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A polycycle and limit cycles in a non-differentiable predator-prey model

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Abstract. For a non-differentiable predator-prey model, we establish conditions for the existence of a heteroclinic orbit which is part of one contractive polycycle and for some values of the parameters, we prove that the heteroclinic orbit is broken and generates a stable limit cycle. In addition, in the parameter space, we prove that there exists a curve such that the unique singularity in the realistic quadrant of the predator-prey model is a weak focus of order two and by Hopf bifurcations we can have at most two small amplitude limit cycles.

Keywords. Stability; limit cycles; bifurcations; predator-prey model.

1. Introduction

We consider the predator-prey model proposed by Rosenzweig [8] given by

$$X_\mu^\alpha\colon \begin{cases} \dot{x} = rx\left(1 - \dfrac{x}{K}\right) - qx^\alpha y \\ \dot{y} = y(pqx^\alpha - c) \end{cases}, \tag{1.1}$$

where $x(t)$ and $y(t)$ are the densities of the prey and the predator respectively at a given time $t \geq 0$. The vector field X_μ^α is defined on the region $\bar{\Omega} = \{(x, y) | x \geq 0, y \geq 0\}$ where $\mu = (r, K, p, q, c) \in \mathbb{R}_+^5$ and $0 < \alpha < 1$ denote the biological parameters and have the following meanings:

(1) r is the intrinsic growth rate or biotic potential of the prey.
(2) q is the maximal predator per capita consumption rate, i.e., the maximum number of prey that can be eaten by predator in each time unit.
(3) p is the conversion prey rate into predator births.
(4) c is the mortality predator rate in the absence of prey.
(5) K is the prey environment carrying capacity.
(6) α is the adaptation parameter that takes into account the effects of non-random search of prey on behalf of the predator.

System (1.1) is a kind of model that justifies the enrichment paradox, i.e 'increasing the supply of limiting nutrients or energy that tends to destroy the steady state of the ecosystem' (see [8]).

In (1.1), the function $f(x) = qx^\alpha$, $0 < \alpha < 1$ corresponds to the functional response of the predators. This kind of function appears in the works of [1, 4, 6] and also is proposed in the bioeconomics literature where it is indicated as a Cobb–Douglas type of function for the harvest rate function instead of the one used in Schaefer's hypothesis [3].

For $0 < \alpha < 1$, the function $f(x)$ is non-differentiable in the y-axis and this restriction influences the dynamics of the model under study, just as it is demonstrated in this work.

In [1] it has been proved that the only singularity in the first quadrant $\mathbb{R}^+ \times \mathbb{R}^+$ is not globally asymptotically stable. The authors show that there exist orbits of (1.1) that reach the axis y in finite time. This behavior is a consequence of the non-differentiability of (1.1) in the axis y as we will show in Lemmas 2 and 4.

In the parameter space, by a topological equivalence, we prove that there exists a manifold such that (r, K, p, q, c, α) is a point of the manifold. The system (1, 1) has a heteroclinic with w-limit $= \{(0, 0)\}$ and α-limit $= \{(K, 0)\}$. The heteroclinic is part of a stable hyperbolic polycycle and contains at least one unstable limit cycle.

If $\Omega = \mathbb{R}^+ \times \mathbb{R}^+$, it is easy to see that X_μ^α in $\bar{\Omega}$ is a continuous and non-differentiable vector field and in Ω it is a C^∞ vector field. In others words, $X_\mu^\alpha \in \mathcal{X}^0(\bar{\Omega}) - \mathcal{X}^1(\bar{\Omega})$ and $X_\mu^\alpha \in \mathcal{X}^\infty(\Omega)$. To describe the bifurcation diagram of X_μ^α in the parameter space in a simple way, it is necessary to reduce system (1.1) to a normal form (see [2, 5, 10]). Let us consider the linear change of coordinates with the change of time.

$$\varphi\colon \mathbb{R}^2 \times \mathbb{R}_0^+ \to \mathbb{R}^2 \times \mathbb{R}_0^+ \text{ such that } \varphi(u, v, \tau) = \left(Ku, \frac{rv}{qK^{\alpha-1}}, \frac{\tau}{r}\right)$$
$$= (x, y, t), \tag{1.2}$$

where $\det D\varphi(u, v, \tau) = \frac{K^{2-\alpha}}{q} > 0$. Renaming the parameters B and C by

$$B = \frac{pqK^\alpha}{r} > 0, \quad C = \frac{c}{pqK^\alpha} > 0, \tag{1.3}$$

we obtain a qualitatively equivalent vector field $Y_\eta^\alpha = \varphi_* X_\mu^\alpha$ which has the form $Y_\eta^\alpha = P^\alpha \frac{\partial}{\partial u} + Q_\eta^\alpha \frac{\partial}{\partial v}$. The associated differential equation system defined on the region $\varphi^{-1}(\bar{\Omega})$ is given by

$$Y_\eta^\alpha\colon \begin{cases} \dot{u} = u(1-u) - u^\alpha v \\ \dot{v} = Bv(u^\alpha - C) \end{cases}, \quad \eta = (B, C) \in \mathbb{R}_+^2 \text{ and } 0 < \alpha < 1. \tag{1.4}$$

As $0 < \alpha < 1$, Y_η^α is a continuous vector field and non differentiable in the $x = 0$ axis, in particular, in the origin of coordinates.

It is easy to see that $Y_\eta^\alpha \in \mathcal{X}^0(\varphi^{-1}(\bar{\Omega})) - \mathcal{X}^1(\varphi^{-1}(\bar{\Omega}))$, $Y_\eta^\alpha \in \mathcal{X}^\infty(\varphi^{-1}(\Omega))$ and the set of singularities in $\varphi^{-1}(\bar{\Omega})$ are

$$\operatorname{Sing}(Y_\eta^\alpha) = (Q_\eta^\alpha)^{-1}(0) \cap (P^\alpha)^{-1}(0) = \{(0, 0), (1, 0), p_C^\alpha\}, \text{ where}$$
$$p_C^\alpha = \left(C^{\frac{1}{\alpha}}, C^{\frac{1-\alpha}{\alpha}}\left(1 - C^{\frac{1}{\alpha}}\right)\right). \tag{1.5}$$

For $C \geq 1$, the dynamics of (1.4) is not interesting, because the vector field has no singularities in $\varphi^{-1}(\Omega)$. Hence, in this work we assume that $0 < C < 1$.

For a simpler description of the existence of the heteroclinic orbit and their dynamic behavior in the parameter space of (1.4), let us consider the set

$$\Delta = \{(\alpha, C, B)|0 < \alpha < 1, 0 < B, 0 < C < 1\}.$$

Let $\Delta_{\bar{B}} = \Delta \cap \{(\alpha, C, B)|B = \bar{B}\}$ with $\bar{B} > 0$ an arbitrary but fixed value. If $\bar{B}$ ranges over the interval $]0, \infty[$, then the family of transversal section $\Delta_{\bar{B}}$ ranges the space Δ and it is enough describe the dynamics of (1.4) in $\Delta_{\bar{B}}$.

We now define the order of a fine focus. For simplicity we assume that the singularity at the origin is a center focus. It is also well-known that there is a function V, analytic in a neighbourhood of the origin, such that the rate of change along orbits, $\dot{V}$, is of the form $\eta_2 r^2 + \eta_4 r^4 + \dots$, where $r^2 = x^2 + y^2$. The focal values are the η_{2k}, and the origin is a centre if and only if they are all zero. However, since they are polynomial functions of the coefficients of the vector field, the ideal they generate has a finite basis, so there is M such that $\eta_{2\ell} = 0$, for $\ell \leq M$ which implies that $\eta_{2\ell} = 0$ for all ℓ. The value of M is not known *a priori*, so it is not clear how many focal values should be calculated.

The computer software *Mathematica* [11], is used to calculate the first few focal values. These are then 'reduced' in the sense that each is computed modulo the ideal generated by the previous ones: that is, the relations $\eta_2 = \eta_4 = \cdots = \eta_{2k} = 0$ are used to eliminate some of the coefficients of the vector field in η_{2k+2}. The reduced focal value η_{2k+2}, with strictly positive factors removed, is known as the *Liapunov quantity* $L(k)$. Common factors of the reduced focal values are removed and the computation proceeds until it can be shown that the remaining expressions cannot be zero simultaneously. The circumstances under which the calculated focal values are zero yield the necessary centre conditions. The origin is a fine focus of *order* k if $L(i) = 0$ for $i = 0, 1, \dots, k-1$ and $L(k) \neq 0$. At most k limit cycles can bifurcate out of a fine focus of order k; these are called *small amplitude* limit cycles [2].

2. Main results

Lemma 1. *For* $(\alpha, B, C) \in \Delta$ *the vector field* (1.4) *has a hyperbolic saddle at the singularity* $(1, 0)$ *where the unstable manifold* W^u *is transversal to the* u*-axis* (*see figure* 1).

Lemma 2. *The vector field* (1.4) *has a non hyperbolic singularity at the origin and a stable manifold* W^s, *as the unique separatrix between a hyperbolic sector and a non Lipschitzian stable parabolic sector* (*see figure* 1).

Lemma 3. *In a parameter space* Δ *there exists a curve* γ *such that if* $(\alpha, B, C) \in \gamma$, *the unique singularity of the vector field* Y_η^α *in the first quadrant* $\varphi^{-1}(\Omega)$, *is a stable weak*

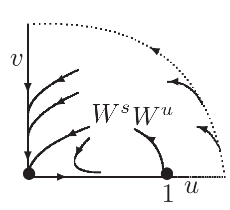


Figure 1.

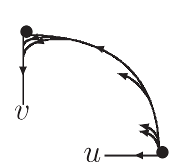


Figure 2.

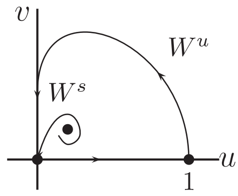


Figure 3.

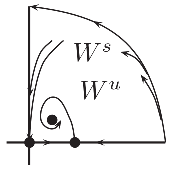


Figure 4.

focus of order two. By the Hopf bifurcations, we have a hyperbolic stable limit cycle that generate from the weak focus and encloses a hyperbolic unstable limit cycle or a semi-stable limit cycle whose interior is stable and unstable.

Lemma 4. *At infinity, in the Poincaré disc the vector field* (1.4) *has a sector of the following form*:

(i) *Only one non Lipschitzian hyperbolic sector in the v-axis direction.*
(ii) *Only one non Lipschitzian parabolic sector in the u-axis direction* (*see figure* 2).

Lemma 5. *For* $0 < \alpha \ll 1$, *if* $(\alpha, C) \in \Delta_{\bar{B}}$, *the unstable manifold W^u of the hyperbolic saddle at the singularity* $(1, 0)$ *of the vector field* (1.4) *in Lemma* 1, *has a contact with some point in the v-axis. The relative position of the manifolds W^u and W^s shown in figure* 1 *is shown qualitatively in figure* 3.

Lemma 6. *For* $0 \ll \alpha \leq 1$, *if* $(\alpha, C) \in \Delta_{\bar{B}}$ *in the first quadrant of the Poincaré disc*, *the manifolds W^u and W^s shown in figure* 1 *has the relative position as shown in figure* 4.

Lemma 7. *If* $(\alpha, C) \in \Delta_{\bar{B}}$, $B > 0$ *and* $0 < \alpha \ll 1$, *the singularity p_C^α of* (1.4) *is not involved by limit cycles.*

Theorem 1. *In the parameter space $\Delta_{\bar{B}}$ there exists a bifurcation curve $H(\alpha, C, \bar{B}) = 0$, such that if $(\alpha, C, \bar{B}) \in H^{-1}(0)$, the vector field* (1.4) *has a heteroclinic orbit* Γ. *This orbit is born as a consequence of the collapse of the manifolds W^u and W^s* (*see figure* 1). *Moreover if* $\frac{1}{2} \leq C < 1$, *the polycycle that form the heteroclinic orbit that connects the singularities at* $(1, 0)$ *and the origin is hyperbolic and stable.*

Theorem 2. *In the parameter space* $\Delta_{\bar{B}}$, *if* $\frac{1}{2} \leq C < 1$ *is an arbitrary but fixed number, there exists α such that the polycycle of* (1.4) *of Theorem* 1 *at least enclosed one unstable limit cycle.*

3. Proof of the main results

Proof of Lemma 1. The proof follows from

$$DY_\eta^\alpha(1, 0) = \begin{pmatrix} -1 & -1 \\ 0 & B(1-C) \end{pmatrix}.$$

□

Proof of Lemma 2. Let us consider $\Gamma = \varphi^{-1}(\Omega)$ and the C^1-diffeomorphism,

$$\psi\colon \mathbb{R}^2_+ \to \mathbb{R}^2_+ \text{ defined as } \psi(x, y) = \left(x^{\frac{2}{1-\alpha}}, y\right) = (u, v),$$

where $\det D\psi(x, y) = \frac{2}{1-\alpha} x^{\frac{1+\alpha}{1-\alpha}} > 0$, for $x > 0$.

In the new coordinates, the vector field (1.4) defined in $(\varphi \circ \psi)^{-1}(\Omega)$ is given by

$$\tilde{Y}^\alpha_\eta\colon \begin{cases} \dot{x} = \frac{1-\alpha}{2}\left[x\left(1-x^{\frac{2}{1-\alpha}}\right) - x^{-1}y\right] \\ \dot{y} = By\left[x^{\frac{2\alpha}{1-\alpha}} - C\right] \end{cases}, \text{ with } \tilde{Y}^\alpha_\eta = (\varphi \circ \psi)_* Y^\alpha_\eta = D^{-1}(\varphi \circ \psi) Y^\alpha_\eta(\varphi \circ \psi). \tag{3.1}$$

Let the change of time be $t \to \frac{2x}{1-\alpha} t$ and the new parameter $A = \frac{2B}{1-\alpha}$. Then we have a differentiable extension of $\tilde{Y}^\alpha_\eta$ to the region $\psi^{-1}(\bar{\Gamma})$ given by

$$\bar{Y}^\alpha_{\bar{\eta}}\colon \begin{cases} \dot{x} = x^2\left(1 - x^{\frac{2}{1-\alpha}}\right) - y \\ \dot{y} = Axy\left[x^{\frac{2\alpha}{1-\alpha}} - C\right] \end{cases}, \quad \text{where } \bar{\eta} = (A, C). \tag{3.2}$$

Clearly $\bar{Y}^\alpha_{\bar{\eta}}(0, y) = -y\frac{\partial}{\partial x}$. Then the orbits of the vector field (3.2) orthogonally cross the $x = 0$ axis ($y > 0$).

For $0 < y_0$, let $(0, y_0)$ be an initial condition and let γ be the orbit of vector field at this point. Then $\gamma^* = \gamma - \{(0, y_0)\}$ is an orbit of system (3.2) and consequently an orbit of (3.1). As ψ is a homeomorphism, systems (1.4) and (3.1) are C^0-equivalent. Hence $\psi(\gamma^*)$ is an orbit of (1.4) and by continuity, $\psi(\gamma)$ is an orbit of (1.4) that is tangent to the vector field Y^α_η at the point $\psi(0, y_0) = (0, y_0)$. Clearly the v-axis ($u = 0$) is an invariant manifold and $Y^\alpha_\eta(0, y_0) = -BC\frac{\partial}{\partial v}$. Thus, for the point $(0, y_0)$, there exist at least two orbits. Since for initial conditions on the v-axis, uniqueness of the orbits does not exist, (1.4) is non Lipschitzian (see figure 1).

As the vector field (3.2) is a differentiable extension of (1.4) to the region $\psi^{-1}(\bar{\Gamma})$, we have

$$\bar{Y}^\alpha_{\bar{\eta}} \in \mathbb{X}^1(\psi^{-1}(\bar{\Gamma})) \quad \text{and} \quad D\bar{Y}^\alpha_{\bar{\eta}}(0, 0) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

In order to desingularize the origin, we consider the horizontal blowing-up

$$\Psi\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad \Psi(u, v) = (u, uv) = (x, y).$$

Hence, we have a new vector field

$$\Psi_*(\bar{Y}^\alpha_{\bar{\eta}}) = (D\Psi)^{-1} \bar{Y}^\alpha_{\bar{\eta}} \Psi = Z^\alpha_{\bar{\eta}}, \tag{3.3}$$

and the associated system is

$$Z^\alpha_{\bar{\eta}}\colon \begin{cases} \dot{u} = u\left[u\left(1 - u^{\frac{2}{1-\alpha}}\right) - v\right] \\ \dot{v} = v\left[Au\left(u^{\frac{2\alpha}{1-\alpha}} - C\right) - u\left(1 - u^{\frac{2}{1-\alpha}}\right) + v\right] \end{cases}, \quad \text{where } DZ^\alpha_{\bar{\eta}}(0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

To desingularize the singularity at the origin of $Z^{\alpha}_{\bar{\eta}}$, we consider the vertical blowing-up

$$\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad \Phi(x, y) = (xy, y) = (u, v).$$

We thus obtain a new vector field

$$\Phi_*(Z^{\alpha}_{\bar{\eta}}) = (D\Phi)^{-1} Z^{\alpha}_{\bar{\eta}} \Phi = \tilde{Z}^{\alpha}_{\bar{\eta}},$$

where the associated system is given by

$$\tilde{Z}^{\alpha}_{\bar{\eta}}\colon \begin{cases} \dot{x} = xy\big[x\big(1 - (xy)^{\frac{2}{1-\alpha}}\big) - 2 - Ax\big((xy)^{\frac{2\alpha}{1-\alpha}} - C\big) + x\big(1 - (xy)^{\frac{2}{1-\alpha}}\big)\big] \\ \dot{y} = y^2\big[Ax\big((xy)^{\frac{2\alpha}{1-\alpha}} - C\big) - x\big(1 - (xy)^{\frac{2}{1-\alpha}}\big) + 1\big] \end{cases}.$$

As the line $x = 0$ is a continuum of singularities of $\tilde{Z}^{\alpha}_{\bar{\eta}}$, to lift these singularities, we consider the extension of the vector field $\bar{Z}^{\alpha}_{\bar{\eta}} = \frac{1}{y}\tilde{Z}^{\alpha}_{\bar{\eta}}$. As $\bar{Z}^{\alpha}_{\bar{\eta}}(x, 0) = x[x(2+AC) - 2]\frac{\partial}{\partial x}$, for $x = 0$ or $x = \frac{2}{2+AC}$, we have $\bar{Z}^{\alpha}_{\bar{\eta}}(x, 0) = 0$.

Moreover,

$$D\bar{Z}^{\alpha}_{\bar{\eta}}(0, 0) = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$D\bar{Z}^{\alpha}_{\bar{\eta}}\left(\frac{2}{2 + AC}, 0\right) = \begin{pmatrix} 2 & 0 \\ 0 & -\frac{AC}{2+AC} \end{pmatrix}.$$

The singularities of $\bar{Z}^{\alpha}_{\bar{\eta}}$, $(0, 0)$ and $\left(\frac{2}{2+AC}, 0\right)$ are hyperbolic saddles. Thus there exists a stable manifold w^s tangent to the straight line $x = \frac{2}{2+AC}$ at the singularity at the origin as shown in figure 5.

By the blowing-down of Φ and Ψ, and by the function ψ we prove the existence of the stable manifold $W^s = (\Phi \circ \Psi \circ \psi)^{-1}(w^s)$ of (1.4).

Furthermore, for $v \ll 1$, the manifold W^s is tangent to $v = \frac{2+AC}{2}u^{1-\alpha}$ at the origin (see figure 1).

As the orbits of system (3.2) orthogonally cross the $x = 0$-axis, the uniqueness of the separatrix W^s follows. □

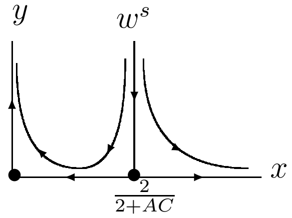


Figure 5.

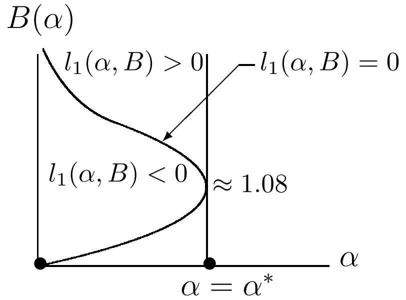


Figure 6.

Proof of Lemma 3. The set of singularities of (1.4), $Y_\eta^\alpha = P^\alpha \frac{\partial}{\partial u} + Q_\eta^\alpha \frac{\partial}{\partial v}$ are

$$\mathrm{Sing}\,(Y_\eta^\alpha) = (P^\alpha)^{-1}(0) \cap (Q_\eta^\alpha)^{-1}(0) = \{(0,0), (1,0), p_C^\alpha\},$$

where for $0 < C < 1$, $p_C^\alpha = \big(C^{\frac{1}{\alpha}}, C^{\frac{1-\alpha}{\alpha}}\big(1 - C^{\frac{1}{\alpha}}\big)\big)$ is the unique singularity of (1.4) in the first quadrant $\varphi^{-1}(\Omega)$. Moreover, it is easy to see that

$$DY_\eta^\alpha(p_C^\alpha) = \begin{pmatrix} 1 - \alpha - C^{\frac{1}{\alpha}}(2-\alpha) & -C \\ \alpha B\big(1 - C^{\frac{1}{\alpha}}\big) & 0 \end{pmatrix}.$$

Then $\det DY_\eta^\alpha(p_C^\alpha) = \alpha BC\big(1 - C^{\frac{1}{\alpha}}\big) > 0$ and the singularity p_C^α is a center-focus.

For $k = 0, 1, 2$, let us denote by L_k the first three Liapunov quantities (see [2]) at the singularity p_C^α of vector field (1.4). With a simple calculation,

$$L_0 = 1 - \alpha - C^{\frac{1}{\alpha}}(2-\alpha).$$

So, $L_0 = 0$ if $C = \big(\frac{1-\alpha}{2-\alpha}\big)^\alpha$ and p_C^α is a weak focus of order at least one. In order to calculate the Liapunov quantities of higher order, we consider the conjugation

$$\psi\colon \mathbb{R}^2 \to \mathbb{R}^2 \text{ such that } \psi(x,y) = \big(-\bar\eta y + C^{\frac{1}{\alpha}}, \alpha B\big(1 - C^{\frac{1}{\alpha}}\big)x + C^{\frac{1-\alpha}{\alpha}}\big(1 - C^{\frac{1}{\alpha}}\big),$$

where $\bar\eta = \big(\frac{\alpha BC}{2-\alpha}\big)^{\frac{1}{2}}$. Then $\psi_*(Y_\eta^\alpha) = (D\psi)^{-1} Y_\eta^\alpha \psi = Z_\eta^\alpha$ and

$$\frac{1}{\bar\eta} Z_\eta^\alpha(x,y) = \left(-y + \sum_{i,j=2}^{5} A_{i,j} x^i y^j + \text{H.O.T.}\right)\frac{\partial}{\partial x} + \left(x + \sum_{i,j=2}^{5} B_{i,j} x^i y^j + \text{H.O.T.}\right)\frac{\partial}{\partial y},$$

where H.O.T. denotes the higher order term and $A_{i,j} = A_{i,j}(\alpha,\eta)$, $B_{i,j} = B_{i,j}(\alpha,\eta)$.

Using the *Mathematica* software [11] for symbolic calculus, we have

$$L_1 = (\alpha, B) = -\frac{1}{16}(1-\alpha)\bar\eta\alpha C^{-\frac{2}{\alpha}} l_1(\alpha, B),$$

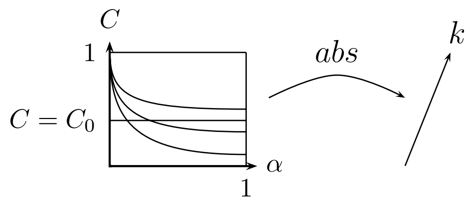
where

$$l_1(\alpha, B, C) = \alpha(2-\alpha)C^2B^2 + C[2\alpha(2-\alpha) - (1-\alpha)^2]B + \alpha(2-\alpha)$$

is a quadratic polynomial with respect to the parameter B. If $\Delta(\alpha, C)$ denotes the discriminant of the above curve, we have

$$\Delta(\alpha, C) = C^2\Delta(\alpha) \text{ with } \Delta(\alpha) = (\alpha-1)^2(1 - 10\alpha + 5\alpha^2).$$

Now $\Delta(\alpha) > 0$ if and only if $0 < \alpha < \alpha^*$, where $\alpha^* = \frac{5-2\sqrt{5}}{5}$ is a solution of the equation $\Delta(\alpha) = 0$. Therefore the quadratic surface has two positive roots that collapse for $\alpha = \alpha^*$.



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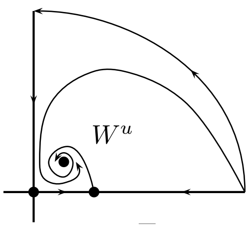

Figure 8.

If $L_0 = 0$, that is, $C = \left(\frac{1-\alpha}{2-\alpha}\right)^{\alpha}$, the equation $l_1(\alpha, B, C) = l_1(\alpha, B) = 0$ defines the implicit relation $B = B(\alpha)$ on the interval $0 \le \alpha \le \alpha^*$ (see figure 6).

Clearly $sg(L_1(\alpha, B)) = -sg(l_1(\alpha, B))$. Again, we use the *Mathematica* software to compute L_2 and we obtain

$$L_2(\alpha, B, C) = -\frac{1}{2304}\frac{(1-\alpha)}{(2-\alpha)}C^{-2\left(1+\frac{2}{\alpha}\right)}\bar{\eta}l_2(\alpha, B, C).$$

If $C = \left(\frac{1-\alpha}{2-\alpha}\right)^{\alpha}$, $l_2(\alpha, B, C) = l_2(\alpha, B)$ and on the αB-plane, we have

$$l_2(\alpha, B)|_{l_1(\alpha,B)=0} = l_2(\alpha) > 0 \quad \text{for } 0 < \alpha < \alpha^*.$$

Therefore, if $(\alpha, B, C) \in L_0^{-1}(0) \cap L_1^{-1}(0)$ and by the fact that all of the factors of $L_2(\alpha, B, C)$ are positive preceded by a negative sign, the singularity p_C^{α} of (1.4) is a weak focus of order two. Perturbing those parameters of such a way inverting the type of stability of p_C^{α}, we have by a consecutive Hopf bifurcation, one stable hyperbolic limit cycle that contains an unstable hyperbolic limit cycle and that it contains a hyperbolic stable focus at singularity. □

Proof of Lemma 4. By the compactification in the Poincaré disc $v = \frac{1}{y}$, $u = \frac{x}{y}$ the vector field (1.4) in the new coordinate xy, in the infinity and in the v-axis direction, has the form

$$\bar{X}_{\eta}^{\alpha}\colon \begin{cases} \dot{x} = \frac{1}{y}(-x^2 + (1+BC)xy - x^{\alpha}y^{1-\alpha} - Bx^{1+\alpha}y^{1-\alpha}) \\ \dot{y} = B(Cy - x^{\alpha}y^{1-\alpha}) \end{cases}.$$

As $\bar{X}_{\eta}^{\alpha}$ is non-differentiable in a neighborhood of the origin, let us consider the rescaling of the axis: $\psi\colon \mathbb{R}_+^2 \to \mathbb{R}_+^2$ such that $\psi(z, w) = \left(z^{\frac{2}{1-\alpha}}, w^{\frac{2}{1-\alpha}}\right) = (x, y)$ and the time rescaling $t \to yzt$ where for $z > 0$ and $w > 0$, $\det D\psi(z, w) = \frac{4}{(1-\alpha)^2}(zw)^{\frac{1+\alpha}{1-\alpha}} > 0$.

The new vector field $\tilde{X}_{\eta}^{\alpha} = (D\psi)^{-1}\bar{X}_{\eta}^{\alpha}\psi$ in the zw coordinates is given by

$$\tilde{X}_{\eta}^{\alpha}\colon \begin{cases} \dot{z} = \dfrac{1-\alpha}{2}\left[(1+BC)z^2w^{\frac{2}{1-\alpha}} - z^{\frac{2(2-\alpha)}{1-\alpha}} - w^2 - Bz^{\frac{2}{1-\alpha}}w^2\right] \\ \dot{w} = \dfrac{1-\alpha}{2}B\left[-z^{\frac{1+\alpha}{1-\alpha}}w^3 + Czw^{\frac{3-\alpha}{1-\alpha}}\right] \end{cases}. \tag{3.4}$$

Moreover, it is clear that

$$D\tilde{X}_{\eta}^{\alpha}(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In order to desingularize the singularity at the origin of $\tilde{X}_\eta^\alpha$, we only consider a horizontal blowing-up. It is not necessary the vertical blowing-up, in fact as $\tilde{X}_\eta^\alpha(0, w) = -\frac{1-\alpha}{2} w^2 \frac{\partial}{\partial z}$ and for $w > 0$ the vector field transversally crosses the axis $z = 0$.

Let us consider the horizontal blowing-up $\Psi \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Psi(x, y) = (x, xy) = (z, w)$ and the time rescaling $t \to xt$. The new vector field in the xy coordinates is given by $\Psi_*(\tilde{X}_\eta^\alpha) = (D\Psi)^{-1} \tilde{X}_\eta^\alpha \Psi = Z_\eta^\alpha$ where the associated system is

$$Z_\eta^\alpha \colon \begin{cases} \dot{x} = \dfrac{1-\alpha}{2}\left[x^{\frac{3-\alpha}{1-\alpha}}\{(1+BC)y^{\frac{2}{1-\alpha}} - By^2 - 1\} - xy^2\right] \\ \dot{y} = -\dfrac{1-\alpha}{2}\left(x^{\frac{2}{1-\alpha}} y^{\frac{3-\alpha}{1-\alpha}} - x^{\frac{3-\alpha}{1-\alpha}} - y^3\right) \end{cases}.$$

Moreover, $Z_\eta^\alpha(0, y) = \frac{1-\alpha}{2} y^3 \frac{\partial}{\partial y}$ and $(0, 0)$ is the only singularity of Z_η^α in the y-axis where

$$DZ_\eta^\alpha(0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

As $Z_\eta^\alpha(x, 0) = \frac{1-\alpha}{2} x^{\frac{3-\alpha}{1-\alpha}} \left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$ the vector field Z_η^α crosses transversely the x-axis. In consequence, there does not exist an orbit in $\mathbb{R}_+ \times \mathbb{R}_+$ with the origin as α-limit, or ω-limit. Then by the blowing-down of Ψ, the function ψ and the time rescaling, the vector field $\bar{X}_\eta^\alpha$ has a saddle point at the origin. Furthermore, if $0 < w_0$ and $(0, w_0)$ is an initial condition and ζ is the orbit of the vector field $\tilde{X}_\eta^\alpha$ for this point, $\zeta^* = \zeta - \{(0, w_0)\}$ is the orbit of (3.4) and the saddle point is non-Lipschitzian.

As ψ is an homeomorphism, $\bar{X}_\eta^\alpha$ and $\tilde{X}_\eta^\alpha$ are C^o-equivalent. Then $\psi(\zeta^*)$ is an orbit of $\bar{X}_\eta^\alpha$ and $\psi(\zeta)$ and by continuity is an orbit of $\bar{X}_\eta^\alpha$. As $\bar{X}_\eta^{\alpha,\bar{\eta}}\left(0, w_0^{\frac{2}{1-\alpha}}\right) = BCy \frac{\partial}{\partial y}$, the y-axis is clearly invariant and the orbit is tangent to the vector field at the point $\left(0, w_0^{\frac{2}{1-\alpha}}\right)$. This proves that for the point $\left(0, w_0^{\frac{2}{1-\alpha}}\right)$, there exist at least two orbits. In the previous argument it is also satisfied that if it is applied in a similar form to an orbit γ of the vector field Z_η^α in $\mathbb{R}_+ \times \mathbb{R}_+$ with the initial condition at the point $(x_0, 0)$, $x_0 > 0$.

Then, by the blowing-down, $\Psi(\gamma)$ is an orbit of (3.4). As $\tilde{X}_\eta^\alpha(z, 0) = -\frac{1-\alpha}{2} z^{\frac{2(2-\alpha)}{1-\alpha}} \frac{\partial}{\partial z}$, the w-axis is invariant and the orbit is tangent to the axis $w = 0$ at the point $(x_0, 0)$. Finally $\psi(\Psi(\gamma))$ is an orbit of the vector field $\bar{X}_\eta^\alpha$ tangent to the x-axis and by the non-uniqueness of the orbits in the axis $u = 0$ and in the axis $v = 0$, the vector field (1.4) at infinity has a non-Lipschitzian saddle point (see figure 2) and this proves part (i) of the Lemma.

In order to prove (ii), let us consider the compactification $u = \frac{1}{x}$, $v = \frac{y}{x}$ in the Poincaré disc. The vector field (1.4) at infinity in the new coordinates xy and in the direction of the u-axis has the form

$$\bar{X}_\eta^\alpha \colon \begin{cases} \dot{x} = 1 - x + x^{1-\alpha} y \\ \dot{y} = y(Bx^{-\alpha} - BC - x + x^2 - x^{2-\alpha}) \end{cases}.$$

By the time rescaling $t \to xt$, we have

$$\tilde{X}_\eta^\alpha \colon \begin{cases} \dot{x} = x - x^2 + x^{2-\alpha} y \\ \dot{y} = y(Bx^{1-\alpha} - BCx - x^2 + x^3 - x^{3-\alpha}) \end{cases}. \tag{3.5}$$

Clearly, $\tilde{X}^\alpha_\eta$ in the neighborhood of the origin is a non-differentiable vector field.

To prove that the vector field $\tilde{X}^\alpha_\eta$ has only one unstable parabolic sector at the origin, in (3.5) we consider the following coordinated rescaling:

$\psi\colon \mathbb{R}^2_+ \to \mathbb{R}^2_+$ such that $\psi(z,w) = \left(z^{\frac{2}{1-\alpha}}, w^{\frac{2}{1-\alpha}}\right) = (x,y)$, and for $z > 0$ and $w > 0$, $\det D\psi(z,w) = \frac{4}{(1-\alpha)^2}(zw)^{\frac{1+\alpha}{1-\alpha}} > 0$.

The vector field in the zw-coordinates is given by $U^\alpha_\eta = (D\psi)^{-1}\tilde{X}^\alpha_\eta\psi$, where

$$U^\alpha_\eta\colon \begin{cases} \dot{z} = \dfrac{1-\alpha}{2} z\left[1 - z^{\frac{2}{1-\alpha}} + z^2 w^{\frac{2}{1-\alpha}}\right] \\ \dot{w} = \dfrac{1-\alpha}{2} w\left[Bz^2 - BCz^{\frac{2}{1-\alpha}} - z^{\frac{4}{1-\alpha}} + z^{\frac{6}{1-\alpha}} - z^{\frac{2(3-\alpha)}{1-\alpha}}\right] \end{cases}.$$

Clearly the vector field U^α_η is differentiable at the origin and $DU^\alpha_\eta(0,0) = \begin{pmatrix} \frac{1-\alpha}{2} & 0 \\ 0 & 0 \end{pmatrix}$. By the time rescaling, we can consider the vector field $\tilde{U}^\alpha_\eta = \frac{1}{z}U^\alpha_\eta$ and we have $\tilde{U}^\alpha_\eta(0,w) = \frac{1-\alpha}{2}\frac{\partial}{\partial z}$. This proves that the infinite is unstable. Moreover, the vector field $\tilde{U}^\alpha_\eta$ orthogonally crosses the w-axis. If $(0, w_0)$ with $w_0 > 0$ is an initial condition and γ the respective orbit in $\mathbb{R}_+ \times \mathbb{R}_+$ of the vector field $\tilde{U}^\alpha_\eta$ through that point, and by the fact that $\tilde{X}^\alpha_\eta(0,y) = By\frac{\partial}{\partial y}$ then, $\psi(\gamma)$ is an orbit of $\tilde{X}^\alpha_\eta$ tangent to the axis $x = 0$ through the point $\left(0, w_0^{\frac{2}{1-\alpha}}\right)$. This proves that in the Poincaré compactification, the vector field (1.4) in the infinity does not have uniqueness of solutions and this complete the proof of Lemma 4. □

Proof of Lemma 5. By (1.5), we can define the abscissa function of the singularity as abs: $]0,1[\times]0,1[\to [0,1]$ where $\mathrm{abs}(\alpha, C) = C^{\frac{1}{\alpha}}$. For $0 < k < 1$ and $B = \bar{B}$ in the parameter space $\Delta_{\bar{B}}$, the family of level curves $\mathrm{abs}(\alpha, C) = k$ in the plane αC, can be continuously extended to the point (0, 1) (see figure 7).

Then for any $0 < C_0 < 1$, $\lim_{(\alpha,C)\to(0,C_0)} \mathrm{abs}(\alpha, C_0) = 0$. Therefore from (1.5),

$$\lim_{(\alpha,C)\to(0,C_0)} \left[\frac{1}{C_0} C^{\frac{1}{\alpha}}(1 - C^{\frac{1}{\alpha}})\right] = 0 \quad \text{and} \quad \lim_{(\alpha,C)\to(0,C_0)} (p^\alpha_C) = 0.$$

For $0 < \alpha \ll 1$, the singularity p^α_C is located in a small neighborhood of the origin and by the existence of the manifold W^s (Lemma 2) and as the w-limit of the manifold W^u (Lemma 4) is not located in the infinity, W^u has contact with the v-axis at some point (uniqueness of solutions does not exist in the v-axis). The relative position of the manifolds W^s and W^u in figure 1, are qualitatively as shown in figure 3, and this proves that W^u with part of the axis forms a heteroclinic orbit with w-limit = (0, 0) and α-limit = (1, 0). □

Proof of Lemma 6. If in (1.4) $\alpha = 1$, Y^1_η is a quadratic vector field and as $B > 0$ in $\varphi^{-1}(\Omega)$, it is well-known that the only singularity $p^1_C = (C, 1-C)$ is a hyperbolic focus which is globally asymptotically stable (see figure 8). If $0 \ll \alpha < 1$, the vector field Y^α_η in the v-axis and in the infinity of the Poincaré disc (Lemmas 2 and 4) is a C^0-small perturbation of Y^1_η with loss of differentiability. In Lemma 2, it is also proved that the existence of the stable manifold W^s for all $(\alpha, C, B) \in \Delta$, then the relative position of the manifold W^s and W^u are as shown in figure 4. □

Proof of Lemma 7. In Lemma 2 we show the existence of the manifold W^s of (1.4) and at the end of the proof, we conclude that W^s is a manifold of (1.4) tangent to the curve

$$v = \frac{1-\alpha+BC}{1-\alpha}u^{1-\alpha}, \quad \text{where } A = \frac{2B}{1-\alpha} \tag{3.6}$$

at the origin. In the phase plane of (1.4), the graph of the curve $\operatorname{div}(Y_\eta^\alpha)(u,v) = 0$, continuously extended to the origin, has the equation

$$v = \frac{1}{\alpha}[(1-BC)u^{1-\alpha} - 2u^{2-\alpha} + Bu]. \tag{3.7}$$

If $0 < \alpha \ll 1$, the graph of (3.6) and (3.7) are tangent in the origin to the straight lines $v = \frac{1-\alpha+BC}{1-\alpha}u$ and $u = 0$ respectively, and by the proof of Lemma 5 with $0 < C < 1$, we know that $\lim_{\alpha\to 0} p_C^\alpha = 0$. In a neighborhood of the origin of the phase plane of (1.4) and in the region below W^s, the divergence does not change sign, $\operatorname{sign}(\operatorname{div}(Y_\eta^\alpha)(u,v)) > 0$ and the singularity p_C^α cannot be surrounded by a limit cycle. □

Proof of Theorem 1. The vector field Y_η^α continuously depend on the parameters $(\alpha, C, \bar{B}) \in \Delta_{\bar{B}}$ and the manifolds $W^u = W^u(\alpha, C, \bar{B})$ and $W^s = W^s(\alpha, C, \bar{B})$ are of class C^0. By the C^0-continuity and by the relative position of these manifolds (Lemmas 5 and 6), for each $C_0 \in (0,1)$, there exists in the parameters space $\alpha_0 \in (0,1)$ such that $W^u(\alpha_0, C_0, \bar{B}) = W^s(\alpha_0, C_0, \bar{B})$. Therefore, in $\Delta_{\bar{B}}$, by the collapse of W^u and W^s, there exists a bifurcation curve $H(\alpha, C, \bar{B}) = 0$ of heteroclinic, then in the phase plane of the vector field (1.4) has a polycycle with vertexes at the non-hyperbolic singularity at the origin the hyperbolic singularity at $(1,0)$.

To study the hyperbolicity rate of the polycycle, let us consider the function $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\psi(x,y) = \big(x^{\frac{1}{\alpha}}, x^{\frac{1-\alpha}{\alpha}}y\big) = (u,v)$. The vector field (1.4) in the new coordinates is given by

$$Z_\eta^\alpha\colon \begin{cases} \dot{x} = \alpha x\big(1 - x^{\frac{1}{\alpha}} - y\big) \\ \dot{y} = y[B(x-C) - (1-\alpha)(1 - x^{\frac{1}{\alpha}} - y)] \end{cases}, \; Z_\eta^\alpha = \psi_* Y_\eta^\alpha. \tag{3.8}$$

If $x > 0$, the function ψ defined in $(\varphi \circ \psi)^{-1}(\Omega)$ with $\det D\psi(x,y) = \frac{1}{\alpha}x^{2\frac{1-\alpha}{\alpha}} > 0$ is a conjugation. Then the systems (3.8) and (1.4) in the open set $(\varphi\circ\psi)^{-1}(\Omega)$ are topologically equivalent.

The vector field (3.8) is a differentiable extension of (1.4) to the region $(\varphi \circ \psi)^{-1}(\bar{\Omega})$ and the set of singularities is given by $\operatorname{Sing}(Z_\eta^\alpha) = \{(0,0), (1,0), \big(0, 1 + \frac{BC}{1-\alpha}\big), \psi^{-1}(p_C^\alpha)\}$.

Let us denote by Γ the heteroclinic of (1.4) which is the collapse of the manifolds W^u and W^s. As the origin (1.4) was unfold into two singularities $(0,0)$, $\big(0, 1 + \frac{\bar{B}C_0}{1-\alpha_0}\big)$, and as the point $(1,0)$ is invariant under the function ψ, $\psi^{-1}(\Gamma)$ is the manifold that joins the singularities $\big(0, 1+\frac{\bar{B}C_0}{1-\alpha_0}\big)$ and $(1,0)$ to form the polycycle with vertex $(0,0)$, $(1,0)$, $\big(0, 1+\frac{\bar{B}C_0}{1-\alpha_0}\big)$.

For the vector field $Z^{\alpha}_{\eta}(x, y)$ we have

$$DZ^{\alpha}_{\eta}(x, y) = \begin{cases} \begin{pmatrix} \alpha_0 & 0 \\ 0 & -(\bar{B}C_0 + 1 - \alpha_0) \end{pmatrix}, & \text{if}(x, y) = (0, 0) \\ \begin{pmatrix} -1 & -\alpha_0 \\ 0 & \bar{B}(1 - C_0) \end{pmatrix}, & \text{if}(x, y) = (1, 0) \\ \begin{pmatrix} \frac{\alpha_0 \bar{B} C_0}{1-\alpha_0} & 0 \\ 0 & \bar{B}C_0 + 1 - \alpha_0 \end{pmatrix}, & \text{if}(x, y) = \left(0, 1 + \dfrac{\bar{B}C_0}{1 - \alpha_0}\right) \end{cases}.$$

Then the vertex of the polycycle are hyperbolic saddles. If $k = 1, 2, 3$, $\lambda_k > 0$ and $\mu_k < 0$ denote the eigenvalues of the hyperbolic saddle points $(0, 0)$, $(1, 0)$ and $\left(0, 1 + \frac{\bar{B}C_0}{1-\alpha_0}\right)$ respectively, and the rate of hyperbolicity (see [7, 9]) is

$$r(\alpha_0, C_0) = \frac{\lambda_1 \lambda_2 \lambda_3}{|\mu_1 \mu_2 \mu_3|} = \frac{(1 - C_0)(1 - \alpha_0)}{C_0}.$$

If $C_0 \in [\frac{1}{2}, 1)$, for all $\alpha \in]o, 1[$, $r(\alpha, C_0) < 1$, $r(\alpha_0, C_0) < 1$ and the polycycle of (3.8) is stable. Therefore the polycycle of (1.4) is stable. □

Proof of Theorem 2. If in (1.4) $\alpha = 1$, the vector field Y^1_{η} is quadratic, and it is known that in $\varphi^{-1}(\Omega)$ the unique singularity $p^1_C = (C, 1 - C)$ is a hyperbolic focus which is globally asymptotically stable as shown in figure 8. By Theorem 1, in the parameter space $\Delta_{\bar{B}}$, for each $C_0 \in [\frac{1}{2}, 1)$ there exists $\alpha_0 \in (0, 1)$ with the hyperbolicity rate $r(\alpha_0, C_0) < 1$, such that the vector field $Y^{\alpha_0}_{\bar{\eta}}$, where $\bar{\eta} = (\bar{B}, C_0)$ has a stable hyperbolic polycycle.

Then there exists in the interior of the polycycle a continuous curve γ close to the polycycle. This curve is transversal to the vector field (1.4) and the sense of the vector field (1.4) is toward the exterior of the region contained by γ. If $\alpha_0 < \alpha \ll 1$ and by the Poincaré–Bendixon theorem, the heteroclinic proved in Theorem 1 is broken toward the interior generating a stable limit cycle (see figure 9). □

4. Computer simulations

In this section we show a numeric simulation of (1.4) made with the Runge Kutta package of the *Mathematica* software. The values of the parameters are given by $\alpha = 0.345$;

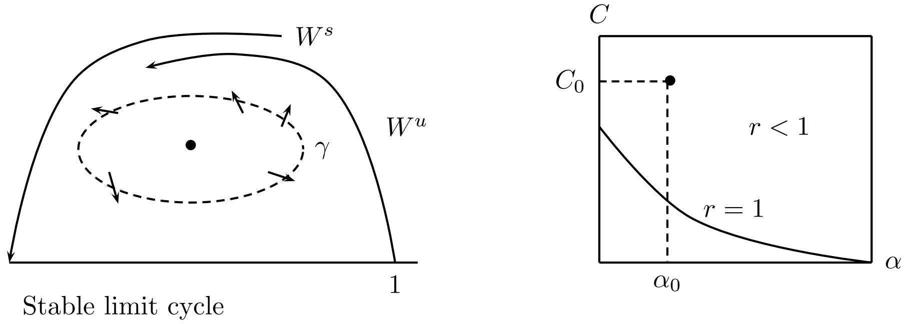

Figure 9.

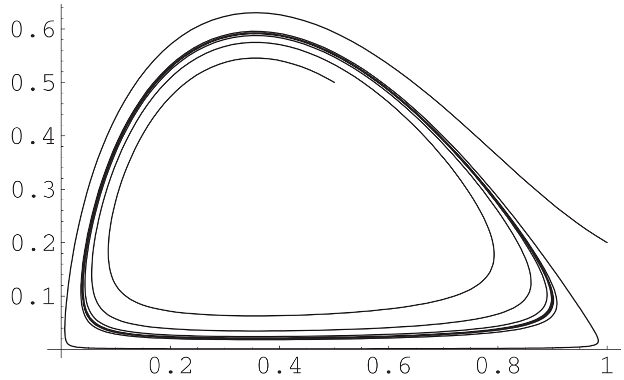

Figure 10.

$C = 0.7$; $B = 2$ and the initial conditions for two orbits $x(0) = 1$, $y(0) = 0.2$ and $x(0) = 0.5$, $y(0) = 0.5$.

In figure 10, we show the stable limit cycle generated by the break of the heteroclinic that joins the stable manifold of the singularities $(0, 0)$ and the unstable manifold of the singularity $(1, 0)$.

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