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Weighted composition operators on weighted Bergman spaces of bounded symmetric domains

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Abstract. In this paper, we study the weighted compositon operators on weighted Bergman spaces of bounded symmetric domains. The necessary and sufficient conditions for a weighted composition operator $W_{\varphi,\psi}$ to be bounded and compact are studied by using the Carleson measure techniques. In the last section, we study the Schatten p-class weighted composition operators.

Keywords. Bergman spaces; bounded symmetric domain; boundedness; Carleson measure; compactness; Schatten p-class operator.

1. Introduction

Let Ω be a bounded symmetric domain in $\mathbf{C}^n$ with Bergman kernel $K(z,\omega)$. We assume that Ω is in its standard representation and the volume measure $\mathrm{d}V$ of Ω is normalised so that $K(z,0)=K(0,\omega)$, for all z and ω in Ω. By Theorem 5.7 of [11] and using the polar coordinates representation, there exists a positive number ε_Ω such that for $\lambda < \varepsilon_\Omega$, we have

$$C_\lambda = \int_\Omega K(z,z)^\lambda \mathrm{d}V(z) < \infty.$$

For each $\lambda < \varepsilon_\Omega$, define $\mathrm{d}V_\lambda(z) = C_\lambda^{-1} K(z,z)^\lambda \mathrm{d}V(z)$. Then $\{\mathrm{d}V_\lambda\}$ defines a weighted family of probability measures on Ω. Also, throughout the paper $\lambda < \varepsilon_\Omega$ is fixed. We define the weighted Bergman spaces $A_\lambda^p(\Omega, \mathrm{d}V_\lambda)$, on Ω, as the set of all holomorphic functions f on Ω so that

$$\|f\|_{A_\lambda^p} = \left(\int_\Omega |f|^p \mathrm{d}V_\lambda\right)^{\frac{1}{p}} < \infty.$$

Note that $A_\lambda^p(\Omega, \mathrm{d}V_\lambda)$ is a closed subspace of $L^p(\Omega, \mathrm{d}V_\lambda)$. For $\lambda = 0$, $A_\lambda^p(\Omega, \mathrm{d}V_\lambda)$ is just the usual Bergman space. For $p=2$, there is an orthogonal projection P_λ from $L^2(\Omega, \mathrm{d}V_\lambda)$ onto $A_\lambda^2(\Omega, \mathrm{d}V_\lambda)$ given by

$$P_\lambda f(z) = \int_\Omega K_\lambda(z,\omega) f(\omega) \mathrm{d}V_\lambda(\omega),$$

where $K_\lambda(z,\omega) = K(z,\omega)^{1-\lambda}$ is the reproducing kernel for $A_\lambda^2(\Omega, \mathrm{d}V_\lambda)$.

Suppose, φ, ψ are holomorphic mappings defined on Ω such that $\varphi(\Omega) \subseteq \Omega$. Then the weighted composition operator $W_{\varphi,\psi}$ is defined as

$$W_{\varphi,\psi} f(z) = \psi(z) f(\varphi(z)), \text{ for all } f \text{ holomorphic in } \Omega, \text{ and } z \in \Omega.$$

For the study of weighted composition operators one can refer to [4, 6, 13, 15] and references therein. Recently, Smith [16] has made a nice connection between the Brennan's conjecture and weighted composition operators. He has shown that Brennan's conjecture is equivalent to the existence of self-maps of unit disk that make certain weighted composition operators compact.

We know that the bounded symmetric domain Ω in its standard representation with normalised volume measure, the kernel function $K(\cdot,\cdot)$ has the following special properties:

(1) $K(0, a) = 1 = K(a, 0)$.
(2) $K(z, a) \neq 0$ (for all z in Ω and a in $\overline{\Omega}$).
(3) $\lim_{a\to\partial\Omega} K(a, a) = \infty$.
(4) $K(z, a)^{-1}$ is a smooth function on $\mathbf{C}^n \times \mathbf{C}^n$. Here $\overline{\Omega}$ denotes the closure of Ω in $\mathbf{C}^n$ and $\partial\Omega$ is the topological boundary, see [8] and [10] for details.

For any a in Ω, let

$$k_a(z) = \frac{K(z, a)}{\sqrt{K(a, a)}}.$$

The $k_a's$ are called normalised reproducing kernels for $A^2(\Omega, \mathrm{d}V)$. They are unit vectors in $A^2(\Omega, \mathrm{d}V)$. Moreover, $k_a^{1-\lambda}$ is a unit vector of $A^2_\lambda(\Omega, \mathrm{d}V_\lambda)$ for any a in Ω.

Let μ be a finite complex Borel measure on Ω. Then the Berezin transform of measure μ, denoted by $\tilde{\mu}_\lambda$, is defined as

$$\tilde{\mu}_\lambda(z) = \int_\Omega |k_z(\omega)|^{2(1-\lambda)} \mathrm{d}\mu(\omega), \quad z \in \Omega.$$

We will denote by $\beta(z, \omega)$ the Bergman distance function on Ω. For $z \in \Omega$ and $r > 0$, let

$$E(z, r) = \{\omega \in \Omega \colon \beta(z, \omega) < r\}.$$

We denote by $|E(z, r)|$ the normalised volume of $E(z, r)$, that is,

$$|E(z, r)| = \int_{E(z,r)} \mathrm{d}V(\omega).$$

Given a finite complex Borel measure μ on Ω, we define a function $\hat{\mu}_r$ on Ω by

$$\hat{\mu}_r(z) = \frac{\mu(E(z, r))}{|E(z, r)|^{1-\lambda}}, \quad z \in \Omega.$$

Theorem 1.1 [17]. *Take $1 \le p < \infty$. Let μ be a finite positive Borel measure on Ω. Then the following conditions are equivalent*:

(1) *The inclusion map $i \colon A^p_\lambda(\Omega, \mathrm{d}V_\lambda) \to L^p(\Omega, \mathrm{d}V_\lambda)$ is bounded.*
(2) *The Berezin transform $\tilde{\mu}_\lambda$ is bounded.*

(3) $\hat{\mu}_r$ *is bounded on* Ω *for all* (*or some*) $r > 0$.
(4) $\{\hat{\mu}_r(a_n)\}$ *is a bounded sequence, where* $\{a_n\}$ *is some bounded sequence in* Ω *independent of* μ.

A positive Borel measure μ satisfying any one of the conditions of Theorem 1.1 is called a Carleson measure on the weighted Bergman space $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$

Theorem 1.2 [17]. *Take* $1 \leq p < \infty$. *Let* μ *be a finite positive Borel measure on* Ω. *Then the following conditions are equivalent*:

(1) *The inclusion map* $i\colon A^p_\lambda(\Omega, \mathrm{d}V_\lambda) \to L^p(\Omega, \mathrm{d}V_\lambda)$ *is compact.*
(2) *The Berezin transform* $\tilde{\mu}_\lambda(z) \to 0$ *as* $z \to \partial\Omega$.
(3) $\hat{\mu}_r(z) \to 0$ *as* $z \to \partial\Omega$ *for all* (*or some*) $r > 0$.
(4) $\hat{\mu}_r(a_n) \to 0$ *as* $n \to \infty$.

A positive Borel measure μ satisfying any one of the conditions of Theorem 1.2 is called a vanishing Carleson measure on the weighted Bergman space $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$.

A positive compact operator T on $A^2_\lambda(\Omega, \mathrm{d}V_\lambda)$ is in the trace class if

$$\mathrm{tr}(T) = \sum_{n=1}^{\infty} \langle Te_n, e_n\rangle < \infty,$$

for some (or all) orthonormal basis $\{e_n\}$ of $A^2_\lambda(\Omega, \mathrm{d}V_\lambda)$.

Take $1 \leq p < \infty$, and T be a compact linear operator on $A^2_\lambda(\Omega, \mathrm{d}V_\lambda)$. Then we say that T belongs to the Schatten p-class S_p if $(T^*T)^{p/2}$ is in the trace class. Also, the S_p norm of T is given as

$$\|T\|_{S_p} = [\mathrm{tr}((T^*T)^{p/2})]^{1/p}.$$

Moreover, S_p is a two-sided ideal of the full algebra $B(A^2_\lambda(\Omega, \mathrm{d}V_\lambda))$ of bounded linear operators on $A^2_\lambda(\Omega, \mathrm{d}V_\lambda)$.

2. Preliminaries

To make the paper self-contained we state the following lemmas.

Lemma 2.1 [12]. *The Bergman kernel* $K(z, \omega)$ *is conjugate symmetric. That is,*

$$K(z, \omega) = \overline{K(\omega, z)};$$

and it is the function of z *in* $A^2_\lambda(\Omega, \mathrm{d}V_\lambda)$.

Lemma 2.2 [17]. *Let* T *be a trace class operator or a positive operator on* $A^2_\lambda(\Omega, \mathrm{d}V_\lambda)$. *Then*

$$\mathrm{tr}(T) = \int_\Omega \langle TK_z^{1-\lambda}, K_z^{1-\lambda}\rangle_{A^2_\lambda} \mathrm{d}\sigma(z),$$

where $\mathrm{d}\sigma(z) = K(z, z)\mathrm{d}V(z)$.

Lemma 2.3 [9]. *Let* Ω *be a bounded symmetric domain in* $\mathbf{C}^n$ *and let* $W_{\varphi,\psi}$ *be a weighted composition operator on* $A^2_\lambda(\Omega, \mathrm{d}V_\lambda)$. *Then*

$$\|W^*_{\varphi,\psi}\|_{A^2_\lambda} = |\psi(z)| \left[\frac{K(\varphi(z), \varphi(z))}{K(z,z)}\right]^{\frac{(1-\lambda)}{2}}.$$

Lemma 2.4 [1]. *Suppose* T *is a positive operator on a Hilbert space* H *and* x *is a unit vector in* H. *Then*

(1) $\langle T^p x, x\rangle \geq \langle Tx, x\rangle^p$ for all $p \geq 1$.
(2) $\langle T^p x, x\rangle \leq \langle Tx, x\rangle^p$ for all $0 < p \leq 1$.

Lemma 2.5 (*Theorem* 18.11(*f*), *p.* 89 *of* [5]). *Suppose* T *is a bounded linear operator on a Hilbert space* H. *If* $T \in S_p$, *then*

$$\mathrm{tr}(|T|^p) = \mathrm{tr}(|T^*|^p).$$

Lemma 2.6 [17]. *Take* $1 \leq p < \infty$. *Suppose* $\mu \geq 0$ *is a finite Borel measure on* Ω. *Then* μ *is a Carleson measure on* $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$ *if and only if* $\mu(E(z,r))/|E(z,r)|^{1-\lambda}$ *is bounded on* Ω (*as a function of* z) *for all* (*or some*) $r > 0$. *Moreover*, *the following quantities are equivalent for any fixed* $r > 0$ *and* $p \geq 1$:

$$\|\hat{\mu}_r\|_\infty = \sup\left\{\frac{\mu(E(z,r))}{|E(z,r)|^{1-\lambda}} : z \in \Omega\right\}$$

and

$$\|\mu\|_p = \sup\left\{\frac{\int_\Omega |f(z)|^p \mathrm{d}\mu(z)}{\int_\Omega |f(z)|^p \mathrm{d}V_\lambda(z)} : f \in A^p_\lambda(\Omega, \mathrm{d}V_\lambda)\right\}.$$

Lemma 2.7 [2]. *For any* $r > 0$, *there exists a constant* C (*depending only on* r) *such that*

$$C^{-1} \leq |E(a,r)||k_a(z)|^2 \leq C,$$

for all $a \in \Omega$ *and* $z \in E(a,r)$.

Note that if we take $z = a$ *in the above estimate*, *then we get*

$$C^{-1} \leq |E(a,r)|K(a,a) \leq C,$$

for all $a \in \Omega$.

Lemma 2.8 [2]. *For any* $r > 0, s > 0, R > 0$, *there exists a constant* C (*depending on* r, s, R) *such that*

$$C^{-1} \leq \frac{|E(a,r)|}{|E(a,s)|} \leq C$$

for all a, b *in* Ω *with* $\beta(a,b) \leq R$.

Lemma 2.9 [3]. *For any* $r > 0$, *there exists a sequence* $\{a_n\}$ *in* Ω *satisfying the following two conditions*:

(1) $\Omega = \bigcup_{n=1}^{\infty} E(a_n, r)$;
(2) *There is a positive integer N such that each point z in Ω belongs to at most N of the sets $E(a_n, 2r)$.*

Lemma 2.10 [3]. *For any $r > 0$ and $p \geq 1$, there exists a constant C (depending only on r) such that*

$$|f(a)|^p \leq \frac{C}{|E(a,r)|} \int_{E(a,r)} |f(z)|^p \mathrm{d}V(z),$$

for all f holomorphic and a in Ω.

3. Bounded and compact weighted composition operators

By using [7], we can prove the following lemma.

Lemma 3.1. *Take $1 \leq p < \infty$. Let Ω be a bounded symmetric domain in $\mathbf{C}^n$, and $\lambda < \varepsilon_\Omega$. Suppose φ, ψ be holomorphic functions defined on Ω such that $\varphi(\Omega) \subseteq \Omega$. Also, suppose that the weighted composition operator $W_{\varphi,\psi}$ is bounded on $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$. Then $W_{\varphi,\psi}$ is compact on $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$ if and only if for any norm bounded sequence $\{f_n\}$ in $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$ such that $f_n \to 0$ uniformly on compact subsets of Ω, then we have $\|W_{\varphi,\psi} f_n\|_{A^p_\lambda} \to 0$.*

Lemma 3.2. *Suppose $\{f_n\}$ is a sequence in $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$ such that $f_n \to 0$ weakly in $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$. Then $\{f_n\}$ is a norm bounded sequence and $f_n \to 0$ uniformly on each compact subsets of Ω.*

Let φ be a holomorphic function defined on Ω such that $\varphi(\Omega) \subseteq \Omega$. Suppose $\psi \in A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$ and $\lambda < \varepsilon_\Omega$. Then the nonnegative measure $\mu_{\varphi,\psi,p}$ is defined as

$$\mu_{\varphi,\psi,p}(E) = \int_{\varphi^{-1}(E)} |\psi|^p \mathrm{d}V_\lambda,$$

where E is a measurable subset of Ω.

Using Lemma 2.1 of [4], we can prove the following result.

Lemma 3.3. *Take $1 \leq p < \infty$. Let Ω be a bounded symmetric domain in $\mathbf{C}^n$, and $\lambda < \varepsilon_\Omega$. Let φ be a holomorphic mapping from Ω into Ω and $\psi \in A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$. Then*

$$\int_\Omega g \mathrm{d}\mu_{\varphi,\psi,p} = \int_\Omega |\psi|^p (g \circ \varphi) \mathrm{d}V_\lambda,$$

where g is an arbitrary measurable positive function on Ω.

Theorem 3.4. *Take $1 \leq p < \infty$. Let Ω be a bounded symmetric domain in $\mathbf{C}^n$, and $\lambda < \varepsilon_\Omega$. Suppose φ, ψ are holomorphic functions defined on Ω such that $\varphi(\Omega) \subseteq \Omega$. Then the weighted composition operator $W_{\varphi,\psi}$ is bounded on $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$ if and only if the measure $\mu_{\varphi,\psi,p}$ is a Carleson measure.*

Proof. Suppose $W_{\varphi,\psi}$ is bounded on $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$. Then there exists a constant $M > 0$ such that

$$\|W_{\varphi,\psi} f\|_{A^p_\lambda} \leq M\|f\|_{A^p_\lambda},$$

for all $f \in A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$.

Take

$$g(\omega) = |k_z(\omega)|^{\frac{2(1-\lambda)}{p}}.$$

From Lemma 2.1 and using the fact that $K(\omega, z) \neq 0$ for any z and ω in Ω, we get that $g \in A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$. Thus, we have

$$\begin{aligned}\int_\Omega |k_z(\omega)|^{2(1-\lambda)}\mathrm{d}\mu_{\varphi,\psi,p}(\omega) &= \int_\Omega |g(\omega)|^p \mathrm{d}\mu_{\varphi,\psi,p}(\omega)\\ &\leq \int_\Omega |\psi(\omega)|^p|g(\varphi(\omega))|^p \mathrm{d}V_\lambda(\omega)\\ &= \|W_{\varphi,\psi} g\|^p_{A^p_\lambda} < \infty.\end{aligned}$$

By Lemma 2.7, there exists a constant $C > 0$ (depending only on r) such that

$$\begin{aligned}\frac{\mu_{\varphi,\psi,p}(E(z,r))}{|E(z,r)|^{1-\lambda}} &= \frac{1}{|E(z,r)|^{1-\lambda}}\int_{E(z,r)} \mathrm{d}\mu_{\varphi,\psi,p}(\omega)\\ &\leq C \int_{E(z,r)} |k_z(\omega)|^{2(1-\lambda)}\mathrm{d}\mu_{\varphi,\psi,p}(\omega) < \infty,\end{aligned}$$

for any $z \in \Omega$.

Thus, by using Lemma 2.6, we get that $\mu_{\varphi,\psi,p}$ is a Carleson measure on $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$.

Conversely, suppose $\mu_{\varphi,\psi,p}$ is a Carleson measure on $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$. Then by Theorem 1.1, there exists a constant $M > 0$ such that

$$\int_\Omega |f(z)|^p \mathrm{d}\mu_{\varphi,\psi,p}(z) \leq M^p\|f\|^p_{A^p_\lambda},$$

for all $f \in A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$.

Hence, using Lemma 3.3, we have

$$\begin{aligned}\|W_{\varphi,\psi} f\|_{A^p_\lambda} &= \int_\Omega |\psi(z)|^p|f(\varphi(z))|^p \mathrm{d}V_\lambda(z)\\ &= \int_\Omega |f(\omega)|^p \mathrm{d}\mu_{\varphi,\psi,p}(\omega)\\ &\leq M^p \int_\Omega |f(\omega)|^p \mathrm{d}V_\lambda(\omega) < \infty,\end{aligned}$$

for all $f \in A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$.

Hence, $W_{\varphi,\psi}$ is a bounded operator on $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$.

Theorem 3.5. *Take $1 \leq p < \infty$. Let Ω be a bounded symmetric domain in $\mathbf{C}^n$, and $\lambda < \varepsilon_\Omega$. Suppose φ, ψ are holomorphic functions defined on Ω such that $\varphi(\Omega) \subseteq \Omega$. Then the weighted composition operator $W_{\varphi,\psi}$ is compact on $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$ if and only if the measure $\mu_{\varphi,\psi,p}$ is a vanishing Carleson measure.*

Proof. Suppose $W_{\varphi,\psi}$ is compact on $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$. Since, $k_z^{1-\lambda} \to 0$ weakly in $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$ as $z \to \partial\Omega$, by Lemmas 3.1 and 3.2, we have

$$\| W_{\varphi,\psi} k_z^{1-\lambda} \|_{A^p_\lambda} \to 0 \quad \text{as} \quad z \to \partial\Omega. \tag{3.1}$$

Again, by using Lemma 2.7, we get a constant $C > 0$ (depending only on r) such that

$$\begin{aligned}
\frac{\mu_{\varphi,\psi,p}(E(z,r))}{|E(z,r)|^{1-\lambda}} &= \frac{1}{|E(z,r)|^{1-\lambda}} \int_{E(z,r)} \mathrm{d}\mu_{\varphi,\psi,p}(\omega) \\
&\leq C \int_{E(z,r)} |k_z(\omega)|^{2(1-\lambda)} \mathrm{d}\mu_{\varphi,\psi,p}(\omega) \\
&\leq C \int_{E(z,r)} |\psi(\omega)|^2 |k_z(\varphi(\omega))|^{2(1-\lambda)} \mathrm{d}\mu_{\varphi,\psi,p}(\omega) \\
&\leq C \| W_{\varphi,\psi} k_z^{1-\lambda} \|_{A^2_\lambda}.
\end{aligned}$$

Thus, condition (3.1) implies that $\mu_{\varphi,\psi,p}$ is a vanishing Carleson measure on $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$.

Conversely, suppose that $\mu_{\varphi,\psi,p}$ is a vanishing Carleson measure on $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$. Then, by Theorem 1.2, we have

$$\lim_{z \to \partial\Omega} \frac{\mu_{\varphi,\psi,p}(E(z,r))}{|E(z,r)|^{1-\lambda}} = 0 \tag{3.2}$$

for all (or some) $r > 0$.

So, from Theorem 3.4, we conclude that $W_{\varphi,\psi}$ is bounded on $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$. We will prove that $W_{\varphi,\psi}$ is a compact operator.

Let $\{f_n\}$ be a sequence in $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$ and $f_n \to 0$ weakly. By Lemma 3.2, f_n is a norm bounded sequence and $f_n \to 0$ uniformly on each compact subsets of Ω.

By Lemmas 2.7 and 2.10, there exists a constant $C > 0$ (depending only on r) such that

$$\begin{aligned}
|f(z)|^p &\leq \frac{C}{|E(z,r)|} \int_{E(z,r)} |f(\omega)|^p \mathrm{d}V(\omega) \\
&= \frac{C C_\lambda}{|E(z,r)|} \int_{E(z,r)} |f(\omega)|^p \frac{\mathrm{d}V_\lambda(\omega)}{K(\omega,\omega)^\lambda} \\
&\leq \frac{C'}{|E(z,r)|} \int_{E(z,r)} |f(\omega)|^p \frac{\mathrm{d}V_\lambda(\omega)}{|E(\omega,r)|^{-\lambda}} \\
&\leq C'' \frac{1}{|E(z,r)|^{1-\lambda}} \int_{E(z,r)} |f(\omega)|^p \mathrm{d}V_\lambda(\omega),
\end{aligned}$$

for all $z \in \Omega$, where the last inequality is obtained by using Lemma 2.8. By combining the above inequality with Lemma 2.8, we get

$$\begin{aligned}\sup\{|f(z)|^p : z \in E(a,r)\} &\leq \sup\left\{\frac{C_1}{|E(z,r)|^{1-\lambda}}\int_{E(a,2r)}|f(\omega)|^p \mathrm{d}V_\lambda(\omega) : z \in E(a,r)\right\}\\ &\leq \frac{C_2}{|E(a,r)|^{1-\lambda}}\int_{E(a,2r)}|f(\omega)|^p \mathrm{d}V_\lambda(\omega),\end{aligned}$$

where C_1 and C_2 are constants depending only on λ and r. Hence, we have

$$\begin{aligned}\|W_{\varphi,\psi} f_n\|_{A^p_\lambda} &= \int_\Omega |\psi(z)|^p |f_n(\varphi(z))|^p \mathrm{d}V_\lambda(\omega)\\ &= \int_\Omega |f_n(\omega)|^p \mathrm{d}\mu_{\varphi,\psi,p}\\ &\leq \sum_{i=1}^{\infty}\int_{E(a_i,r)} |f_n(\omega)|^p \mathrm{d}\mu_{\varphi,\psi,p}\\ &\leq \sum_{i=1}^{\infty}\mu_{\varphi,\psi,p}((E(a_i,r))\sup\{|f_n(\omega)|^p : \omega \in E(a_i,r)\}\\ &\leq C\sum_{i=1}^{\infty}\frac{\mu_{\varphi,\psi,p}((E(a_i,r))}{|E(a_i,r)|^{1-\lambda}}\int_{E(a_i,2r)}|f_n(\omega)|^p \mathrm{d}V_\lambda(\omega),\end{aligned}$$

where the sequence $\{a_n\}$ is the same as in Lemma 2.9.

From condition (3.2), for every $\epsilon > 0$, there exists a $\delta > 0$ $(0 < \delta < 1)$ such that

$$\frac{\mu_{\varphi,\psi,p}(E(z,r))}{|E(z,r)|^{1-\lambda}} < \epsilon,$$

whenever $d(z, \partial\Omega) < \delta$.

Take

$$\Omega_1 = \{\omega \in \Omega : d(z, \partial\Omega) \geq \delta, \quad \beta(z,\omega) \leq r\}.$$

Then, Ω_1 is a compact subset of Ω (as $0 < 2r < \frac{\delta}{2}$). Thus, we have

$$\begin{aligned}\int_\Omega |W_{\varphi,\psi} f_n(z)|^p \mathrm{d}V_\lambda(z) &= \int_\Omega |f_n(\omega)|^p \mathrm{d}\mu_{\varphi,\psi,p}(\omega)\\ &\leq C\sum_{i=1}^{\infty}\frac{\mu_{\varphi,\psi,p}((E(a_i,r))}{|E(a_i,r)|^{1-\lambda}}\int_{E(a_i,2r)}|f_n(\omega)|^p \mathrm{d}V_\lambda(\omega)\\ &\leq C\sum_{d(a,\partial\Omega)\geq\delta}\frac{\mu_{\varphi,\psi,p}((E(a_i,r))}{|E(a_i,r)|^{1-\lambda}}\int_{E(a_i,2r)}|f_n(\omega)|^p \mathrm{d}V_\lambda(\omega)\\ &\quad + C\sum_{d(a,\partial\Omega)<\delta}\frac{\mu_{\varphi,\psi,p}((E(a_i,r))}{|E(a_i,r)|^{1-\lambda}}\int_{E(a_i,2r)}|f_n(\omega)|^p \mathrm{d}V_\lambda(\omega)\end{aligned}$$

$$\leq CN \sum_{d(a,\partial\Omega)\geq\delta} \int_{E(a_i,2r)} |f_n(\omega)|^p \mathrm{d}V_\lambda(\omega)$$

$$+ C\epsilon \sum_{d(a,\partial\Omega)<\delta} \int_{E(a_i,2r)} |f_n(\omega)|^p \mathrm{d}V_\lambda(\omega)$$

$$\leq CNK \int_\Omega |f_n(\omega)|^p \mathrm{d}V_\lambda(\omega) + C\epsilon K \int_\Omega |f_n(\omega)|^p \mathrm{d}V_\lambda(\omega),$$

where the constant N is the same as in Lemma 2.9. Since, f_n is a norm bounded sequence and f_n converges to zero uniformly on compact subsets of Ω, we have $\|W_{\varphi,\psi} f_n\|_{A^p_\lambda} \to 0$ as $n \to \infty$.

The following theorem follows by combining Theorems 1.1, 1.2, 3.4 and 3.5.

Theorem 3.6. *Take $1 \leq p < \infty$ and let $\lambda < \varepsilon_\Omega$. Suppose φ, ψ be holomorphic functions defined on Ω such that $\varphi(\Omega) \subseteq \Omega$. Then*

(1) *The weighted composition operator $W_{\varphi,\psi}$ is bounded on $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$ if and only if*

$$\sup_{z\in\Omega} \int_\Omega |k_z(\omega)|^{2(1-\lambda)} \mathrm{d}\mu_{\varphi,\psi,p}(\omega) < \infty.$$

(2) *The weighted composition operator $W_{\varphi,\psi}$ is compact on $A^p_\lambda(\Omega, \mathrm{d}V_\lambda)$ if and only if*

$$\lim_{z\in\partial\Omega} \int_\Omega |k_z(\omega)|^{2(1-\lambda)} \mathrm{d}\mu_{\varphi,\psi,p}(\omega) = 0.$$

4. Schatten class weighted composition operators

Theorem 4.1. *Take $1 \leq p < \infty$ and let $\lambda < \varepsilon_\Omega$. Suppose φ, ψ are holomorphic functions defined on Ω such that $\varphi(\Omega) \subseteq \Omega$. Let $W_{\varphi,\psi}$ be compact on $A^2_\lambda(\Omega, \mathrm{d}V_\lambda)$. Then $W_{\varphi,\psi} \in S_p$ if and only if the Berezin symbol of measure $\mu_{\varphi,\psi,p}$ belongs to $L^{p/2}(\Omega, \mathrm{d}\sigma)$, where $\mathrm{d}\sigma(z) = K(z,z)\mathrm{d}V(z)$.*

Proof. For any $f, g \in A^2_\lambda(\Omega, \mathrm{d}V_\lambda)$, we have

$$\begin{aligned}\langle (W_{\varphi,\psi})^*(W_{\varphi,\psi})f,\ g\rangle_{A^2_\lambda} &= \langle (W_{\varphi,\psi})f,\ (W_{\varphi,\psi})g\rangle_{A^2_\lambda} \\ &= \int_\Omega f(\varphi(z))\overline{g(\varphi(z))}|\psi(z)|^2 \mathrm{d}V_\lambda(z) \\ &= \int_\Omega f(\omega)\overline{g(\omega)}|\psi(z)|^2 \mathrm{d}\mu_{\varphi,\psi,p}(\omega).\end{aligned}$$

Consider the Toeplitz operator

$$T_{\mu_{\varphi,\psi,p}} f(z) = \int_\Omega f(\omega) K_\lambda(z,\omega)\ \mathrm{d}\mu_{\varphi,\psi,p}(\omega).$$

Then, by using Fubini's theorem, we have

$$
\begin{aligned}
\langle T_{\mu_{\varphi,\psi,p}} f, g\rangle_{A^2_\lambda} &= \int_\Omega \int_\Omega f(\omega) K_\lambda(z,\omega) \mathrm{d}\mu_{\varphi,\psi,p}(\omega) \overline{g(z)}\, \mathrm{d}V_\lambda(z) \\
&= \int_\Omega f(\omega) \overline{\int_\Omega g(z) K_\lambda(\omega, z) \mathrm{d}V_\lambda(z)}\, \mathrm{d}\mu_{\varphi,\psi,p}(\omega) \\
&= \int_\Omega f(\omega)\overline{g(\omega)}\, \mathrm{d}\mu_{\varphi,\psi,p}(\omega).
\end{aligned}
$$

Thus, $T_{\mu_{\varphi,\psi,p}} = (W_{\varphi,\psi})^*(W_{\varphi,\psi})$. Also, by definition an operator $T \in S_p$ if and only if $(T^*T)^{p/2}$ is in the trace class, and this is equivalent to saying that $T^*T \in S_{p/2}$ (Lemma 1.4.6, p. 18 of [18]). Thus $W_{\varphi,\psi} \in S_p$ if and only if $T_{\mu_{\varphi,\psi,p}} \in S_{p/2}$. Again, by Theorem C of [17], $T_{\mu_{\varphi,\psi,p}} \in S_{p/2}$ if and only if the Berezin symbol of the measure $\mu_{\varphi,\psi,p}$ belongs to $L^{p/2}(\Omega, \mathrm{d}\sigma)$.

Theorem 4.2. *Take $2 \leq p < \infty$ and let $\lambda < \varepsilon_\Omega$. Take $W_{\varphi,\psi}$ the weighted composition operator on $A^2_\lambda(\Omega, \mathrm{d}V_\lambda)$. If $W_{\varphi,\psi} \in S_p$, then*

$$
\int_\Omega |\psi(z)| \left[\frac{K(\varphi(z), \varphi(z))}{K(z,z)}\right]^{\frac{p(1-\lambda)}{2}} \mathrm{d}\sigma(z) < \infty,
$$

where $\mathrm{d}\sigma(z) = K(z,z)\mathrm{d}V(z)$.

Proof. By using Lemmas 2.2–2.5, we get

$$
\begin{aligned}
\operatorname{tr}(|W_{\varphi,\psi}|^p) &= \operatorname{tr}(|W^*_{\varphi,\psi}|^p) = \operatorname{tr}(|W_{\varphi,\psi} W^*_{\varphi,\psi}|)^{p/2} \\
&= \int_\Omega \langle (W_{\varphi,\psi} W^*_{\varphi,\psi})^{p/2} k_z^{1-\lambda}, k_z^{1-\lambda}\rangle_{A^2_\lambda} \mathrm{d}\sigma(z) \\
&\geq \int_\Omega \langle (W_{\varphi,\psi} W^*_{\varphi,\psi}) k_z^{1-\lambda}, k_z^{1-\lambda}\rangle^{p/2}_{A^2_\lambda} \mathrm{d}\sigma(z) \\
&= \int_\Omega \| W^*_{\varphi,\psi} k_z^{1-\lambda}\|^p_{A^2_\lambda} \mathrm{d}\sigma(z) \\
&= \int_\Omega |\psi(z)| \left[\frac{K(\varphi(z), \varphi(z))}{K(z,z)}\right]^{\frac{p(1-\lambda)}{2}} \mathrm{d}\sigma(z).
\end{aligned}
$$

Theorem 4.3. *Take $0 < p < 2$ and let $\lambda < \varepsilon_\Omega$. Take $W_{\varphi,\psi}$ the weighted composition operator $W_{\varphi,\psi}$ on $A^2_\lambda(\Omega, \mathrm{d}V_\lambda)$. If*

$$
\int_\Omega |\psi(z)| \left[\frac{K(\varphi(z), \varphi(z))}{K(z,z)}\right]^{\frac{p(1-\lambda)}{2}} \mathrm{d}\sigma(z) < \infty,
$$

where $\mathrm{d}\sigma(z) = K(z,z)\mathrm{d}V(z)$, *then* $W_{\varphi,\psi} \in S_p$.

Proof. By using Lemmas 2.2–2.5, we get

$$\begin{aligned}\operatorname{tr}(|W_{\varphi,\psi}|^p) &= \int_\Omega \langle (W_{\varphi,\psi} W_{\varphi,\psi}^*)^{p/2} k_z^{1-\lambda}, k_z^{1-\lambda}\rangle_{A_\lambda^2} \mathrm{d}\sigma(z)\\ &\geq \int_\Omega \langle (W_{\varphi,\psi} W_{\varphi,\psi}^*) k_z^{1-\lambda}, k_z^{1-\lambda}\rangle_{A_\lambda^2}^{p/2} \mathrm{d}\sigma(z)\\ &= \int_\Omega \|W_{\varphi,\psi}^* k_z^{1-\lambda}\|_{A_\lambda^2}^p \mathrm{d}\sigma(z)\\ &= \int_\Omega |\psi(z)| \left[\frac{K(\varphi(z),\varphi(z))}{K(z,z)}\right]^{\frac{p(1-\lambda)}{2}} \mathrm{d}\sigma(z) < \infty.\end{aligned}$$

Thus, $W_{\varphi,\psi} \in S_p$.

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