Proc. Indian Acad. Sci. (Math. Sci.) Vol. 117, No. 2, May 2007, pp. 197–203.

Some further remarks on good sets

K GOWRI NAVADA

Department of Mathematics, Periyar University, Salem 636 011, India
E-mail: gnavada@yahoo.com

MS received 24 January 2006; revised 19 August 2006

Abstract. We show that in n-fold cartesian product, $n \geq 4$, a related component need not be a full component. We also prove that when $n \geq 4$, uniform boundedness of lengths of geodesics is not a necessary condition for boundedness of solutions of (1) for bounded function f.

Keywords. Good set; full set; full component; related component; geodesic; boundary of a good set.

1. Introduction and preliminaries

The purpose of this note is to answer two questions about good sets raised in [4] and [5].

Let $X_1, X_2, \ldots, X_n$ be nonempty sets and let $\Omega = X_1 \times X_2 \times \cdots \times X_n$ be their cartesian product. We will write $\vec{x}$ to denote a point $(x_1, x_2, \ldots, x_n) \in \Omega$. For each $1 \leq i \leq n$, Π_i denotes the canonical projection of Ω onto X_i.

A subset $S \subset \Omega$ is said to be *good*, if every complex-valued function f on S is of the form:

$$f(x_1, x_2, \ldots, x_n) = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n), \ \ (x_1, x_2, \ldots, x_n) \in S, \tag{1}$$

for suitable functions $u_1, u_2, \ldots, u_n$ on $X_1, X_2, \ldots, X_n$ respectively ([4], p. 181).

For a good set S, a subset $B \subset \bigcup_{i=1}^{n} \Pi_i S$ is said to be a *boundary set of* S, if for any complex-valued function U on B and for any $f\colon S \longrightarrow \mathbb{C}$, eq. (1) subject to

$$u_i|_{B \cap \Pi_i S} = U|_{B \cap \Pi_i S}, \quad 1 \leq i \leq n,$$

admits a unique solution. For a good set there always exists a boundary set ([4], p. 187).

A subset $S \subset \Omega$ is said to be *full*, if S is a maximal good set in $\Pi_1 S \times \Pi_2 S \times \cdots \times \Pi_n S$.

A set $S \subset \Omega$ is full if and only if it has a boundary consisting of $n - 1$ points ([4], Theorem 3, page 185).

If a set S is good, maximal full subsets of S form a partition of S. They are called *full components* of S ([4], p. 183).

Two points $\vec{x}, \vec{y}$ in a good set S are said to be *related*, denoted by $\vec{x} R \vec{y}$, if there exists a finite subset of S which is full and contains both $\vec{x}$ and $\vec{y}$. R is an equivalence relation, whose equivalence classes are called *related components* of S. The related components of S are full subsets of S (ref. [4]).

2. Example of a full set which is not related

For a good set in two dimensions, full components are same as related components and are called *linked components* ([3], p. 60). In p. 190 of [4], the question whether full components are the same as related components for $n \geq 3$, is raised. Theorem 1 of [2] answers the question partially, where it is proved that a full set with finitely many related components is itself related.

Here we prove that when the dimension n is ≥ 4, a full component need not be a related component, by giving an example of a full set with infinitely many related components.

Our example, in a four-fold cartesian product, will consist of countable number of points $S = \{\vec{y_1}, \vec{y_2}, \vec{y_3}, \dots\}$ such that for each n the subset S_n of first n points of this set will be a good set and have a boundary consisting of four or five points depending on whether n is even or odd. Moreover, any boundary of S_n will necessarily contain two or one point from the coordinates of $\vec{y_n}$, depending on whether n is odd or even, so that eventually these points of the boundary disappear and S will have boundary with only three points. So S will be full. On the other hand, S will have related components consisting of single points.

Let $\{x_1, x_2, x_3, x_4, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n, \dots\}$ be a set of distinct symbols. The jth term of this sequence will be called the jth symbol. Then x_i will be ith symbol, $1 \leq i \leq 4$, while α_j will be the $(j+4)$th symbol for $j \geq 1$.

The countable infinite set S in four dimensions is defined as $S = \{\vec{y_1}, \vec{y_2}, \vec{y_3}, \dots\}$, where

$$\vec{y_1} = (x_1, x_2, x_3, x_4),$$

$$\vec{y_2} = (x_1, x_2, \alpha_1, \alpha_2),$$

$$\vec{y_3} = (\alpha_3, \alpha_4, x_3, \alpha_2),$$

$$\vec{y_4} = (\alpha_3, \alpha_4, \alpha_1, x_4),$$

$$\vec{y_5} = (x_1, \alpha_4, \alpha_5, \alpha_6),$$

$$\vec{y_6} = (\alpha_3, x_2, \alpha_5, \alpha_6),$$

and so on. In general, for $n \geq 2$,

$$\vec{y}_{4n-3} = (x_1, \alpha_{4n-4}, \alpha_{4n-3}, \alpha_{4n-2}), \quad \vec{y}_{4n-2} = (\alpha_{4n-5}, x_2, \alpha_{4n-3}, \alpha_{4n-2}),$$

$$\vec{y}_{4n-1} = (\alpha_{4n-1}, \alpha_{4n}, x_3, \alpha_{4n-2}), \quad \vec{y}_{4n} = (\alpha_{4n-1}, \alpha_{4n}, \alpha_{4n-3}, x_4).$$

We prove that this set is full and singletons are its related components.

Let $S_n = \{\vec{y_1}, \vec{y_2}, \dots, \vec{y_n}\}$. Consider the matrix M_n corresponding to the set S_n, called the *matrix* of S_n (p. 58 of [3]), whose rows correspond to the points $\vec{y_i}$, $1 \leq i \leq n$, and whose columns correspond to the symbols occurring in the points of S_n. The ijth entry of this matrix is 1, if the jth symbol occurs in the point $\vec{y_i}$. Otherwise, the ijth entry is 0.

If n is odd, say $n = 2m - 1$, then M_{2m-1} consists of $2m - 1$ rows and $2m + 4$ columns corresponding to the symbols $\{x_1, x_2, x_3, x_4, \alpha_1, \alpha_2, \dots, \alpha_{2m-1}, \alpha_{2m}\}$. If n is even, $n = 2m$, then there are $2m$ rows and $2m+4$ columns corresponding to $\{x_1, x_2, x_3, x_4, \alpha_1, \alpha_2, \dots, \alpha_{2m-1}, \alpha_{2m}\}$ in M_{2m}. The matrix of S is defined similarly.

We will prove by induction that for each n, S_{2n} is a good set and has boundary $\{x_1, x_2, x_3, \alpha_{2n}\}$ (or $\{x_1, x_2, x_3, \alpha_{2n-1}\}$). Clearly the statement holds for $n = 1$. Assume that the statement holds for $n = m - 1$. We have to show that the statement holds for $n = m$.

Since S_{2m-2} is good and has a boundary consisting of $\{x_1, x_2, x_3, \alpha_{2m-2}\}$ it is clear that S_{2m-1} is also good and has a boundary consisting of $\{x_1, x_2, x_3, \alpha_{2m-1}, \alpha_{2m}\}$. (Note that $\vec{y}_{2m-1}$ has two new coordinates α_{2m-1} and α_{2m} not occurring in $\vec{y}_1, \ldots, \vec{y}_{2m-2}$). For any f on S_{2m-1}, a solution u_1, u_2, u_3, u_4 satisfying

$$f(z_1, z_2, z_3, z_4) = u_1(z_1) + u_2(z_2) + u_3(z_3) + u_4(z_4),$$
$$(z_1, z_2, z_3, z_4) \in S_{2m-1},$$

is uniquely determined once we fix the values of u_i, $1 \le i \le 4$, on the boundary points.

We drop the columns corresponding to the symbols $x_1, x_2, x_3, \alpha_{2m-1}, \alpha_{2m}$ from M_{2m-1} and get an invertible $(2m-1) \times (2m-1)$-matrix N_{2m-1} with entries zeros and ones given below:

$$N_{2m-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & . & . & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & . & . & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & . & . & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ . & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ . & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

To show that S_{2m} is good and has a boundary consisting of $\{x_1, x_2, x_3, \alpha_{2m}\}$, consider the matrix of S_{2m}.

If we drop from the matrix M_{2m} the columns corresponding to $x_1, x_2, x_3, \alpha_{2m}$, we get a $2m \times 2m$-matrix given below.

$$N_{2m} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & . & . & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & . & . & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & . & . & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & 0 & . & . & . & 0 & 1 & 1 & 1 & 0 \\ . & 0 & . & . & . & 1 & 0 & 1 & 1 & 0 \\ . & 0 & . & . & . & 0 & 0 & 0 & 1 & 1 \\ . & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Its initial $(2m-1) \times (2m-1)$ matrix is the matrix N_{2m-1} obtained above which is invertible, and hence has its rows linearly independent. It is clear from this that rows of

N_{2m} are also linearly independent, so that N_{2m} is invertible. This proves that S_{2m} is good and $\{x_1, x_2, x_3, \alpha_{2m}\}$ is its boundary. We can see similarly that $\{x_1, x_2, x_3, \alpha_{2m-1}\}$ is also a boundary for S_{2m}.

We now prove that $\{x_1, x_2, x_3, \alpha_{2m}\}$ (or $\{x_1, x_2, x_3, \alpha_{2m-1}\}$) is not a boundary of S_{2n} for any $m < n$. Indeed, if $\{x_1, x_2, x_3, \alpha_{2m}\}$ is a boundary of S_{2n} for some $m < n$, then for any f on S_{2m+2} there is a solution u_1, u_2, u_3, u_4 of

$$f(z_1, z_2, z_3, z_4) = u_1(z_1) + u_2(z_2) + u_3(z_3) + u_4(z_4),$$
$$(z_1, z_2, z_3, z_4) \in S_{2m+2},$$

where values of u_1, u_2, u_3 and an appropriate u_i are preassigned on $x_1, x_2, x_3, \alpha_{2m}$ respectively.

Since $\{x_1, x_2, x_3, \alpha_{2m}\}$ is a boundary of S_{2m}, $u_4(x_4)$ and the value of appropriate u_i is determined on α_{2m-1} by the values of the function on the points of S_{2m}.

Now if $2m \equiv 0 (\text{mod } 4)$, then

$$f(\vec{y}_{2m+1}) = u_1(x_1) + u_2(\alpha_{2m}) + u_3(\alpha_{2m+1}) + u_4(\alpha_{2m+2}),$$
$$f(\vec{y}_{2m+2}) = u_1(\alpha_{2m-1}) + u_2(x_2) + u_3(\alpha_{2m+1}) + u_4(\alpha_{2m+2}),$$

which clearly do not hold together if

$$f(\vec{y}_{2m+2}) \neq f(\vec{y}_{2m+1}) - u_1(x_1) - u_2(\alpha_{2m}) + u_1(\alpha_{2m-1}) + u_2(x_2).$$

The case $2m \equiv 2 (\text{mod } 4)$ can be treated similarly and we see that the set $\{x_1, x_2, x_3, \alpha_{2m}\}$ cannot be a boundary of S_{2n} for $n > m$. The case of $\{x_1, x_2, x_3, \alpha_{2m-1}\}$ is similar. The set S is good because every finite subset of it is good. To show that it is full it is enough to show that any boundary B of S consists of three points. Take a function f on S and fix the values of u_i on $\Pi_i S \cap B$, $1 \leq i \leq 4$. Then there exists a unique solution u_i on $\Pi_i S$, $i = 1, 2, 3, 4$ of (1). This gives a solution of (1) on S_{2n} with $f = f|_{S_{2n}}$, for any $n \in \mathbb{Z}_+$. Fix values of u_i on $B \cap \Pi_i S_{2n}$, $1 \leq i \leq 4$. Then there is a solution of (1) with $f = f|_{S_{2n}}$, and u_i prescribed on $B \cap \Pi_i S_{2n}$, $1 \leq i \leq 4$, as above. If $|B| > 3$, then for n large enough we get $|B \cap (\cup_{i=1}^4 \Pi_i S_{2n})| \geq 4$, and this will give a boundary for S_{2n} with, either more than four points or, a four-point boundary which does not contain α_{2n-1} and α_{2n}. But this is not possible. This shows that $|B| = 3$ which means S is full.

No finite subset K of S, other than singleton, is full. We prove this by showing that there is a point $\vec{y} = (y_1, y_2, y_3, y_4) \notin K$ with $y_i \in \Pi_i K$ for $i = 1, 2, 3, 4$ such that $K \cup \{\vec{y}\}$ is also good. Any two-point subset of S is not full as any two points of S differ in at least two coordinates. So we can assume that $|K| \geq 3$. Let n be the least integer such that $K \subseteq S_{2n}$. Then either $\vec{y}_{2n-1}$ or $\vec{y}_{2n} \in K$. If $\vec{y}_{2n} \in K$, then let $\vec{z} = \vec{y}_{2n}$; otherwise let $\vec{z} = \vec{y}_{2n-1}$. Then α_{2n-1} and α_{2n} are the coordinates of $\vec{z}$.

Let $\vec{y}$ be same as $\vec{z}$ except that the coordinate α_{2n} is replaced by some other corresponding coordinate of a point in K. Then $\vec{y} \notin K$. Further $K \cup \{\vec{y}\}$ is good: for which it is enough to show that $S_{2n} \cup \{\vec{y}\}$ is good.

Consider the $(2n + 1) \times (2n + 1)$-matrix whose columns correspond to the symbols $\{x_4, \alpha_1, \ldots, \alpha_{2n-1}, \alpha_{2n}\}$ and rows correspond to the points of $\{\vec{y}_1, \ldots, \vec{y}_{2n}, \vec{y}\}$.

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & . & . & . & . & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & . & . & . & . & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & . & . & . & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & . & . & . & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & . & . & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & . & 0 \\
. & . & . & . & . & . & . & . & . & . & . \\
. & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
. & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
. & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
. & . & . & . & . & . & . & . & . & 1 & 0
\end{bmatrix},$$

where the last row has a 1 at the column corresponding to α_{2n-1} and a 0 in the column corresponding to α_{2n}. The first $2n$ rows of this matrix are linearly independent as the set $S_{2n} = \{\vec{y}_1, \ldots, \vec{y}_{2n}\}$ is good. Further, it is clear that the last row is not a linear combination of the first $2n$ rows. Therefore $S_{2n} \cup \{\vec{y}\}$ is good. This proves that the related components of S are singletons.

3. A set with unbounded geodesic length which has bounded solutions for bounded functions *f*

A modification of this example can be used to answer another question.

Let S be a related set in n-fold cartesian product $X_1 \times X_2 \times \cdots \times X_n$.

Given two points $\vec{x}, \vec{y} \in S$ the smallest related subset of S containing $\vec{x}$ and $\vec{y}$ is called the *geodesic* between $\vec{x}$ and $\vec{y}$ ([4], p. 190). In [5], it is shown that if there is an upper bound for lengths of geodesics in S, then the solutions $u_1, \ldots, u_n$ of (1) are bounded whenever f is bounded. Further, the author surmises that the converse may be true. Namely, that boundedness of geodesic lengths is a necessary condition for bounded solutions $u_1, \ldots, u_n$, for bounded f. This is true for $n = 2$. However, for $n \geq 4$ we prove that this is not a necessary condition.

Consider the set $S_{4n} \cup \{\vec{z}\}$ where the point $\vec{z} = (\alpha_{4n-1}, x_2, \alpha_{4n-3}, x_4)$. We prove that $S_{4n} \cup \{\vec{z}\}$ is full. First we note that the infinite matrix with rows corresponding to $\{\vec{y}_2, \vec{y}_3, \ldots\}$ and columns corresponding to $\{\alpha_1, \alpha_2, \ldots\}$ has an inverse given by

$$\begin{bmatrix}
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & . & . & . & . & . & . \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & . & . & . & . & . & . \\
-\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & . & . & . & . \\
-\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & . & . & . & . \\
\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & . & . \\
\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & . & . \\
-\frac{1}{16} & \frac{1}{16} & \frac{1}{16} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & . \\
-\frac{1}{16} & \frac{1}{16} & \frac{1}{16} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & . \\
. & . & . & . & . & . & . & . & . & . & . & . \\
\frac{1}{2^{2n-1}} & -\frac{1}{2^{2n-1}} & -\frac{1}{2^{2n-1}} & \frac{1}{2^{2n-2}} & . & . & . & . & . & -\frac{1}{2} & \frac{1}{2} & . \\
\frac{1}{2^{2n-1}} & -\frac{1}{2^{2n-1}} & -\frac{1}{2^{2n-1}} & \frac{1}{2^{2n-2}} & . & . & . & . & . & \frac{1}{2} & -\frac{1}{2} & . \\
. & . & . & . & . & . & . & . & . & . & . & .
\end{bmatrix}.$$

Now consider the $4n \times 4n$-matrix A_{4n} with rows corresponding to the points $\{\vec{y}_2, \vec{y}_3, \dots, \vec{y_{4n}}, \vec{z}\}$ and columns corresponding to $\{\alpha_1, \dots, \alpha_{4n}\}$:

$$A_{4n} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & . & . & . & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & . & . & . & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & . & . & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & . & . & . & . & . & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & 1 & 0 & 1 & 0 \end{bmatrix}.$$

This matrix is invertible and the inverse is

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & . & . & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & . & . & 0 \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & . & . & 0 \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & . & . & 0 \\ \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{2} & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ \frac{1}{2^{2n-1}} & -\frac{1}{2^{2n-1}} & -\frac{1}{2^{2n-1}} & \frac{1}{2^{2n-2}} & . & . & . & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2^{2n-1}} & -\frac{1}{2^{2n-1}} & -\frac{1}{2^{2n-1}} & \frac{1}{2^{2n-2}} & . & . & . & \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2^{2n-1}} & \frac{1}{2^{2n-1}} & \frac{1}{2^{2n-1}} & -\frac{1}{2^{2n-2}} & . & . & . & \frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 & . & . & 0 & 1 & -1 \end{bmatrix}.$$

This proves that $S_{4n} \cup \{\vec{z}\}$ is good and $\{x_1, x_2, x_3\}$ is its boundary. So this set is also full. Note that the sums of the absolute values of entries in a row is less than or equal to 3, for any row, independent of n.

We prove that the geodesic between $\vec{y}_1$ and $\vec{y}_{4n}$ in $S_{4n} \cup \{\vec{z}\}$ is the whole set $S_{4n} \cup \{\vec{z}\}$. If $K \subset S_{4n} \cup \{\vec{z}\}$ is a full set containing $\{\vec{y}_1, \vec{y}_{4n}\}$ then we have to show that $K = S_{4n} \cup \{\vec{z}\}$.

First we note that $\vec{z} \in K$. If $K \neq S_{4n} \cup \{\vec{z}\}$, let i be the number of points of S_{4n} which are not in K. These i points should have at least i symbols occurring in them which are not in $\cup_{i=1}^{4} \Pi_i K$ because K is full and, when we add these i points to K the set remains good. These symbols are from $\{\alpha_1, \dots, \alpha_{4n-2}\}$ because $x_1, x_2, x_3, x_4, \alpha_{4n-1}$ and α_{4n} occur as co-ordinates in the points of K. Let $\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_i}$ where $j_1 < j_2 < \cdots < j_i$ be these symbols. If we prove that these symbols are used by at least $i+1$ points of S_{4n}, we get a contradiction because these $i+1$ points cannot be in K.

For this, we show that the i columns of the matrix A_{4n} corresponding to $\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_i}$ have nonzero entries in at least $i+1$ rows of A_{4n}. Let us first take the case when $\alpha_{j_s} \neq \alpha_1$

or α_2 for $s = 1, 2, \ldots, i$. Then these i columns contain exactly $3i$ nonzero entries in them. Any row of A_{4n} contains at most three nonzero entries in it. But observe that the row containing the last nonzero entry of the column corresponding to α_{j_i} has only one nonzero entry in these i columns. So we need at least $i + 1$ rows to cover all the nonzero entries of these i columns. Now assume that $\alpha_{j_s} = \alpha_1$ (or α_2) for some s. Then the total number of nonzero entries in these i columns is $3(i - 1) + 2$. As in the previous case, there is a row containing only one nonzero entry of these i columns. So again it is easy to see that we need at least $i + 1$ rows to cover all the nonzero entries in these i columns. In the last case, when $\alpha_{j_s} = \alpha_1$ and $\alpha_{j_t} = \alpha_2$ for some $1 \leq s,\ t \leq i$, the total number of nonzero entries in these columns is $3(i - 2) + 2 + 2$. But in this case the first row contains only two nonzero entries of these i columns. As before the row containing the last nonzero entry of the column corresponding to α_{j_i} has only one nonzero entry of these columns. So, again we need at least $i + 1$ rows to cover the nonzero entries of these columns. This contradiction proves that $K = S_{4n} \cup \{\vec{z}\}$.

Since the bound on the absolute row sums of A_{4n}^{-1} is independent of n, it is clear how to construct a full set S in which geodesic lengths are not bounded, but solutions $u_1, \ldots, u_n$ of (1) are bounded for bounded f.

A similar kind of construction is possible for the case $n = 3$ also and it will be communicated shortly.

Acknowledgement

The author would like to thank Prof. M G Nadkarni for fruitful discussions and encouragement and for suggesting the problem. The author also thanks IMSc for short visiting appointments which made this work possible.

References

[1] Cowsik R C, Kłopotowski A and Nadkarni M G, When is $f(x, y) = u(x) + v(y)$?, *Proc. Indian Acad. Sci. (Math. Sci.)* **109** (1999) 57–64

[2] Gowri Navada K, Some remarks on good sets, *Proc. Indian Acad. Sci. (Math. Sci.)* **114** (2004) 389–397

[3] Kłopotowski A, Nadkarni M G and Bhaskara Rao K P S, When is $f(x_1, x_2, \ldots, x_n) = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n)$?, *Proc. Indian Acad. Sci. (Math. Sci.)* **113** (2003) 77–86

[4] Kłopotowski A, Nadkarni M G and Bhaskara Rao K P S, Geometry of good sets in n-fold cartesian products, *Proc. Indian Acad. Sci. (Math. Sci.)* **114** (2004) 181–197

[5] Nadkarni M G, Kolmogorov's superposition theorem and sums of algebras, *J. Anal.* **12** (2004) 21–67