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On the cohomology of orbit space of free $\mathbb{Z}_p$-actions on lens spaces

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Abstract. Let $G = \mathbb{Z}_p$, p an odd prime, act freely on a finite-dimensional CW-complex X with mod p cohomology isomorphic to that of a lens space $L^{2m-1}(p; q_1, \ldots, q_m)$. In this paper, we determine the mod p cohomology ring of the orbit space X/G, when $p^2 \nmid m$.

Keywords. Lens space; free action; cohomology algebra; spectral sequence.

1. Introduction

Let p be an odd prime and $m > 1$ an integer. Consider the $(2m-1)$-sphere $S^{2m-1} \subset \mathbb{C} \times \cdots \times \mathbb{C}$ (m-times). Given integers $q_1, \ldots, q_m$ relatively prime to p, the map $(\xi_1, \ldots, \xi_m) \to (\zeta^{q_1}\xi_1, \ldots, \zeta^{q_m}\xi_m)$, where $\zeta = e^{2\pi\iota/p^2}$, defines a free action of $G = \langle\zeta\rangle$ on S^{2m-1}. The orbit spaces of G and the subgroup $N = \langle\zeta^p\rangle$ are the lens spaces $L^{2m-1}(p^2; q_1, \ldots, q_m)$ and $L^{2m-1}(p; q_1, \ldots, q_m)$, respectively. Thus, we have a free action of $\mathbb{Z}_p$ on $L^{2m-1}(p; q_1, \ldots, q_m)$ with the orbit space $L^{2m-1}(p^2; q_1, \ldots, q_m)$. By a mod p cohomology lens space, we mean a space X whose Čech cohomology $H^*(X; \mathbb{Z}_p)$ is isomorphic to that of a lens space $L^{2m-1}(p; q_1, \ldots, q_m)$. We will write $X \sim_p L^{2m-1}(p; q_1, \ldots, q_m)$ to indicate this fact. If $G = \mathbb{Z}_p$ acts on a mod p cohomology lens space X, then the fixed point set of G on X has been investigated by Su [4]. In this paper, we determine the cohomology ring (mod p) of the orbit space X/G, when G acts freely on X. The following theorem is established.

Theorem. *Let $G = \mathbb{Z}_p$ act freely on a finite-dimensional CW-complex $X \sim_p L^{2m-1}(p; q_1, \ldots, q_m)$. If $p^2 \nmid m$, then $H^*(X/G; \mathbb{Z}_p)$ is one of the following graded commutative algebras*:

(i) $\mathbb{Z}_p[x, y, z, u_1, u_3, \ldots, u_{2p-3}]/I$, *where I is the homogeneous ideal*

$$\langle x^2, y^p, z^n, u_h y - A_h x y^{(h+1)/2}, u_h u_{2p-h}, u_h u_{h'} - B_{hh'} x u_{h+h'-1} - C_{hh'} y^{(h+h')/2}, u_h u_{h'} - B'_{hh'} z x u_{h+h'-2p-1} - C'_{hh'} z y^{(h+h'-2p)/2}\rangle,$$

$m = np$, $\deg x = 1$, $\deg y = 2$, $\deg z = 2p$, $\deg u_h = h$, $\beta_p(x) = y$, *and* $0 = B_{hh'} = C_{hh'} = B'_{hh'} = C'_{hh'}$ *when* $h = h'$. *(β_p is the mod-p Bockstein homomorphism associated with the sequence* $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$).

(ii) $\mathbb{Z}_p[x, z]/\langle x^2, z^m\rangle$, *where* $\deg x = 1$ *and* $\deg z = 2$.

2. Preliminaries

In this section, we recall some known facts which will be used in the proof of our theorem. Given a G-space X, there is an associated fibration $X \xrightarrow{i} X_G \xrightarrow{\pi} B_G$, and a map $\eta\colon X_G \to X/G$, where $X_G = (E_G \times X)/G$ and $E_G \to B_G$ is the universal G-bundle. When G acts freely on X, $\eta\colon X_G \to X/G$ is homotopy equivalance, so the cohomology rings $H^*(X_G)$ and $H^*(X/G)$ (with coefficients in a field) are isomorphic. To compute $H^*(X_G)$, we exploit the Leray–Serre spectral sequence of the map $\pi\colon X_G \to B_G$. The E_2-term of this spectral sequence is given by

$$E_2^{k,l} \cong H^k(B_G; \mathcal{H}^l(X))$$

(where $\mathcal{H}^l(X)$ is a locally constant sheaf with stalk $H^l(X)$ and group G) and it converges to $H^*(X_G)$, as an algebra. The cup product in E_{r+1} is induced from that in E_r via the isomorphism $E_{r+1} \cong H^*(E_r)$. When $\pi_1(B_G)$ operates trivially on $H^*(X)$, the system of local coefficients is simple (constant) so that

$$E_2^{k,l} \cong H^k(B_G) \otimes H^l(X)\,.$$

In this case, the restriction of the product structure in the spectral sequence to the subalgebras $E_2^{*,0}$ and $E_2^{0,*}$ gives the cup products on $H^*(B_G)$ and $H^*(X)$ respectively. The edge homomorphisms

$$H^p(B_G) = E_2^{p,0} \to E_3^{p,0} \to \cdots \to E_{p+1}^{p,0} = E_\infty^{p,0} \subseteq H^p(X_G) \text{ and}$$
$$H^q(X_G) \to E_\infty^{0,q} = E_{q+1}^{0,q} \subset \cdots \subset E_2^{0,q} = H^q(X)$$

are the homomorphisms

$$\pi^*\colon H^p(B_G) \to H^p(X_G) \quad \text{and} \quad \iota^*\colon H^q(X_G) \to H^q(X),$$

respectively.

The above results about spectral sequences can be found in [3]. We also recall that

$$H^*(B_G; \mathbb{Z}_p) = \mathbb{Z}_p[s,t]/(s^2) = \Lambda(s) \otimes \mathbb{Z}_p[t],$$

where $\deg s = 1$, $\deg t = 2$ and $\beta_p(s) = t$.

The following fact will be used without mentioning it explicitly.

PROPOSITION

Suppose that $G = \mathbb{Z}_p$ acts on a finite-dimensional CW-complex space X with the fixed point set F. If $H^j(X; \mathbb{Z}_p) = 0$ for $j > n$, then the inclusion map $F \to X$ induces an isomorphism

$$H^j(X_G; \mathbb{Z}_p) \to H^j(F_G; \mathbb{Z}_p)$$

for $j > n$ (see Theorem 1.5 in Chapter VII of [1]).

3. Proof

The example of free action of $G = \mathbb{Z}_p$ on the lens space $L^{2m-1}(p; q_1, \dots, q_m)$ described in the introduction is a test case for the general theorem. All cohomology groups in the proof should be considered to have coefficients in $\mathbb{Z}_p$. Since $\pi_1(B_G) = G$ acts trivially on $H^*(X)$, the fibration $X \xrightarrow{i} X_G \xrightarrow{\pi} B_G$ has a simple system of local coefficients on B_G. So the spectral sequence has

$$E_2^{k,l} \cong H^k(B_G) \otimes H^l(X).$$

Let $a \in H^1(X)$ and $b \in H^2(X)$ be generators of the cohomology ring $H^*(X)$. As there are no fixed points of G on X, the spectral sequence does not collapse at the E_2-term. Consequently, we have either $d_2(1 \otimes a) = t \otimes 1$ and $d_2(1 \otimes b) = 0$ or $d_2(1 \otimes a) = 0$ and $d_2(1 \otimes b) = t \otimes a$.

If $d_2(1 \otimes a) = 0$ and $d_2(1 \otimes b) = t \otimes a$, then we have $d_2(1 \otimes b^q) = qt \otimes ab^q$ and $d_2(1 \otimes ab^q) = 0$ for $1 \le q \le m-1$. So $0 = d_2[(1 \otimes b^{m-1}) \cup (1 \otimes b)] = mt \otimes ab^{m-1}$, which is true iff $p \mid m$. Suppose that $m = np$. Then

$$d_2\colon E_2^{k,l} \to E_2^{k+2,l-1}$$

is an isomorphism if l is even and $2p \nmid l$, and $d_2 = 0$ if l is odd or $2p \mid l$. So $E_3^{k,l} = E_2^{k,l}$ for all k if $l = 2qp$ or $2(q+1)p - 1$, where $0 \le q < n$; $k = 0, 1$ if l is odd and $2p \nmid (l+1)$, and $E_3^{k,l} = 0$, otherwise. It is easily seen that $d_3 = 0$, for example, if $u \in E_3^{0,2(q+1)p+1}$ and $d_3(u) = A[st \otimes ab^{(q+1)p-1}](A \in \mathbb{Z}_p)$, then, for $v = [t \otimes 1] \in E_3^{2,0}$, we have $0 = d_3(u \cup v) = A[st^2 \otimes ab^{(q+1)p-1}] \Rightarrow A = 0$. A similar argument shows that the differentials $d_4, \dots, d_{2p-1}$ are all trivial. If

$$d_{2p}\colon E_{2p}^{0,2p-1} \to E_{2p}^{2p,0}$$

is also trivial, then

$$d_{2p}\colon E_{2p}^{k,l} \to E_{2p}^{k+2p,l-2p+1}$$

is trivial for every k and l, because every element of $E_{2p}^{k,2(q+1)p-1}$ can be written as the product of an element of $E_{2p}^{k,2qp}$ by $1 \otimes ab^{p-1} \in E_{2p}^{0,2p-1}$ and

$$d_{2p}\colon E_{2p}^{k,2(q+1)p} \to E_{2p}^{k+2p,2qp+1}$$

is obviously trivial. If $n = 1$, then $E_\infty = E_3$, where the top and bottom lines survive. This contradicts our hypothesis; so $n > 1$. If $d_{2p+1}[1 \otimes b^p] = [st^p \otimes 1]$, then it can be easily verified that

$$d_{2p+1}[1 \otimes b^{qp}] = q[st^p \otimes b^{(q-1)p}] \quad \text{and}$$

$$d_{2p+1}[1 \otimes ab^{(q+1)p-1}] = q[st^p \otimes ab^{qp-1}]$$

for $1 \le q < n$. Consequently,

$$0 = d_{2p+1}[(1 \otimes ab^{np-1}) \cup (1 \otimes b^p)] = n(st^p \otimes ab^{np-1}),$$

which is not true for $(n, p) = 1$. On the other hand, if

$$d_{2p+1}\colon E_{2p}^{0,2p} \to E_{2p}^{2p+1,0}$$

is trivial, then

$$d_{2p+1}\colon E_{2p}^{k,l} \to E_{2p}^{k+2p+1,l-2p}$$

is also trivial for every k and l, as above. Now $d_r = 0$ for every $r > 2p+1$, so several lines of the spectral sequence survive to infinity. This contradicts our hypothesis. Therefore,

$$d_{2p}\colon E_{2p}^{0,2p-1} \to E_{2p}^{2p,0}$$

must be non trivial. Assume that $d_{2p}[1 \otimes ab^{p-1}] = [t^p \otimes 1]$. Then

$$d_{2p}\colon E_{2p}^{k,l+2p-1} \to E_{2p}^{k+2p,l}$$

is an isomorphism for $l = 2qp$, $0 \le q < n$, and is trivial homomorphism for other values of l. Accordingly, we have $E_\infty = E_{2p+1}$, and hence

$$H^j(X_G) = \begin{cases} \mathbb{Z}_p, & j = 2qp,\ 2(q+1)p - 1,\ 0 \le q < n; \\ \mathbb{Z}_p \oplus \mathbb{Z}_p, & 2qp < j < 2(q+1)p - 1,\ 0 \le q < n; \text{ and} \\ 0, & j > 2np - 1. \end{cases}$$

The elements $1 \otimes b^p \in E_2^{0,2p}$ and $1 \otimes ab^{(h-1)/2} \in E_2^{0,h}$, for $h = 1, 3, \ldots, 2p-3$ are permanent cocyles; so they determine elements $z \in E_\infty^{0,2p}$ and $w_h \in E_\infty^{0,h}$, respectively. Obviously, $\iota^*(z) = b^p$, $z^n = 0$ and $w_h w_{h'} = 0$. Let $x = \pi^*(s) \in E_\infty^{1,0}$ and $y = \pi^*(t) \in E_\infty^{2,0}$. Then $x^2 = 0$, $y^p = 0$, and, by the naturality of the Bockstein homomorphism β_p, we have $\beta_p(x) = y$ and $y w_h = 0$ but $x w_h \neq 0$. It follows that the total complex Tot $E_\infty^{*,*}$ is the graded commutative algebra

$$\text{Tot } E_\infty^{*,*} = \mathbb{Z}_p[x, y, z, w_1, w_3, \ldots, w_{2p-3}]/\langle x^2, y^p, z^n, w_h w_{h'}, w_h y\rangle,$$

where $h, h' = 1, 3, \ldots, 2p-3$. Choose $u_h \in H^h(X_G)$ representing w_h for $h = 1, 3, \ldots, 2p-3$. Then $\iota^*(u_h) = ab^{(h-1)/2}$, $u_h^2 = 0$ and $u_h u_{2p-h} = 0$. It follows that

$$H^*(X_G) = \mathbb{Z}_p[x, y, z, u_1, u_3, \ldots, u_{2p-3}]/I,$$

where I is the ideal generated by the homogenous elements

$$x^2, y^p, z^n, y u_h - A_h x y^{(h+1)/2}, u_h u_{2p-h}, u_h u_{h'} - B_{hh'} x u_{h+h'-1} - C_{hh'} y^{(h+h')/2}$$

and $u_h u_{h'} - B'_{hh'} z x u_{h+h'-2p-1} - C'_{hh'} z y^{(h+h'-2p)/2}$.

Here $\deg x = 1$, $\deg y = 2$, $\deg z = 2p$, $\deg u_h = h$ and, when $h = h'$, $0 = B_{hh'} = C_{hh'} = B'_{hh'} = C'_{hh'}$.

If $p \nmid m$, then we must have $d_2(1 \otimes a) = t \otimes 1$, $d_2(1 \otimes b) = 0$. It can be easily observed that

$$d_2\colon E_2^{k,l} \to E_2^{k+2,l-1}$$

is a trivial homomorphism for l even and an isomorphism for l odd. It is now easy to see that $d_r = 0$ for every $r > 2$. So $E_\infty^{k,l} = E_3^{k,l} = \mathbb{Z}_p$ for $k < 2$ and $l = 0, 2, 4, \ldots, 2m-2$. Therefore, we have

$$H^j(X_G) = \begin{cases} \mathbb{Z}_p\,, & 0 \le j \le 2m-1; \\ 0\,, & \text{otherwise.} \end{cases}$$

If $x \in H^1(X_G)$ is determined by $s \otimes 1 \in E_2^{1,0}$, then $x^2 \in E_\infty^{2,0} = 0$. The multiplication by x

$$x \cup (\cdot)\colon E_\infty^{0,l} \to E_\infty^{1,l}$$

is an isomorphism for l even. The element $1 \otimes b \in E_2^{0,2}$ is a permanent cocycle and determines an element $z \in E_\infty^{0,2} = H^2(X_G)$. We have $\iota^*(z) = b$ and $z^m = 0$. Therefore, the total complex Tot $E_\infty^{*,*}$ is the graded commutative algebra

$$\operatorname{Tot} E_\infty^{*,*} = \mathbb{Z}_p[x, z]/\langle x^2, z^m \rangle\,.$$

Notice that $H^j(X_G)$ is $E_\infty^{0,j}$ for j even and $E_\infty^{1,j-1}$ for j odd. Hence,

$$H^*(X_G) = \mathbb{Z}_p[x, z]/\langle x^2, z^m \rangle,$$

where $\deg x = 1$ and $\deg z = 2$. This completes the proof. $\square$

4. Example

We realize here the second case of our theorem. Recall that $G = \mathbb{Z}_p$ acts freely on $L^{2m-1}(p; q_1, \ldots, q_m)$ with the orbit space $L^{2m-1}(p^2; q_1, \ldots, q_m) = K$. We claim that

$$H^*(K; \mathbb{Z}_p) = \mathbb{Z}_p[x, z]/\langle x^2, z^m \rangle,$$

where $\deg x = 1$, $\deg z = 2$. It is known that K is a CW-complex with 1-cell of each dimension $i = 0, 1, \ldots, 2m-1$ and the cellular chain complex of K is

$$0 \to C_{2m-1} \xrightarrow{0} C_{2m-2} \xrightarrow{\times p^2} C_{2m-3} \longrightarrow \cdots \longrightarrow C_2 \xrightarrow{\times p^2} C_1 \xrightarrow{0} C_0,$$

where each $C_i = \mathbb{Z}$. Accordingly, the co-chain complex of K with coefficients in $\mathbb{Z}_p$ is

$$0 \to \mathbb{Z}_p \to \mathbb{Z}_p \to \cdots \to \mathbb{Z}_p \to \mathbb{Z}_p \to 0\,,$$

where each coboundary operator is the trivial homomorphism. Therefore

$$H^j(K; \mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p\,, & \text{for } 0 \le j \le 2m-1; \\ 0\,, & \text{for } j \ge 2m. \end{cases}$$

To determine the cup product in $H^j(K; \mathbb{Z}_p)$, we first observe that the inclusion $K^{(2i-1)} \to K^{(2i+1)}$ induces isomorphism $H^j(K^{(2i-1)}; \mathbb{Z}_p) \cong H^j(K^{(2i+1)}; \mathbb{Z}_p)$ for $j \le 2i-1$ so that we can identify them. For $j < 2i-1$, this follows from the cohomology exact sequence of the pair $(K^{(2i+1)}, K^{(2i-1)})$. The exact cohomology sequence of the pairs

$(K^{(2i+1)}, K^{(2i)})$ and $(K^{(2i)}, K^{(2i-1)})$ show that $H^{2i-1}(K^{(2i+1)}; \mathbb{Z}_p) \cong H^{2i-1}(K^{(2i)}; \mathbb{Z}_p)$ and $H^{2i-1}(K^{(2i)}; \mathbb{Z}_p) \cong H^{2i-1}(K^{(2i-1)}; \mathbb{Z}_p)$; the latter because the homomorphism $H^{2i}(K^{(2i)}, K^{(2i-1)}; \mathbb{Z}_p) \to H^{2i}(K^{(2i)}; \mathbb{Z}_p)$ is surjective.

Now, we choose generators $x \in H^1(K; \mathbb{Z}_p)$ and $z \in H^2(K; \mathbb{Z}_p)$. Then obviously $x^2 = 0$ and $z^m = 0$. We can assume, by induction, that z^i and xz^i generate $H^{2i}(K; \mathbb{Z}_p)$ and $H^{2i+1}(K; \mathbb{Z}_p)$, respectively, for $i \leq m-2$. Then, there is an element kxz^{m-2} such that $z \cup kxz^{m-2} = kxz^{m-1}$ generates $H^{2m-1}(K; \mathbb{Z}_p)$ (see Corollary 3.39 of [2]). We must have $(k, p) = 1$, otherwise the order of kxz^{m-1} would be less than p. Thus xz^{m-1} generates $H^{2m-1}(K; \mathbb{Z}_p)$, and this is true only if z^{m-1} generates $H^{2m-2}(K; \mathbb{Z}_p)$. Hence our claim.

5. Remarks

(i) It is clear from the proof of the theorem that if $p \nmid m$, then only the second possibility of the theorem holds. Furthermore, if $X \sim_p L^{2m-1}(p; q_1, \ldots, q_m)$ and $\pi_1(X) = \mathbb{Z}_p$, then there exists a simply connected space Y with a free action of $\Delta = \mathbb{Z}_p$ such that $Y \sim_p S^{2m-1}$ and $Y/\Delta \approx X$ (Theorems 3.11 and 2.6 of [4]). If $G = \mathbb{Z}_p$ acts freely on X, then the liftings of transformations (on X) induced by the elements of G form a group Γ of order p^2 which acts freely on Y and hence Γ must be cyclic. It is clear that Γ contains the group Δ of deck transformations of the covering $Y \to X$, and $G = \Gamma/\Delta$. So $X/G \approx Y/\Gamma$. Since $Y \sim_p S^{2m-1}$, the mod p cohomology algebra of Y/Γ is a truncation of $H^*(B_\Gamma; \mathbb{Z}_p)$. Thus, in this case also, only the second possibility of the theorem holds irrespective of the condition whether or not $p|m$.

(ii) We recall that a paracompact Hausdorff space X is called finitistic if every open covering of X has a finite dimensional open refinement (see p. 133 of [1]). Our theorem and its proof go through for finitistic spaces.

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