x}{2t}, y \rangle} e^{i\frac{|y|^2}{4t}} f(y) \, dy. \quad (2.1)$$

Now set $h(y) = e^{i\frac{|y|^2}{4t}} f(y)$. Then from (2.1) we get

$$u(x, t) = \frac{c_n e^{i\frac{|x|^2}{4t}}}{t^{n/2}} \hat{h}\left(\frac{x}{2t}\right).$$

Now we take $t = t_0$ and apply Hardy's theorem. From the hypothesis on $u(x, t_0)$, we have

$$\left| \hat{h}\left(\frac{x}{2t_0}\right) \right| \leq Be^{-b|x|^2}.$$

Thus

$$|\hat{h}(x)| \leq Be^{-4bt_0^2|x|^2}.$$

Clearly $|h(x)| \leq Ce^{-a|x|^2}$. Since $16abt_0^2 > 1$, Hardy's theorem applies and we conclude $h \equiv 0$. It then follows that $f \equiv 0$ and hence $u(\cdot, t) \equiv 0$ for all t .

We now wish to extend Theorem 2 to a complex semi-simple Lie group. We need to introduce some notation.

Let G denote a complex, connected semi-simple Lie group. Let K denote a fixed maximal compact sub-group of G . Let \mathcal{G} and \mathcal{K} denote the Lie algebras of G and K respectively. Let B denote the Cartan–Killing form on \mathcal{G} , and the Cartan decomposition is given by

$\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$. The restriction of B to $\mathcal{P} \times \mathcal{P}$ is strictly positive definite and hence defines a norm. Let \mathcal{A} be a fixed maximal Abelian subspace of \mathcal{P} . Let Σ denote the set of non-zero roots corresponding to the pair $(\mathcal{G}, \mathcal{A})$, and Σ_+ the set of positive roots α for some ordering. W will denote the Weyl group associated to Σ . Let \mathcal{A}_+ be the positive Weyl chamber associated to Σ . Let A denote the analytic sub-group with Lie algebra \mathcal{A} . We have the map, $\exp: \mathcal{A} \rightarrow A$. Likewise, we have $A_+ = \exp(\mathcal{A}_+)$. Then one has the polar decomposition

$$G = K \overline{A_+} K. \quad (2.2)$$

The Haar measure on A can then be written for this polar decomposition by a formula of Harish-Chandra [5] as

$$dx = \left(\sum_{s \in W} (\det s) e^{s\rho(H)} \right)^2 dH = \phi(H)^2 dH, \quad H \in \mathcal{A}. \quad (2.3)$$

As usual $\rho = \frac{1}{2} \sum_{\Sigma_+} \alpha$. The spectral variables are elements of \mathcal{A}^* and will be denoted by λ . On G/K we have a G invariant Riemannian metric obtained through the Killing form and we can form a Laplace–Beltrami operator Δ using this metric (see [7]). Note now, that there is a unique element $H_\lambda \in \mathcal{A}$, such that

$$\lambda(H) = B(H_\lambda, H), \quad H \in \mathcal{A}. \quad (2.4)$$

The norm of elements $H \in \mathcal{A}$ will be denoted by

$$|H|^2 = B(H, H).$$

Lastly a function f on G is said to be K bi-invariant if and only if

$$f(k_1 a k_2) = f(a), \quad a \in A.$$

From the polar decomposition we may view the function as only depending on its values on A_+ , or by using the inverse exponential map we may also view f as a complex-valued function on \mathcal{A} solely determined by its values on \mathcal{A}_+ . We are ready to state our theorem.

Theorem 3. *Let G denote a connected, complex, semi-simple Lie group. Let f be a bi-invariant function such that*

$$|f(H)| \leq A e^{-a|H|^2}.$$

Consider the initial value problem

$$-iu_t = \Delta u, \quad u|_{t=0} = f.$$

Then u is also bi-invariant. Furthermore, if

$$|u(H, t_0)| \leq B e^{-b|H|^2},$$

and $16abt_0^2 > 1$, then necessarily $u \equiv 0$ for all $t \geq 0$.

Proof. The proof requires the use of well-known facts about spherical functions. First we need to recall Theorem 5.7, p. 432 of [7] which states that on a complex semi-simple group, the elementary spherical functions are given by (where $\phi(H)$ is as in (2.3)),

$$\phi_\lambda(H) = c(\lambda) \frac{\sum_{s \in W} (\det s) e^{is\lambda(H)}}{\phi(H)}. \quad (2.5)$$

Here $c(\lambda)$ is the celebrated c -function of Harish-Chandra [5, 6] (see also, [3]). Furthermore, we also have from [7] that

$$\Delta \phi_\lambda(H) = -(|\lambda|^2 + |\rho|^2) \phi_\lambda(H). \quad (2.6)$$

The spherical transform of f is given by

$$\hat{f}(\lambda) = \int_{\mathcal{A}} \phi_{-\lambda}(H) f(H) \phi^2(H) dH. \quad (2.7)$$

Thus the solution of our initial value problem in view of (2.6) is

$$u(H, t) = \int_{\mathcal{A}^*} e^{-it(|\lambda|^2 + |\rho|^2)} \phi_\lambda(H) \hat{f}(\lambda) |c(\lambda)|^{-2} d\lambda. \quad (2.8)$$

Putting (2.5) and (2.7) into (2.8) we get

$$\begin{aligned} u(H_1, t) \phi(H_1) &= \int_{\mathcal{A}} \left(\int_{\mathcal{A}^*} e^{-it(|\lambda|^2 + |\rho|^2)} \left(\sum_{s \in W} (\det s) e^{is\lambda(H_1)} \right) \right. \\ &\quad \times \left. \left(\sum_{s' \in W} (\det s') e^{-is'\lambda(H_2)} \right) d\lambda \right) f(H_2) \phi(H_2) dH_2. \end{aligned} \quad (2.9)$$

The inner integral in λ may be re-written in view of (2.4) as

$$\sum_{s, s'} (\det s) (\det s') e^{-it|\rho|^2} e^{i \frac{|sH_1 - s'H_2|^2}{4t}} \int_{\mathcal{A}} e^{-it|H_\lambda + \frac{1}{2t}(sH_1 - s'H_2)|^2} dH_\lambda. \quad (2.10)$$

Evaluating the integral in (2.10) we get from (2.9), ($l = \dim \mathcal{A}$),

$$u(H_1, t) \phi(H_1) = c_l \frac{e^{-it|\rho|^2}}{t^{l/2}} \int_{\mathcal{A}} \left(\sum_{s, s'} (\det s) (\det s') e^{i \frac{|sH_1 - s'H_2|^2}{4t}} \right) f(H_2) \phi(H_2) dH_2. \quad (2.11)$$

Set $g(H_2) = f(H_2) \phi(H_2)$. Now note that because f is bi-invariant and $\phi(H)$ is odd under the action of the Weyl group, $g(sH) = (\det s)g(H)$. We proceed to re-write (2.11) as in the Euclidean case and we will make use of the Euclidean Fourier transform. Re-writing (2.11) we get

$$\begin{aligned} u(H_1, t) \phi(H_1) &= c_l \frac{e^{-it|\rho|^2} e^{i \frac{|H_1|^2}{4t}}}{t^{l/2}} \sum_{s, s'} (\det s) (\det s') \\ &\quad \times \int_{R^l} e^{-iB\left(\frac{ss'H_1}{2t}, H_2\right)} e^{i \frac{|H_2|^2}{4t}} g(H_2) dH_2. \end{aligned} \quad (2.12)$$

Here l is the rank of G . Set

$$R(H_2) = e^{i \frac{|H_2|^2}{4t}} g(H_2). \quad (2.13)$$

Then (2.12) states that

$$u(H_1, t)\phi(H_1) = c_l \frac{e^{-i\left(t|\rho|^2 - \frac{|H_1|^2}{4t}\right)}}{t^{l/2}} \sum_{s, s'} (\det s)(\det s') \hat{R}\left(\frac{ss'H_1}{2t}\right). \quad (2.14)$$

Since $R(H)$ is odd under reflection by the Weyl group, we finally can write (2.14) as

$$u(H_1, t)\phi(H_1) = c_l |W|^2 \frac{e^{-i\left(t|\rho|^2 - \frac{|H_1|^2}{4t}\right)}}{t^{l/2}} \hat{R}\left(\frac{H_1}{2t}\right). \quad (2.15)$$

Noting further from (2.3) that $|\phi(H)| \leq ce^{c|H|}$, we see right away from (2.15) and the hypothesis that at $t = t_0$,

$$|\hat{R}(H_1)| \leq ce^{-4b't_0^2|H_1|^2}, \quad b' < b$$

and also

$$|R(H_2)| \leq ce^{-a'|H_2|^2}, \quad a' < a.$$

Thus by Theorem 1, we again conclude $R \equiv 0$. This implies $f \equiv 0$ and hence the theorem follows.

Lemma 1. The results of Theorems 2 and 3 are sharp. One cannot relax the condition $16abt_0^2 > 1$.

Proof. We only display the proof for R^n . For theorem 3 one can check the validity of the lemma by doing an explicit computation on $SL(2, C)$. For R^n , choose initial data

$$f(x) = e^{-|x|^2 - i \frac{|x|^2}{4}}.$$

An elementary computation using (2.1) shows that

$$u(x, 1) = c_n e^{\frac{i|x|^2}{4}} e^{-\frac{|x|^2}{16}}.$$

Thus we have $16abt_0^2 = 1$ and uniqueness fails.

3. The Heisenberg group

We may ask the uniqueness question above for the sub-Laplacian Δ_b on the Heisenberg group. That is, we consider the Schrödinger equation

$$iu_t = \Delta_b u, \quad u|_{t=0} = f(x). \quad (3.1)$$

However there is a difficulty in writing the fundamental solution to this operator due to the presence of closed loops in the contact plane. To see this, we proceed heuristically. First of all, the fundamental solution to the heat equation on the Heisenberg group is given by

$$K(x, u, \xi, t) = \int_{-\infty}^{\infty} e^{-i\lambda\xi} e^{-t\lambda^2} \frac{\lambda}{\sinh(\lambda t)} e^{-\frac{1}{4}\lambda \coth(\lambda t)(x^2+u^2)} d\lambda, \quad (3.2)$$

where the Heisenberg group is viewed as R^3 and points on it written as (x, u, ξ) . To try to get a solution operator for (3.1), we perform a change of variables in (3.2) by letting $t \rightarrow -it$. The integrand in (3.2) becomes

$$\frac{\lambda}{\sin(\lambda t)} e^{-\frac{i}{4}\lambda \cot(\lambda t)(x^2+u^2)}. \quad (3.3)$$

Thus we note that the putative solution operator for fixed t is singular at $\lambda = k\pi/t$. In fact, (3.3) converges to a Dirac delta at $u = x = 0$ as $\lambda \rightarrow k\pi/t$. This phenomena is attributable to the geodesics of the Heisenberg group projecting onto circles (closed loops) in the contact plane at the origin and having a cut-locus at $k\pi/t$. The geodesics on the Heisenberg group are well-known and a formula is found in [8]. They are given by

$$\begin{aligned} x(s) &= \frac{\cos \beta (1 - \cos(ts)) + \sin \beta \sin(ts)}{t}, \\ u(s) &= \frac{-\sin \beta (1 - \cos(ts)) + \cos \beta \sin(ts)}{t}, \\ \xi(s) &= 2 \frac{ts - \sin(ts)}{t^2}. \end{aligned}$$

Now notice that the geodesics are not closed but their projection into the $x - u$ plane, the contact plane, are circles given by

$$\left(x(s) - \frac{\cos \beta}{t}\right)^2 + \left(u(s) + \frac{\sin \beta}{t}\right)^2 = \frac{1}{t^2}.$$

The cut-locus emerges exactly at a distance of $k\pi/t$ from $u = x = 0$ along the circle of radius $1/t$, and it is exactly there the integrand of the solution operator has its singularities. A general construction of the heat kernel on CR manifolds in [1] exhibits the same phenomena.

4. Strichartz estimates and decay estimates on complex Lie groups

We will use the results of our computation in §2 on complex semi-simple groups to obtain various estimates on the solution $u(H, t)$. The estimates fall into two categories. One where we integrate over a fixed time slice and another where we integrate over both space and time, the Strichartz estimates. We have the following.

Theorem 4. *Under the assumptions of Theorem 3, the solution $u(H, t)$ for bi-invariant initial data f satisfies*

$$\|u|\phi|^{1-\frac{2}{q}}\|_{L^q(G)} \leq ct^{-l(\frac{1}{p}-\frac{1}{2})} \|f|\phi|^{1-\frac{2}{p}}\|_{L^p(G)}, \quad 1 \leq p \leq 2, \quad \frac{1}{p} + \frac{1}{q} = 1. \quad (4.1)$$

Assume that

$$-iu_t - \Delta u = \psi(H, t), \quad u|_{t=0} = f,$$

where ψ is also bi-invariant for every fixed t . Then for $p = 2(l+2)/(l+4)$, $q = 2(l+2)/l$, we have

$$\|u|\phi|^{1-\frac{2}{q}}\|_{L^q(G \times (0, \infty))} \leq c(\|f\|_{L^2(G)} + \|\psi|\phi|^{1-\frac{2}{p}}\|_{L^p(G \times (0, \infty))}). \quad (4.2)$$

Proof. The proofs follow from the identity (2.11). It follows from (2.11) that the function $u(H, t)\phi(H)$ is obtained by applying the Euclidean fundamental solution of the Schrödinger equation to the data given by $g(H)$. Thus from the Euclidean estimates

$$\|u\phi\|_{L^q(R^l)} \leq ct^{-l(\frac{1}{p}-\frac{1}{2})}\|g\|_{L^p(R^l)},$$

where p, q is as in eq. (4.1) above. Re-writing the last inequality using (2.3) we get eq. (4.1) of our theorem.

A similar computation as in Corollary 1 of [10] gives eq. (4.2).

Remark. The methods of Theorem 4 and [10] also extend to the wave equation on complex semi-simple Lie groups with bi-invariant functions as data. Strichartz and decay estimates are easily obtained by following the methods developed in §2 and Theorem 4.

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