

## A sharp upper bound for the first eigenvalue of the Laplacian of compact hypersurfaces in rank-1 symmetric spaces

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**Abstract.** Let  $M$  be a closed hypersurface in a simply connected rank-1 symmetric space  $\bar{M}$ . In this paper, we give an upper bound for the first eigenvalue of the Laplacian of  $M$  in terms of the Ricci curvature of  $\bar{M}$  and the square of the length of the second fundamental form of the geodesic spheres with center at the center-of-mass of  $M$ .

**Keywords.** Hypersurface; center-of-mass; rank-1 symmetric space; Laplacian; eigenvalue.

### 1. Introduction

Let  $(\mathbb{M}(\kappa), ds^2)$  denote the simply connected space form of constant curvature  $\kappa$  where  $\kappa = 0, 1$  or  $-1$  and dimension  $n \geq 2$ . Let  $M$  be a closed hypersurface of  $\mathbb{M}(\kappa)$ . When  $\kappa = 0$  and  $M$  a closed hypersurface of  $\mathbb{R}^n$ , Bleecker–Weiner [2] proved that the first eigenvalue  $\lambda_1(M)$  of the Laplace operator of  $M$  satisfies the inequality:  $\lambda_1(M) \leq \frac{1}{\text{vol}(M)} \int_M |A|^2$ , where  $|A|^2$  denotes the square of the length of the second fundamental form of the hypersurface  $M$ . In [10], Reilly improved this inequality to show that  $\lambda_1(M) \leq \frac{n-1}{\text{vol}(M)} \int_M |H|^2$ , where  $H$  is the mean curvature of the hypersurface  $M$ . These inequalities of Bleecker–Weiner and Reilly are also sharp for geodesic spheres in  $\mathbb{R}^n$ . Since then, Reilly's inequality has been extended to hypersurfaces in other simply connected space forms (see [7] and [8] for details and related results).

While trying to understand these results, we noticed that one can obtain a similar sharp upper bound for the first eigenvalue  $\lambda_1(M)$ , of closed hypersurfaces  $M$  in rank-1 symmetric spaces. Namely  $\lambda_1(M) \leq \frac{1}{\text{vol}(M)} \int_M \lambda_1(S(r))$  where  $\lambda_1(S(r))$  is the first eigenvalue of the geodesic sphere  $S(r)$  with center at a point  $p_0$  called the center-of-mass of the hypersurface  $M$  and radius  $r(q) = d(p_0, q)$ .

We refer to [4] and [9] for the basic Riemannian geometry used in this paper.

#### 1.1 Statement of results

To state our results we need the notion of center-of-mass and result on the existence of center-of-mass for measurable subsets of  $\bar{M}$ .

Let  $(M, g)$  be a complete Riemannian manifold. For a point  $p \in M$ , we denote by  $c(p)$ , the convexity radius of  $(M, g)$  at  $p$ . For a subset  $A \subseteq B(q, c(q))$  for some  $q \in M$ , we let  $CA$  denote the convex hull of  $A$ . Let  $\exp_q: T_q M \rightarrow M$  be the exponential map

and  $X = (x_1, x_2, \dots, x_n)$  be the normal coordinates centered at  $q$ . We identify  $CA$  with  $\exp_q^{-1}(CA)$  and we also denote  $g_q(X, X)$  as  $\|X\|_q^2$  for  $X \in T_q M$ .

We now state and prove the center-of-mass theorem (see also [6] and [1]).

**Theorem 1.** *Let  $A$  be a measurable subset of  $(M, g)$  contained in  $B(q_0, c(q_0))$  for some point  $q_0 \in M$ . Let  $G: [0, 2c(q_0)] \rightarrow \mathbb{R}$  be a continuous function such that  $G$  is positive on  $(0, 2c(q_0))$ . Then there exists a point  $p \in CA$  such that*

$$\int_A G(\|X\|_p) X dV = 0,$$

where  $X = (x_1, x_2, \dots, x_n)$  denotes the geodesic normal coordinate system centred at  $p$ .

*Proof.* For  $q \in CA$ , we define

$$v(q) := \int_A G(\|X\|_q) X dV,$$

where  $X = (x_1, x_2, \dots, x_n)$  is a geodesic normal coordinate system centred at  $q$ .

We shall now show that the continuous vector field  $v$  points inward along the boundary  $\partial CA$  of  $CA$ . Then the theorem follows from the Brouwer's fixed point theorem.

Since  $CA$  is convex, it is contained in the half-space  $H_q := \{X \in T_q M: g(X, v(q)) \leq 0\}$  for every  $q \in \partial CA$ , where  $v(q)$  denotes the outward normal to  $\partial CA$  at  $q$ . This implies that  $g(v(q), v(q)) < 0$  for all  $q \in \partial CA$ . Thus  $v$  points inward along the boundary of  $CA$ .

We can find a  $\delta > 0$  such that  $\exp_q(\delta v(q)) \in CA$  for every  $q \in CA$ . Then the continuous map  $f_v: CA \rightarrow CA$  defined by

$$f_v(q) := \exp_q(\delta v(q))$$

has a fixed point  $p \in CA$  by the Brouwer's fixed point theorem. Hence  $v(p) = 0$ . This completes the proof of the theorem.  $\square$

## DEFINITION 1

The point  $p$  in the theorem is called a center-of-mass of the measurable subset  $A$  with respect to the mass distribution function  $G$ .

Before we state our results, we fix some notations that we will be using throughout this paper.

We let  $(\bar{M}, ds^2)$  denote any one of the following rank-1 symmetric spaces: the round sphere  $S^n$  with constant sectional curvature  $\frac{1}{4}$ , complex projective space  $\mathbb{C}P^n$ , quaternionic projective space  $\mathbb{H}P^n$  and the Cayley projective plane  $CaP^2$  with sectional curvature  $\frac{1}{4} \leq K_{\bar{M}} \leq 1$  or their non-compact duals with sectional curvature  $-1 \leq K_{\bar{M}} \leq -\frac{1}{4}$ . We also write  $\dim \bar{M} = kn$ , where  $k = \dim_{\mathbb{R}} \mathbb{K}$ ;  $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$  or  $Ca$ . We also let  $\text{Ric}_{\bar{M}}$  denote the Ricci curvature of  $\bar{M}$  and also remark that  $\text{Ric}_{\bar{M}}$  is constant. The round sphere  $S^n$  with constant sectional curvature 1 is denoted by  $(S^n, Can)$ .

Given a point  $p \in \bar{M}$ , we let  $S(r)$  denote the geodesic sphere of radius  $r$  with center  $p$ . Let  $\Delta_{S(r)}$  denote the Laplacian of the geodesic sphere  $S(r)$  with respect to the induced metric and  $\lambda_1(S(r))$  denote the first eigenvalue of  $\Delta_{S(r)}$ .

**Theorem 2.** Let  $M$  be a closed hypersurface in a simply connected rank-1 symmetric space  $(\bar{M}, ds^2)$  of compact type. Assume that  $M$  is contained in a ball of radius  $\pi/2$ . Let  $p_0$  be the center-of-mass of  $M$  with respect to the mass distribution function  $G(t) = 1/t$ . Then

$$\begin{aligned}\lambda_1(M) &\leq \frac{1}{\text{vol}(M)} \int_M \lambda_1(S(r)) \\ &= \frac{1}{\text{vol}(M)} \int_M (|A(r)|^2 + \text{Ric}_{\bar{M}}),\end{aligned}\quad (1)$$

where  $r(x) := d(p_0, x)$  is the distance from the point  $p_0$  to the point  $x$ . Furthermore, equality holds in the above inequality iff  $M$  is a geodesic sphere with center  $p_0$ .

**Theorem 3.** Let  $M$  be a closed hypersurface in a simply connected rank-1 symmetric space  $(\bar{M}, ds^2)$  of non-compact type. Let  $p_0$  be the center-of-mass of  $M$  with respect to the mass distribution function  $G(t) = 1/t$ . Then

$$\begin{aligned}\lambda_1(M) &\leq \frac{1}{\text{vol}(M)} \int_M \lambda_1(S(r)) \\ &= \frac{1}{\text{vol}(M)} \int_M (|A(r)|^2 + \text{Ric}_{\bar{M}}),\end{aligned}\quad (2)$$

where  $r(x) := d(p_0, x)$  is the distance from the point  $p_0$  to the point  $x$ . Furthermore, equality holds in the above inequality iff  $M$  is a geodesic sphere with center  $p_0$ .

## 2. Preliminaries

Let  $0 < r < i(\bar{M})$ . Let  $S(r)$  be a geodesic sphere of radius  $r$  in  $(\bar{M}, ds^2)$  centred at a point  $m \in M$ . We identify  $S(r)$  with the inverse image  $\exp^{-1}(S(r))$  with the metric  $\exp_m^*(ds^2|_{S(r)})$ . Let  $\Delta_{S(r)}$  denote the Laplacian of  $S(r)$  and  $\lambda_1(S(r))$  the first eigenvalue of  $\Delta_{S(r)}$ .

### 2.1 First eigenvalue of $\Delta_{S(r)}$

In this section we study the first eigenvalue  $\lambda_1(S(r))$  of  $\Delta_{S(r)}$ . This is also done in [3] and [1].

**2.1.1  $(\bar{M}, ds^2)$  of compact type.** If  $k = 1$ , the geodesic sphere  $S(r)$  is isometric to  $(\bar{M}_{n-1} = S^{n-1}, 4 \sin^2(r/2) \text{Can})$ . Therefore the first eigenvalue of  $S(r)$  is  $\lambda_1(S(r)) = \frac{n-1}{4 \sin^2(r/2)}$ .

We know that there is a canonical Riemannian submersion

$$\Pi : S(r) \rightarrow (\bar{M}_{n-1}, 4 \sin^2(r/2) ds^2)$$

with connected totally geodesic fibres, where  $\bar{M}_{n-1}$  is the simply connected compact rank-1 symmetric space of dimension  $k(n-1)$ .

If  $k \geq 2$ , we can decompose the Laplacian  $\Delta_{S(r)}$  of  $S(r)$  as

$$\Delta_{S(r)} = \frac{1}{4 \cos^2(r/2)} \Delta_{(S^{k-1}, \text{Can})} + \frac{1}{4 \sin^2(r/2)} \Delta_{(S^{kn-1}, \text{Can})}.\quad (3)$$

We also know that the Euclidean co-ordinate functions  $X_i$ 's, for  $1 \leq i \leq kn$ , are the first eigenfunctions of  $\Delta_{(S^{kn-1}, Can)}$  corresponding to the first eigenvalue  $kn - 1$  (see [5] for details). Since the fibres are all totally geodesic, when we restrict these eigenfunctions to each fibre, they become eigenfunctions of

$$\frac{1}{4 \cos^2(r/2)} \Delta_{(S^{k-1}, Can)}$$

with eigenvalue

$$\frac{k-1}{4 \cos^2(r/2)}.$$

Hence we get

$$\Delta_{S(r)} X_i = \left( \frac{kn-1}{4 \sin^2(r/2)} + \frac{k-1}{4 \cos^2(r/2)} \right) X_i$$

for  $1 \leq i \leq kn$ .

Let  $\Delta_H$  denote the horizontal Laplacian of the Riemannian submersion. Since

$$\Delta_H |_{\Pi^* C^\infty(\tilde{M}_{n-1})} = \Pi^* \Delta_{(\tilde{M}_{n-1}, 4 \sin^2(r/2) ds^2)}$$

all the eigenfunctions of  $\Delta_{(\tilde{M}_{n-1}, 4 \sin^2(r/2) ds^2)}$  are also eigenfunctions of  $\Delta_{S(r)}$  with the same eigenvalues. In particular, the first non-zero eigenvalue

$$\frac{2kn}{4 \sin^2(r/2)}$$

occurs as an eigenvalue of  $\Delta_{S(r)}$  also. Now

$$\frac{kn-1}{\sin^2(r/2)} + \frac{k-1}{\cos^2(r/2)} < \frac{2kn}{\sin^2(r/2)}$$

iff

$$\tan(r/2) < \sqrt{\frac{kn+1}{k-1}}.$$

Since  $k \geq 1$  and  $n \geq 2$ , we see that  $\sqrt{\frac{kn+1}{k-1}} > 1$ . Therefore  $\tan(r/2) < \sqrt{\frac{kn+1}{k-1}}$  for  $r < \pi/2$  and consequently the functions  $X_i$ , for  $1 \leq i \leq kn$ , are all first eigenfunctions of  $\Delta_{S(r)}$  with the first eigenvalue

$$\lambda_1(S(r)) = \frac{kn-1}{4 \sin^2(r/2)} + \frac{k-1}{4 \cos^2(r/2)}.$$

2.1.2  $(\tilde{M}, ds^2)$  of non-compact type. We denote by  $(\tilde{M})^*$  the compact dual of  $\tilde{M}$ .

If  $k = 1$ , the geodesic sphere  $S(r)$  is isometric to  $((\tilde{M}_{n-1})^* = S^{n-1}, 4 \sinh^2(r/2) Can)$  and hence the first eigenvalue  $\lambda_1(S(r))$  of  $S(r)$  is  $\frac{n-1}{4 \sinh^2(r/2)}$ .

We have the canonical Riemannian submersion

$$\Pi : (S(r), ds^2|_{S(r)}) \rightarrow ((\tilde{M}_{n-1})^*, 4 \sinh^2(r/2) ds^2)$$

with connected totally geodesic fibres.

If  $k \geq 2$ , we can decompose the Laplacian  $\Delta_{S(r)}$  as

$$\Delta_{S(r)} = \frac{-1}{4 \cosh^2(r/2)} \Delta_{(S^{k-1}, Can)} + \frac{1}{4 \sinh^2(r/2)} \Delta_{(S^{kn-1}, Can)}.$$

We know that the euclidean coordinate functions  $X_i$ 's, for  $1 \leq i \leq kn$ , are eigenfunctions of  $\Delta_{S(r)}$  with eigenvalue

$$\lambda_1(S(r)) = \frac{kn - 1}{4 \sinh^2(r/2)} - \frac{k - 1}{4 \cosh^2(r/2)}.$$

Since

$$\Delta_H|_{\Pi^* C^\infty((\tilde{M}_{n-1})^*)} = \Pi^* \Delta_{((\tilde{M}_{n-1})^*, 4 \sinh^2(r/2) ds^2)}$$

all the eigenfunctions of  $\Delta_{((\tilde{M}_{n-1})^*, 4 \sinh^2(r/2) ds^2)}$  are also eigenfunctions of  $\Delta_{S(r)}$  with the same eigenvalues. In particular, the first non-zero eigenvalue

$$\frac{2kn}{4 \sinh^2(r/2)}$$

occurs as an eigenvalue of  $\Delta_{S(r)}$  also. Now

$$\frac{kn - 1}{4 \sinh^2(r/2)} - \frac{k - 1}{4 \cosh^2(r/2)}$$

will be the first non-zero eigenvalue of  $\Delta_{S(r)}$  so long as

$$\frac{kn - 1}{4 \sinh^2(r/2)} - \frac{k - 1}{4 \cosh^2(r/2)} < \frac{2kn}{4 \sinh^2(r/2)}.$$

Since the inequality above is valid for all  $r > 0$ , we get that

$$\lambda_1(S(r)) = \frac{kn - 1}{4 \sinh^2(r/2)} - \frac{k - 1}{4 \cosh^2(r/2)}$$

for all  $r > 0$ .

**2.1.3 Geometry of  $S(r)$ .** We will now relate  $\lambda_1(S(r))$  of  $S(r)$  with the square of the length of the second fundamental form of  $S(r)$  and the Ricci curvature of  $\tilde{M}$ .

Let  $\gamma$  be a geodesic starting at the point  $p$ . Let  $R_{\gamma'(t)}: T_{\gamma'(t)}\tilde{M} \rightarrow T_{\gamma'(t)}\tilde{M}$  be the symmetric endomorphism defined by  $R_{\gamma'(t)}(v) := R(v, \gamma'(t))\gamma'(t)$ , where  $R$  is the curvature tensor of  $\tilde{M}$ .

For  $0 < t < i(\tilde{M})$ , we let  $A(t)$  denote the Weingarten map  $A(\gamma(t))$  of the smooth hypersurface  $S(t)$  at the point  $\gamma(t)$ . It is known that these family of symmetric endomorphisms  $A(t)$  satisfy the Riccati equation

$$A' + A^2 + R_{\gamma'} = 0$$

along the geodesic  $\gamma$ . Therefore, by taking the trace of these endomorphisms we get

$$\begin{aligned} -\text{Tr}(A'(r)) &= \text{Tr}(A^2(r)) + \text{Tr}(R_{\gamma'(r)}) \\ &= |A(r)|^2 + \text{Ric}_{\tilde{M}}(\gamma', \gamma') \end{aligned}$$

where  $|A(r)|^2$  is the square of the length of the second fundamental form of the geodesic sphere  $S(r)$  and  $\text{Ric}_{\bar{M}}$  is the Ricci curvature of  $\bar{M}$ .

Let  $E_2, E_3, \dots, E_{kn}$  be an orthonormal basis of  $T_{\gamma(r)}S(r)$  such that the vectors  $E_i$ , for  $2 \leq i \leq k$ , are tangent to the fibre of the canonical Riemannian submersion

$$\Pi: S(r) \rightarrow \begin{cases} (\bar{M}_{n-1}, ds^2) & \text{if } (\bar{M}, ds^2) \text{ is of compact type} \\ (\bar{M}_{n-1}^*, ds^2) & \text{if } (\bar{M}, ds^2) \text{ is of non-compact type.} \end{cases}$$

Then an easy Jacobi field computation shows that

$$A(r)E_i = \begin{cases} \cot r E_i & \text{if } (\bar{M}, ds^2) \text{ is of compact type} \\ \coth r E_i & \text{if } (\bar{M}, ds^2) \text{ is of non-compact type} \end{cases}$$

for  $2 \leq i \leq k$  and

$$A(r)E_i = \begin{cases} \frac{1}{2} \cot(r/2) E_i & \text{if } (\bar{M}, ds^2) \text{ is of compact type} \\ \frac{1}{2} \coth(r/2) E_i & \text{if } (\bar{M}, ds^2) \text{ is of non-compact type} \end{cases}$$

for  $k+1 \leq i \leq kn$ . Therefore

$$\text{Tr}(A(r)) = \begin{cases} \frac{k(n-1)}{2} \cot(r/2) + (k-1) \cot r & \text{if } (\bar{M}, ds^2) \text{ is of compact type} \\ \frac{k(n-1)}{2} \coth(r/2) + (k-1) \coth r & \text{if } (\bar{M}, ds^2) \text{ is of non-compact type} \end{cases}$$

and  $-\text{Tr}A'(r) = \lambda_1(S(r))$ .

### 3. Proof of Theorems 2 and 3

We will now prove Theorems 2 and 3.

Let  $M$  be a closed hypersurface in  $\bar{M}$  contained in a ball of radius  $i(\bar{M})/2$  where  $i(\bar{M}) = \infty$  if  $\bar{M}$  is non-compact.

By the center-of-mass theorem (Theorem 1), there exists a point  $p_0$  such that  $\int_M \frac{1}{r} X = 0$ . Since  $X = (x_1, x_2, \dots, x_{kn})$ , we see that  $\int_M \frac{1}{r} x_i = 0$  for  $1 \leq i \leq kn$ .

Let  $f_i := \frac{x_i}{r}$  for  $1 \leq i \leq kn$ . Since  $\int_M f_i = 0$ , we can use these functions  $f_i$ 's as test functions in the Rayleigh quotient. Therefore

$$\lambda_1(M) \int_M f_i^2 \leq \int_M \|\nabla^M f_i\|^2$$

for  $1 \leq i \leq kn$ , where  $\nabla^M$  denotes the gradient in  $M$ . Since  $\sum_{i=1}^{kn} x_i^2 = r^2$ , we see that  $\sum_{i=1}^{kn} f_i^2 = 1$ . Hence

$$\lambda_1(M) \text{vol}(M) \leq \sum_{i=1}^{kn} \int_M \|\nabla^M f_i\|^2.$$

If we now show that

$$\sum_{i=1}^{kn} \int_M \|\nabla^M f_i\|^2 \leq \int_M \lambda_1(S(r))$$

then we are through.

By Green's identity,  $\|\nabla^M f_i\|^2 = f_i \Delta_M f_i - \Delta_M(f_i^2)$  for  $1 \leq i \leq kn$ . Therefore,

$$\int_M \|\nabla^M f_i\|^2 = \int_M f_i \Delta_M f_i - \int_M \Delta_M(f_i^2).$$

Since the boundary of  $M$  is empty, it follows from the divergence theorem that  $\int_M \Delta_M(f_i^2) = 0$  for  $1 \leq i \leq kn$ . Thus  $\int_M \|\nabla^M f_i\|^2 = \int_M f_i \Delta_M f_i$ .

Let us now recall that  $\Delta = -\frac{\partial^2}{\partial v^2} - \text{Tr}(A)\frac{\partial}{\partial v} + \Delta_M$ , where  $\Delta$  is the Laplacian of  $(\bar{M}, ds^2)$ ,  $A$  is the Weingarten map of the hypersurface  $M$  and  $v$  is the unit outward normal to  $M$ . Using this identity, we write that  $\Delta_M f_i = \Delta f_i + \frac{\partial^2}{\partial v^2} f_i + \text{Tr}(A)\langle \nabla f_i, v \rangle$ .

We will now compute  $\Delta f_i$ ,  $\langle \nabla f_i, v \rangle$  and  $\partial^2 f_i / \partial v^2$ .

We decompose the Laplacian  $\Delta$  of  $\bar{M}$  as

$$\Delta = -\frac{\partial^2}{\partial r^2} - \text{Tr}(A(r))\frac{\partial}{\partial r} + \Delta_{S(r)}$$

along the radial geodesics starting at  $p_0$ . Now  $\frac{\partial}{\partial r}(\frac{x_i}{r}) = 0$  for  $1 \leq i \leq kn$ . Therefore  $\Delta f_i = \Delta_{S(r)} f_i$ . In §2, we have shown that  $\Delta_{S(r)} f_i = \lambda_1(S(r)) f_i$  for  $0 < r < i(\bar{M})/2$ .

Now

$$\begin{aligned} f_i \frac{\partial^2 f_i}{\partial v^2} &= \frac{1}{2} \frac{\partial}{\partial v} \langle \nabla(f_i^2), v \rangle - \langle \nabla f_i, v \rangle^2 \\ &= \frac{1}{2} \frac{\partial^2}{\partial v^2} (f_i^2) - \left( \frac{\partial f_i}{\partial v} \right)^2. \end{aligned}$$

Therefore

$$\sum_{i=1}^{kn} f_i \frac{\partial^2 f_i}{\partial v^2} = - \sum_{i=1}^{kn} \left( \frac{\partial f_i}{\partial v} \right)^2$$

and

$$\sum_{i=1}^{kn} \int_M f_i \frac{\partial^2 f_i}{\partial v^2} = - \sum_{i=1}^{kn} \int_M \left( \frac{\partial f_i}{\partial v} \right)^2.$$

Similarly,

$$\begin{aligned} \sum_{i=1}^{kn} \int_M \text{Tr}(A) f_i \langle \nabla f_i, v \rangle &= \frac{1}{2} \int_M \text{Tr}(A) \frac{\partial}{\partial v} \left( \sum_{i=1}^{kn} f_i^2 \right) \\ &= 0. \end{aligned}$$

We substitute these quantities in  $\sum_{i=1}^{kn} \int_M f_i \Delta_M f_i$  to get

$$\begin{aligned} \sum_{i=1}^{kn} \int_M f_i \Delta_M f_i &= \sum_{i=1}^{kn} \int_M f_i \Delta f_i + \int_M f_i \frac{\partial^2 f_i}{\partial v^2} + \sum_{i=1}^{kn} \int_M \text{Tr}(A) f_i \langle \nabla f_i, v \rangle \\ &= \int_M \lambda_1(S(r)) \sum_{i=1}^{kn} f_i^2 - \sum_{i=1}^{kn} \int_M \left( \frac{\partial f_i}{\partial v} \right)^2 \end{aligned}$$

$$\begin{aligned}
&= \int_M \lambda_1(S(r)) - \sum_{i=1}^{kn} \int_M \left( \frac{\partial f_i}{\partial v} \right)^2 \\
&\leq \int_M \lambda_1(S(r)) \\
&= \int_M (|A(r)|^2 + \text{Ric}_{\tilde{M}}).
\end{aligned} \tag{4}$$

This completes the proof of the inequality.

For the equality part of the proof, we notice that equality holds in the above inequality iff  $\frac{\partial f_i}{\partial v} = 0$ , for  $1 \leq i \leq kn$ , at all points in  $M$ . This is true iff the unit outward normal field is same as the radial vector field  $\nabla r$ . Hence the equality holds iff  $M$  is a geodesic sphere of radius  $d(p_0, M)$  with center  $p_0$ . This completes the proof of theorems 2 and 3.

*Remark 1.* The analogue of Bleecker–Weiner [2] result is not known in rank-1 symmetric spaces. However we have analogous results of Theorems 2 and 3 in  $\mathbb{R}^n$ .

**Theorem 4.** *Let  $M$  be a closed hypersurface in  $\mathbb{R}^n$ . Let  $p_0$  be the center-of-mass of  $M$  with respect to the mass distribution function  $G(t) = 1/t$ . Then*

$$\begin{aligned}
\lambda_1(M) &\leq \frac{1}{\text{vol}(M)} \int_M \lambda_1(S(r)) \\
&= \frac{1}{\text{vol}(M)} \int_M |A(r)|^2 \\
&= \frac{n-1}{\text{vol}(M)} \int_M \frac{1}{r^2(x)},
\end{aligned} \tag{5}$$

where  $r(x) := d(p_0, x)$  is the distance from the point  $p_0$  to the point  $x$ . Furthermore, equality holds in the above inequality iff  $M$  is a geodesic sphere with center  $p_0$ .

*Proof.* Let us first observe that the inequality 4 in the proof of theorems 2 and 3 is valid in all Riemannian manifolds in which the functions  $x_i/r$  are first eigenfunctions of the geodesic sphere  $S(r)$ . This is true in  $\mathbb{R}^n$ . Further, the first eigenvalue  $\lambda_1(S(r))$  of the geodesic sphere  $S(r)$  in  $\mathbb{R}^n$  is  $\frac{n-1}{r^2}$ . Therefore

$$\begin{aligned}
\sum_{i=1}^{kn} \int_M f_i \Delta_M f_i &\leq \int_M \lambda_1(S(r)) \\
&= (n-1) \int_M \frac{1}{r^2(x)}.
\end{aligned}$$

Hence

$$\lambda_1(M) \leq \frac{n-1}{\text{vol}(M)} \int_M \frac{1}{r^2(x)}. \quad \square$$

*Remark 2.* The analogue of Bleecker–Weiner [2] result in rank-1 symmetric spaces and their relation with the results of Theorems 2, 3 and 4 will be discussed in a subsequent paper.



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