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On the stability of Jensen's functional equation on groups

VALERIĬ A FAĬZIEV and PRASANNA K SAHOO*

Tver State Agricultural Academy, Tver Sakharovo, Russia
*Department of Mathematics, University of Louisville, Louisville, KY 40292, USA
E-mail: vfaiz@tvcom.ru; sahoo@louisville.edu

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Abstract. In this paper we establish the stability of Jensen's functional equation on some classes of groups. We prove that Jensen equation is stable on noncommutative groups such as metabelian groups and $T(2, K)$, where K is an arbitrary commutative field with characteristic different from two. We also prove that any group A can be embedded into some group G such that the Jensen functional equation is stable on G.

Keywords. Additive mapping; Banach spaces; Jensen equation; Jensen function; metabelian group; metric group; pseudoadditive mapping; pseudo-Jensen function; pseudocharacter; quasiadditive map; quasicharacter; quasi-Jensen function; semigroup; direct product of groups; stability of functional equation, wreath product of groups.

1. Introduction

Given an operator T and a solution class $\{u\}$ with the property that $T(u) = 0$, when does $\|T(v)\| \leq \varepsilon$ for an $\varepsilon > 0$ imply that $\|u - v\| \leq \delta(\varepsilon)$ for some u and for some $\delta > 0$? This problem is called the stability of the functional transformation. A great deal of work has been done in connection with the ordinary and partial differential equations. If f is a function from a normed vector space into a Banach space, and $\|f(x+y)-f(x)-f(y)\| \leq \varepsilon$, Hyers in 1941 proved that there exists an additive map A such that $\|f(x) - A(x)\| \leq \varepsilon$. If $f(x)$ is a real continuous function of x over $\mathbb{R}$, and $|f(x + y) - f(x) - f(y)| \leq \varepsilon$, it was shown by Hyers and Ulam in 1952 that there exists a constant k such that $|f(x) - kx| \leq 2\varepsilon$. Taking these results into account, we say that the additive Cauchy equation $f(x + y) = f(x) + f(y)$ is stable in the sense of Hyers and Ulam. For more on stability of homomorphisms, the interested reader is referred to [22, 8–11] and [1]. After Hyers's 1941 result a great number of papers on the subject have been published, generalizing Ulam's problem and Hyers's theorem in various directions (see [7], [10–14] and [20]).

In this paper we study the stability of Jensen's functional equation

$$f(xy) + f(xy^{-1}) = 2f(x)$$

on some classes of noncommutative groups. This Jensen's equation was studied in the papers [2], [3] and [19]. The question of stability of this equation was investigated in [16, 15, 17, 21] and [18]. In all these papers domain of f is either an abelian group or some of its subsets.

2. Auxiliary results

Suppose that G is an arbitrary group and E is an arbitrary real Banach space. In this sequel, we will write the arbitrary group G in multiplicative notation so that 1 will denote the identity element of G.

DEFINITION 2.1

We will say that a function $f: G \to E$ is a $(G; E)$-*Jensen function* if for any $x, y \in G$ we have

$$f(xy) + f(xy^{-1}) - 2f(x) = 0. \tag{2.1}$$

We denote the set of all $(G; E)$-Jensen functions by $J(G; E)$.

Denote by $J_0(G; E)$ the subset of $J(G; E)$ consisting of functions f such that $f(1) = 0$. Obviously $J_0(G; E)$ is a subspace of $J(G; E)$ and $J(G; E) = J_0(G; E) \oplus E$.

DEFINITION 2.2

We will say that a function $f: G \to E$ is a $(G; E)$-quasi-Jensen function if there is $c > 0$ such that for any $x, y \in G$ we have

$$\|f(xy) + f(xy^{-1}) - 2f(x)\| \le c. \tag{2.2}$$

It is clear that the set of $(G; E)$-quasi-Jensen functions is a linear real space. Denote it by $KJ(G; E)$. From (2.2) we obtain

$$\|f(y) + f(y^{-1}) - 2f(1)\| \le c.$$

Therefore

$$\|f(y) + f(y^{-1})\| \le c_1, \tag{2.3}$$

where $c_1 = c + \|2f(1)\|$. Now letting x for y in (2.2), we get

$$\|f(x^2) + f(1) - 2f(x)\| \le c.$$

Hence

$$\|f(x^2) - 2f(x)\| \le c_2, \tag{2.4}$$

where $c_2 = c + \|f(1)\|$. Again substitution of $y = x^2$ in (2.2) yields

$$\|f(x^3) + f(x^{-1}) - 2f(x)\| \le c.$$

Thus taking into account (2.3) we obtain

$$\|f(x^3) - 3f(x)\| \le c_3, \tag{2.5}$$

where $c_3 = c + c_1$.

Let c be as in (2.2) and define the set C as follows: $C = \{c_m \mid m \in \mathbb{N}\}$, where $c_1 = c + 2\|f(1)\|$, $c_2 = c + \|f(1)\|$, $c_3 = c + c_1$ and $c_m = c + c_1 + c_{m-2}$, if $m > 3$.

Lemma 2.3. *Let $f \in KJ(G; E)$ such that*

$$\| f(xy) + f(xy^{-1}) - 2f(x) \| \le c.$$

Then for any $x \in G$ and any $m \in \mathbb{N}$ the following relation holds:

$$\|f(x^m) - mf(x)\| \le c_m. \tag{2.6}$$

Proof. The proof is by induction on m. For $m = 3$, the Lemma is established. Suppose that for m the lemma has been already established, let us verify it for $m + 1$. Letting $y = x^m$ in (2.2), we have

$$\|f(x^{m+1}) + f(x^{-(m-1)} - 2f(x)\| \le c.$$

From (2.3) we obtain

$$\|f(x^{m+1}) - f(x^{m-1}) - 2f(x)\| \le c + c_1.$$

By induction hypothesis we have

$$\|f(x^{m-1}) - (m-1)f(x)\| \le c_{m-1}$$

and hence,

$$\|f(x^{m+1}) - (m+1)f(x)\| \le c_{m+1} = c + c_1 + c_{m-1}.$$

Now the lemma is proved. □

Lemma 2.4. *Let $f \in KJ(G; E)$. For any $m > 1$, $k \in \mathbb{N}$ and $x \in G$ we have*

$$\|f(x^{m^k}) - m^k f(x)\| \le c_m(1 + m + \cdots + m^{k-1}) \tag{2.7}$$

and

$$\left\| \frac{1}{m^k} f(x^{m^k}) - f(x) \right\| \le c_m. \tag{2.8}$$

Proof. The proof will be based on induction on k. If $k = 1$, then (2.7) follows from (2.6). Suppose that (2.7) for k is true, let us verify it for $k + 1$. Substituting x^m for x in (2.7) implies

$$\|f(x^{m^{k+1}}) - m^k f(x^m)\| \le c_m(1 + m + \cdots + m^{k-1}).$$

Now using (2.6) we obtain

$$\|m^k f(x^m) - m^{k+1} f(x)\| \le c_m m^k$$

and hence

$$\|f(x^{m^{k+1}}) - m^{k+1} f(x)\| \le c_m(1 + m + \cdots + m^k).$$

The latter implies

$$\left\| \frac{1}{m^{k+1}} f(x^{m^{k+1}}) - f(x) \right\| \le c_m(1 + m + \cdots + m^k) \frac{1}{m^{k+1}} \le c_m.$$

This completes the proof of the lemma. □

From (2.8) it follows that the set

$$\left\{\frac{1}{m^k} f(x^{m^k}) \mid \quad k \in \mathbb{N}\right\}$$

is bounded.

Substituting x^{m^n} in place of x in (2.8), we obtain

$$\left\| \frac{1}{m^k} f(x^{m^{n+k}}) - f(x^{m^n}) \right\| \leq c_m,$$

$$\left\| \frac{1}{m^{n+k}} f(x^{m^{n+k}}) - \frac{1}{m^n} f(x^{m^n}) \right\| \leq \frac{c_m}{m^n} \to 0, \quad \text{as} \quad n \to \infty.$$

From the latter it follows that the sequence

$$\left\{\frac{1}{m^k} f(x^{m^k}) \mid \quad k \in \mathbb{N}\right\}$$

is a Cauchy sequence. Since the space E is complete, the above sequence has a limit and we denote it by $\varphi_m(x)$. Thus

$$\varphi_m(x) = \lim_{k\to\infty} \frac{1}{m^k} f(x^{m^k}).$$

From (2.8) it follows that

$$\|\varphi_m(x) - f(x)\| \leq c_m, \quad \forall x \in G. \tag{2.9}$$

Lemma 2.5. *Let $f \in KJ(G; E)$ such that*

$$\|f(xy) + f(xy^{-1}) - 2f(x)\| \leq c, \qquad \forall x, y \in G.$$

Then for any $m \in \mathbb{N}$ we have $\varphi_m \in KJ(G; E)$.

Proof. Indeed, by (2.9)

$$\begin{aligned}
&\|\varphi_m(xy) + \varphi_m(xy^{-1}) - 2\varphi_m(x)\| \\
&\quad = \|\varphi_m(xy) - f(xy) + \varphi_m(xy^{-1}) - f(xy^{-1}) - 2\varphi_m(x) + 2f(x) \\
&\qquad + f(xy) + f(xy^{-1}) - 2f(x)\| \\
&\quad \leq \|\varphi_m(xy) - f(xy)\| + \|\varphi_m(xy^{-1}) - f(xy^{-1})\| \\
&\qquad + 2\|\varphi_m(x) - f(x)\| + \|f(xy) + f(xy^{-1}) - 2f(x)\| \\
&\quad \leq 4c_m + c.
\end{aligned}$$

This completes the proof of the lemma. □

For any $x \in G$ we have the relation

$$\varphi_m(x^{m^k}) = m^k \varphi_m(x). \tag{2.10}$$

Indeed

$$\begin{aligned}\varphi_m(x^{m^k}) &= \lim_{\ell\to\infty} \frac{1}{m^\ell} f((x^{m^k})^{m^\ell})\\ &= \lim_{\ell\to\infty} \frac{m^k}{m^{k+\ell}} f(x^{m^{k+\ell}})\\ &= m^k \lim_{p\to\infty} \frac{1}{m^p} f(x^{m^p})\\ &= m^k \varphi_m(x).\end{aligned}$$

Lemma 2.6. *If $f \in KJ(G; E)$, then $\varphi_2 = \varphi_m$ for any $m \geq 2$.*

Proof. By Lemma 2.5 we have $\varphi_2, \varphi_m \in KJ(G; E)$. Hence the function

$$g(x) = \lim_{k\to\infty} \frac{1}{m^k} \varphi_2(x^{m^k})$$

is well-defined and is a $(G; E)$-quasi-Jensen function.

It is clear that $g(x^{m^k}) = m^k g(x)$ and $g(x^{2^k}) = 2^k g(x)$ for any $x \in G$ and any $k \in \mathbb{N}$. From (2.9) it follows that there are $d_1, d_2 \in \mathbb{R}_+$ such that for all $x \in G$,

$$\|\varphi_2(x) - g(x)\| \leq d_1 \quad \text{and} \quad \|\varphi_m(x) - g(x)\| \leq d_2. \tag{2.11}$$

Hence $g \equiv \varphi_2$ and $g \equiv \varphi_m$ and we obtain $\varphi_2 \equiv \varphi_m$. $\square$

DEFINITION 2.7

By $(G; E)$-pseudo-Jensen function we will mean a $(G; E)$-quasi-Jensen function f such that $f(x^n) = nf(x)$ for any $x \in G$ and any $n \in \mathbb{Z}$.

Lemma 2.8. *For any $f \in KJ(G; E)$ the function*

$$\hat{f}(x) = \lim_{k\to\infty} \frac{1}{2^k} f(x^{2^k})$$

is well-defined and is a $(G; E)$-pseudo-Jensen function such that for any $x \in G$,

$$\|\hat{f}(x) - f(x)\| \leq c_2.$$

Proof. By Lemma 2.5, $\hat{f}$ is a $(G; E)$-quasi-Jensen function. Now by Lemma 2.6, we have $\hat{f}(x^m) = \varphi_m(x^m) = m\varphi_m(x) = m\hat{f}(x)$. Thus $\varphi_m(x) = \hat{f}(x)$ and hence $\varphi_2(x) = \hat{f}(x)$ by Lemma 2.6. From equality $\hat{f} = \varphi_2$ we have $\|\hat{f}(x) - f(x)\| = \|\varphi_2(x) - f(x)\| \leq c_2$. $\square$

Remark 2.9. If $f \in PJ(G; E)$, then

1. $f(x^{-n}) = -nf(x)$ for any $x \in G$ and $n \in \mathbb{N}$;
2. if $y \in G$ is an element of finite order then $f(y) = 0$;
3. if f is a bounded function on G, then $f \equiv 0$.

Proof. For some $c > 0$ the following relation holds:

$$\|f(xy) + f(xy^{-1}) - 2f(x)\| \leq c.$$

From (2.3) it follows that

$$\|f(y^k) + f(y^{-k})\| \leq c_1 \quad \forall y \in G,\ \forall k \in \mathbb{N}.$$

The last inequality is equivalent to $k\|f(y) + f(y^{-1})\| \leq c_1$ or $\|f(y) + f(y^{-1})\| \leq \frac{c_1}{k}$ for all $y \in G$ and all $k \in \mathbb{N}$. The latter implies $f(y^{-1}) = -f(y)$. Thus for any $n \in \mathbb{N}$ we have $f(y^{-n}) = f((y^n)^{-1}) = -f(y^n) = -nf(y)$. Hence, the assertion 1 is established.

Similarly we verify assertions 2 and 3. $\square$

We denote by $B(G; E)$ the space of all bounded functions on a group G that take values in E.

Theorem 2.10. *For an arbitrary group G the following decomposition holds*:

$$KJ(G; E) = PJ(G; E) \oplus B(G; E).$$

Proof. It is clear that $PJ(G; E)$ and $B(G; E)$ are subspaces of $KJ(G; E)$, and $PJ(G; E) \cap B(G; E) = \{0\}$. Hence the subspace of $KJ(G; E)$ generated by $PJ(G; E)$ and $B(G; E)$ is their direct sum. That is $PJ(G; E) \oplus B(G; E) \subseteq KJ(G; E)$. Let us verify that $KJ(G; E) \subseteq PJ(G; E) \oplus B(G; E)$. Indeed, if $f \in KJ(G; E)$, then by Lemma 2.8 we have $\hat{f} \in PJ(G; E)$ and $\hat{f} - f \in B(G; E)$. $\square$

DEFINITION 2.11

Let E be a Banach space and G be a group. A mapping $f : G \to E$ is said to be a $(G; E)$-*quasiadditive mapping* of a group G if set $\{f(xy) - f(x) - f(y) | x, y \in G\}$ is bounded.

DEFINITION 2.12

By a $(G; E)$-pseudoadditive mapping of a group G we mean its $(G; E)$-quasiadditive mapping f that satisfies $f(x^n) = nf(x)$ for all $x \in G$ and all $n \in \mathbb{Z}$.

DEFINITION 2.13

A quasicharacter of a group G is a real-valued function f on G such that the set $\{f(xy) - f(x) - f(y) | x, y \in G\}$ is bounded.

DEFINITION 2.14

By a pseudocharacter of a group G we mean its quasicharacter f that satisfies $f(x^n) = nf(x)$ for all $x \in G$ and all $n \in \mathbb{Z}$.

The set of all $(G; E)$-quasiadditive mappings is a vector space (with respect to the usual operations of addition of functions and their multiplication by numbers), which will be denoted by $KAM(G; E)$. The subspace of $KAM(G; E)$ consisting of $(G; E)$-pseudoadditive mappings will be denoted by $PAM(G; E)$ and the subspace consisting of additive mappings from G to E will be denoted by $\mathrm{Hom}(G; E)$. We say that a $(G; E)$-pseudoadditive mapping φ of the group G is *nontrivial* if $\varphi \notin \mathrm{Hom}(G; E)$.

The space of quasicharacters will be denoted by $KX(G)$, the space of pseudocharacters will be denoted by $PX(G)$, and the space of real additive characters on G will be denoted by $X(G)$.

Remark 2.15. If a group G has nontrivial pseudocharacter, then for any Banach space E there is nontrivial $(G; E)$-pseudoadditive mapping.

Proof. Let f be a nontrivial pseudocharacter of the group G and $e \in E$ such that $e \neq 0$. Consider a mapping $\varphi\colon G \to E$ such that $\varphi(x) = f(x) \cdot e$. It easy to see that φ is a nontrivial $(G; E)$-additive mapping. □

In [5] and [6] some classes of groups having nontrivial pseudocharacters are considered.

Theorem 2.16. *For any group G the following relations hold*:

1. $KAM(G; E) \subseteq KJ(G; E)$, $PAM(G; E) \subseteq PJ(G; E)$, $\mathrm{Hom}(G; E) \subseteq J_0(G; E)$.
2. *If $f \in PJ(G; E)$ and for any $x, y \in G$ $f(xy) = f(yx)$, then $f \in PAM(G; E)$.*

Proof.

1. Let $f \in KAM(G; E)$ and $c > 0$ such that $\|f(xy) - f(x) - f(y)\| \leq c$ for all $x, y \in G$. Then we have
$$\begin{aligned}
&\|f(xy) + f(xy^{-1}) - 2f(x)\| \\
&\quad = \|f(xy) - f(x) - f(y) + f(xy^{-1}) - f(x) - f(y^{-1}) \\
&\qquad + 2f(x) + f(y) + f(y^{-1}) - 2f(x)\| \\
&\quad = \|f(xy) - f(x) - f(y) + f(xy^{-1}) - f(x) - f(y^{-1})\| \\
&\quad \leq \|f(xy) - f(x) - f(y)\| + \|f(xy^{-1}) - f(x) - f(y^{-1})\| \\
&\quad \leq 2c,
\end{aligned}$$
that is, $KAM(G; E) \subseteq KJ(G; E)$. Hence, $PAM(G; E) \subseteq PJ(G; E)$.
2. Let $f \in PJ(G; E)$, $c > 0$ such that $\|f(xy) + f(xy^{-1}) - 2f(x)\| \leq c$ and $f(xy) = f(yx)$ for all $x, y \in G$. Then we have
$$\begin{aligned}
&2\|f(xy) - f(x) - f(y)\| \\
&\quad = \|f(xy) + f(xy^{-1}) - 2f(x) + f(xy) + f(yx^{-1}) - 2f(y)\| \\
&\quad \leq \|f(xy) + f(xy^{-1}) - 2f(x)\| \\
&\qquad + \|f(xy) + f(yx^{-1}) - 2f(y)\| \leq 2c.
\end{aligned}$$
Hence $\|f(xy) - f(x) - f(y)\| \leq c$ and $f \in PAM(G; E)$. □

COROLLARY 2.17

If G is an abelian group, then $PJ(G; E) = \mathrm{Hom}(G; E)$.

Proof. By Theorem 2.16 we have $PJ(G; E) = PAM(G; E)$. Let $f \in PAM(G; E)$. Then for some $c > 0$ and for any $n \in \mathbb{N}$, and $a, b \in G$ we have

$$\begin{aligned} n\|f(ab) - f(a) - f(b)\| &= \| f((ab)^n) - f(a^n) - f(b^n) \| \\ &= \|f(a^n b^n) - f(a^n) - f(b^n)\| \\ &\le c. \end{aligned}$$

The latter is possible only if $f \in \mathrm{Hom}(G; E)$. □

3. Stability

Suppose that G is a group and E is a real Banach space.

DEFINITION 3.1

We shall say that eq. (2.1) is stable for the pair $(G; E)$ if for any $f: G \to E$ satisfying functional inequality

$$\|f(xy) + f(xy^{-1}) - 2f(x)\| \le c \quad \forall x, y \in G$$

for some $c > 0$ there is a solution j of the functional equation (2.1) such that the function $j(x) - f(x)$ belongs to $B(G; E)$.

It is clear that eq. (2.1) is stable on G if and only if $PJ(G; E) = J_0(G; E)$. From Corollary 2.17 it follows that eq. (2.1) is stable on any abelian group. We will say that a $(G; E)$-pseudo-Jensen function f in nontrivial if $f \notin J_0(G; E)$.

Theorem 3.2. *Let* E_1, E_2 *be a Banach space over reals. Then eq.* (2.1) *is stable for the pair* $(G; E_1)$ *if and only if it is stable for the pair* $(G; E_2)$.

Proof. Let E be a Banach space and $\mathbb{R}$ be the set of reals. Suppose that eq. (2.1) is stable for the pair $(G; E)$. Suppose that (2.1) is not stable for the pair $(G, \mathbb{R})$, then there is a nontrivial real-valued pseudo-Jensen function f on G. Now let $e \in E$ and $\|e\| = 1$. Consider the function $\varphi: G \to E$ given by the formula $\varphi(x) = f(x) \cdot e$. It is clear that φ is a nontrivial pseudo-Jensen E-valued function, and we obtain a contradiction.

Now suppose that eq. (2.1) is stable for the pair $(G, \mathbb{R})$, that is, $PJ(G, \mathbb{R}) = J_0(G, \mathbb{R})$. Denote by E^* the space of linear bounded functionals on E endowed by functional norm topology. It is clear that for any $\psi \in PJ(G, H)$ and any $\lambda \in H^*$ the function $\lambda \circ \psi$ belongs to the space $PJ(G, \mathbb{R})$. Indeed, let for some $c > 0$ and any $x, y \in G$ we have $\|\psi(xy) + \psi(xy^{-1}) - 2\psi(x)\| \le c$. Hence

$$\begin{aligned} &|\lambda \circ \psi(xy) + \lambda \circ \psi(xy^{-1}) - \lambda \circ \psi(2x)| \\ &\quad = |\lambda(\psi(xy) + \psi(xy^{-1}) - 2\psi(x))| \le c\|\lambda\|. \end{aligned}$$

Obviously, $\lambda \circ \psi(x^n) = n\lambda \circ \psi(x)$ for any $x \in G$ and for any $n \in \mathbb{N}$. Hence the function $\lambda \circ \psi$ belongs to the space $PJ(G, \mathbb{R})$. Let $f: G \to H$ be a nontrivial pseudo-Jensen mapping. Then $x, y \in G$ such that $f(xy)+f(xy^{-1})-2f(x) \neq 0$. Hahn–Banach theorem implies that there is a $\ell \in H^*$ such that $\ell(f(xy)+f(xy^{-1})-2f(x)) \neq 0$, and we see that $\ell \circ f$ is a nontrivial pseudo-Jensen real-valued function on G. This contradiction proves the theorem. □

In what follows the space $KJ(G, \mathbb{R})$ will be denoted by $KJ(G)$, the space $PJ(G, \mathbb{R})$ will be denoted by $PJ(G)$, the space $J(G, \mathbb{R})$ will be denoted by $J(G, \mathbb{R})$, and the space $J_0(G, \mathbb{R})$ will be denoted by $J_0(G)$.

COROLLARY 3.3

Equation (2.1) *over a group G is stable if and only if* $PJ(G) = J_0(G)$.

Due to the previous theorem we may simply say that eq. (2.1) is stable or not stable.

Remark 3.4. For any group G and any Banach space E the following relation $PAM(G; E) \cap J(G; E) = \mathrm{Hom}(G; E)$ holds.

Proof. It is clear that $\mathrm{Hom}(G; E) \subseteq PAM(G; E) \cap J(G; E)$.

Suppose that $f \in PAM(G; E) \cap J(G; E)$. Then by Lemma 1 from [4] we have $f(xy) = f(yx)$. Since $f \in J(G; E)$, the map f satisfies

$$f(xy) + f(xy^{-1}) - 2f(x) = 0. \tag{3.1}$$

Interchanging x with y in (3.1), we have

$$f(yx) + f(yx^{-1}) - 2f(y) = 0$$

which is

$$f(xy) - f(xy^{-1}) - 2f(y) = 0. \tag{3.2}$$

Adding (3.1) and (3.2) we obtain $2f(xy) - 2f(x) - 2f(y) = 0$. Hence $f(xy) = f(x) + f(y)$ and $f \in \mathrm{Hom}(G; E)$. Thus we obtain

$$PAM(G; E) \cap J(G; E) = \mathrm{Hom}(G; E) \tag{3.3}$$

and the proof is complete. □

Remark 3.5. If a group G has nontrivial pseudocharacter, then eq. (2.1) is not stable on G.

Proof. Let let φ be a nontrivial pseudocharacter of G. Suppose that there is $j \in J_0(G)$ such that the function $\varphi - j$ is bounded. Then there is $c > 0$ such that $|\varphi(x) - j(x)| \leq c$ for any $x \in G$. Hence for any $n \in \mathbb{N}$ we have $c \geq |\varphi(x^n) - j(x^n)| = n|\varphi(x) - j(x)|$ and we see that the latter is possible if $\varphi(x) = j(x)$. So, $\varphi \in PX(G) \cap J_0(G)$. Hence, $f \in X(G)$ and we come to a contradiction with the assumption about f. □

Let G be an arbitrary group. For $a, b, c \in G$, we set $[a, b] = a^{-1}b^{-1}ab$ and $[a, b, c] = [[a, b], c]$.

DEFINITION 3.6

We shall say that G is *metabelian* if for any $x, y, z \in G$ we have $[[x, y], z] = 1$.

It is clear that if $[x, y] = 1$ then $[[x, y], z] = 1$, and hence any abelian group is metabelian.

Our next goal is to prove a stability theorem for any metabelian group. Consider the group H over two generators a, b and the following defining relations:

$$[b, a]a = a[b, a], \quad b[b, a] = [b, a]b.$$

If we set $c = [b, a]$ we get the following representation of H in terms of generators and defining relations:

$$H = \langle a, b, \| c = [b, a], \quad [c, a] = [c, b] = 1\rangle. \tag{3.4}$$

It is well-known that each element of H can be uniquely represented as $g = a^m b^n c^k$, where $m, n, k \in \mathbb{Z}$. The mapping

$$g = a^m b^n c^k \to \begin{bmatrix} 1 & n & k \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix}$$

is an isomorphism between H and $UT(3, \mathbb{Z})$.

Lemma 3.7. *Let $f \in PJ(H)$ and $f(c) = 0$. Then $f \in PX(H) = X(H)$.*

Proof. Let $x = a^m b^n c^k$ and $y = a^{m_1} b^{n_1} c^{k_1}$ be two elements from H. Then from the representation (3.4) it follows that

$$xy = a^{m+m_1} b^{n+n_1} c^{m_1 n+k+k_1}, \quad yx = a^{m+m_1} b^{n+n_1} c^{mn_1+k+k_1}.$$

Hence

$$f(xy) = f(a^{m+m_1} b^{n+n_1}) + f(c^{m_1 n+k+k_1}) = f(a^{m+m_1} b^{n+n_1}),$$

$$f(yx) = f(a^{m+m_1} b^{n+n_1}) + f(c^{mn_1+k+k_1}) = f(a^{m+m_1} b^{n+n_1}).$$

Thus $f(xy) = f(yx)$ for any $x, y \in G$. By Theorem 2.16 we obtain that $f \in PX(G)$. From the representation (3.4) it follows that the subgroup of H generated by element c is the commutator subgroup of H. Lemma 2 from [4] establishes that if $\varphi \in PX(G)$ such that $f\big|_{G'} \equiv 0$, then $\varphi \in X(G)$. Hence, $f \in X(H)$ and $PX(G) = X(G)$. □

Lemma 3.8. *Let $f \in PJ(H)$ and $f(a) = f(b) = f(c) = 0$. Then $f \equiv 0$.*

Proof. By Lemma 3.7 we have $f(a^m b^n c^k) = f(a^m) + f(b^n) + f(c^k) = 0$. □

Lemma 3.9. *A function ϕ defined by the formula $\phi(a^m b^n c^k) = mn - 2k$ is an element of $J_0(G)$.*

Proof. It is clear that $\phi(1) = 0$. Now let $x = a^m b^n c^k$, $y = a^{m_1} b^{n_1} c^{k_1}$, then $xy^{-1} = a^m b^n c^k c^{-k_1} b^{-n_1} a^{-m_1} = a^{m-m_1} b^{n-n_1} c^{m_1 n_1 - m_1 n + k - k_1}$. Hence

$$\begin{aligned} & f(xy) + f(xy^{-1}) - 2f(x) \\ &\quad = (m + m_1)(n + n_1) - 2(m_1 n + k + k_1) + (m - m_1)(n - n_1) \\ &\qquad - 2(m_1 n_1 - m_1 n + k - k_1) - 2(mn - 2k) \\ &\quad = 0 \end{aligned}$$

and the proof of the lemma is now complete. □

Lemma 3.10. $PJ(H) = J_0(H)$.

Proof. Let $g \in PJ(H)$ and $g(a) = \alpha$, $g(b) = \beta$, $g(c) = \gamma$. Then there are $\psi \in X(H)$ and $\lambda \in \mathbb{R}$ such that $\psi(a) = \alpha$, $\psi(b) = \beta$, and $\lambda\phi(c) = \gamma$. Furthermore, we have $j = \psi + \lambda\phi \in J_0(H)$ and $(g - j)(a) = (g - j)(b) = (g - j)(c) = 0$. By Lemma 3.8 we get $(g - j) \equiv 0$. Hence $g = j$ and $g \in J_0(H)$. □

Theorem 3.11. *Equation* (2.1) *is stable on any metabelian group.*

Proof. Let G be a metabelian group and $f \in PJ(G)$. Let $x, y \in G$. Then there is a homomorphism τ of H into G such that $\tau(a) = x$ and $\tau(b) = y$. Obviously, the function $f^*(g) = f(\tau(g))$ belongs to $PJ(H)$. Now if $f(xy) + f(xy^{-1}) - 2f(x) \neq 0$, then $f^*(ab) + f^*(ab^{-1}) - 2f^*(a) \neq 0$ and we arrive at a contradiction with the previous lemma. Thus $f \in J_0(G)$, $PJ(G) = J_0(G)$ and the eq. (2.1) is stable on G. □

DEFINITION 3.12

Let G be a group, $f \in PJ(G; E)$, and b an automorphism of G. We will say that f is invariant relative to b if for any $x \in G$ the relation $f(x^b) = f(x)$ holds. If the latter relation is valid for any $b \in B$, where B is a group of automorphism of G, then we will say that f is invariant relative to B.

Lemma 3.13. *Let f be an element from $PJ(G; E)$ and b an element of order two from G. Then f is invariant relative to inner automorphism of G corresponding to element b.*

Proof. Let $\|f(xy) + f(xy^{-1}) - 2f(x)\| \leq c$ for some $c > 0$ and for any $x, y \in G$. Then we have

$$\|f(bxb) + f(bb^{-1}x^{-1}) - 2f(b)\| \leq c$$

$$\|f(bxb) + f(x^{-1}) - 2f(b)\| \leq c$$

$$\|f(x^b) + f(x^{-1})\| \leq c$$

$$\|f(x^b) - f(x)\| \leq c.$$

From the latter we obtain $\|f(x^{nb}) - f(x^n)\| \leq c$ for any $n \in \mathbb{N}$. Therefore $n\|f(x^b) - f(x)\| \leq c$ and we get $f(x^b) = f(x)$. □

Let K be an arbitrary commutative field. Let K^* be the set of nonzero elements of K with operation of multiplication. Denote by G the group $T(2, K)$ consisting of matrices of the form

$$\begin{bmatrix} \alpha & t \\ 0 & \beta \end{bmatrix}; \quad \alpha, \beta \in K^*; \quad t \in K.$$

Denote by T, E, D the subgroups of $G = T(2, K)$ consisting of matrices

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \quad \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

respectively, where $a, b \in K^*$ and $t \in K$.

It is clear that $T \lhd G$ and we have the following semidirect products, $G = D \cdot T$. Subgroup C of G generated by T and E is a semidirect product $C = E \cdot T$. Now we prove a stability theorem on the noncommutative group $T(2, K)$.

Theorem 3.14. *Let K be an arbitrary commutative field. If the characteristic of K is not equal to two, then the Jensen functional equation is stable on G.*

Proof. Let $f \in PJ(G)$. Every element of E has order two. Hence, by Lemma 3.13 we have $f^e = f$ for any $e \in E$. Here f^e denotes $f(x^e)$ for $x \in G$, and x^e denotes $e^{-1}xe$. Now from the relation

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix},$$

it follows that if $e = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $v = \left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right]$, then

$$f(v) = f^e(v) = f(v^{-1}) = -f(v).$$

Hence $f\big|_T \equiv 0$. It is clear that the map

$$\tau\colon \begin{bmatrix} \alpha & t \\ 0 & \beta \end{bmatrix} \to \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

is a homomorphism of G onto D. Let $\varphi = f\big|_D$. Then we can extend φ onto G by the rule $\varphi(g) = \varphi(g^\tau)$. It clear that $\varphi \in PJ(G)$ and for any $d \in D$ we have the following relation $\varphi(d) = f(d)$.

Now let $\omega(x) = f(x) - \varphi(x)$. So $\omega \in PJ(G)$ and $\omega_{D \cup T} \equiv 0$. Let us show that $\omega \equiv 0$ on G. For some nonnegative number δ, we have

$$|\omega(xy) + \omega(xy^{-1}) - 2\omega(x)| \le \delta \tag{3.5}$$

for any $x, y \in G$. Let $x = au$, $y = bv$, where $a, b \in D$ and $u, v \in T$. Then

$$xy = abu^b v, \quad xy^{-1} = ab^{-1}(uv^{-1})^{a^{-1}}.$$

Now from (3.5) it follows

$$|\omega(abu^b v) + \omega(ab^{-1}(uv^{-1})^{a^{-1}}) - 2\omega(au)| \le \delta.$$

If we put $a = b$ and $u = 1$, then from the last relation we get

$$|\omega(a^2 v) + \omega((v^{-1})^{a^{-1}}) - 2\omega(a)| = |\omega(a^2 v)| \leq \delta.$$

Taking into account equality $(a^2 v)^n = (a^n)^2 v^{a^{2(n-1)}} v^{a^{2(n-2)}} \cdots v^{a^2} v$ we get

$$n|\omega(a^2 v)| = |\omega((a^2 v)^n)| = |\omega((a^n)^2 v^{a^{2(n-1)}} v^{a^{2(n-2)}} \cdots v^{a^2} v)| \leq \delta,$$

that is

$$|\omega(a^2 v)| \leq \frac{1}{n}\delta, \text{ for any } n \in \mathbb{N}.$$

It follows that

$$\omega(a^2 v) \equiv 0 \text{ for any } a \in D \text{ and any } v \in T. \tag{3.6}$$

Now let $x = bu$ be an arbitrary element of G. Then $x^2 = b^2 u^b u$. Therefore $\omega(x) = \frac{1}{2}\omega(x^2) = 0$. So, $\omega \equiv 0$ on G, and $PJ(G) = PJ(D) = J_0(D)$. The proof of the theorem is now complete. □

4. The theorem of embedding

In this setion, we prove that any group A can be embedded into some group G such that the Jensen functional equation is stable on G. From now on, the set of pseudo-Jensen functions on G invariant relative to B will be denoted by $PJ(G, B; E)$ and if $E = \mathbb{R}$, then the space $PJ(G, B; \mathbb{R})$ will be denoted $PJ(G, B)$.

Let A and B be arbitrary groups. For each $b \in B$ denote by $A(b)$ a group that is isomorphic to A under isomorphism $a \to a(b)$. Denote by $D = A^{(B)} = \prod_{b \in B} A(b)$ the direct product of groups $A(b)$. It is clear that if $a_1(b_1)a_2(b_2) \cdots a_k(b_k)$ is an element of D, then for any $b \in B$, the mapping

$$b^*\colon a_1(b_1)a_2(b_2) \cdots a_k(b_k) \to a_1(b_1 b)a_2(b_2 b) \cdots a_k(b_k b)$$

is an automorphism of D and $b \to b^*$ is an embedding of B into Aut D.

Hence, we can form a semidirect product $G = B \cdot D$. This group is called *the wreath product* of the groups A and B, and will be denoted by $G = A \wr B$. We will identify the group A with subgroup $A(1)$ of D, where $1 \in B$. Hence, we can assume that A is a subgroup of D.

Let us denote, by C, the group $\prod_{i \in \mathbb{N}} C_i$, where C_i is a group of order two with generator b_i.

Theorem 4.1. *Let A be an arbitrary group. Then A can be embedded into a group G such that* $PJ(G) = J_0(G) = X(G)$. *Hence eq.* (2.1) *is stable on group G.*

Proof. Let C be a group as described above. Let us verify that eq. (2.1) is stable on $G = A \wr C$. Denote by D the subgroup of G generated by $A(b)$, $b \in C$. By Lemma 3.13 we have that if $f \in PJ(G)$, then $f\big|_D \in PJ(D, C)$. Let b_i, $i = 1, 2, \ldots, k$ be distinct elements from C. Then for any a_i, $i = 1, 2, \ldots, k$ the subgroup of D generated by $a_i(b_i)$, $i = 1, 2, \ldots, k$

is abelian. Hence if $u = a_1(b_1)a_2(b_2)\cdots a_k(b_k)$, $v = \alpha_1(b_1)\alpha_2(b_2)\cdots\alpha_k(b_k) \in D$ and $f \in PJ(D, C)$, then by Corollary 2.17

$$|f(uv) + f(uv^{-1}) - 2f(u)|$$
$$= \left|\sum_{i=1}^{k}[f(a_i\alpha_i(b_i)) + f(a_i\alpha_i^{-1}(b_i)) - 2f(a_i(b_i))]\right|.$$

Let b_i for $i \in \mathbb{N}$ be distinct elements from C. Let $a, \alpha \in A$. Consider elements $u_k = a(b_1)a(b_2)\cdots a(b_k)$ and $v_k = \alpha(b_1)\alpha(b_2)\cdots\alpha(b_k)$. Then by Corollary 2.17, for any $k \in \mathbb{N}$, we have

$$|f(u_kv_k) + f(u_kv_k^{-1}) - 2f(u_k)|$$
$$= \left|\sum_{i=1}^{k}[f(a\alpha(b_i)) + f(a\alpha^{-1}(b_i)) - 2f(a(b_i))]\right|.$$

By Lemma 3.13 we have $f(d(b_i)) = f(d(1))$ for any $d \in A$ and for any $i \in \mathbb{N}$. Let $r = f(a\alpha(b_i)) + f(a\alpha^{-1}(b_i)) - 2f(a(b_i))$. Hence

$$|f(u_kv_k) + f(u_kv_k^{-1}) - 2f(u_k)|$$
$$= \left|\sum_{i=1}^{k}[f(a\alpha(b_i)) + f(a\alpha^{-1}(b_i)) - 2f(a(b_i))]\right|.$$
$$= |k[f(a\alpha(1)) + f(a\alpha^{-1}(1)) - 2f(a(1))]|.$$
$$= k \cdot |r|.$$

Further we have

$$|f(u_kv_k) + f(u_kv_k^{-1}) - 2f(u_k)| \le c.$$

Hence

$$k|r| \le c,$$

and

$$|r| \le c\,\frac{1}{k} \quad \forall k \in \mathbb{N}.$$

The latter is possible only if $r = 0$. Thus $f \in J_0(D, B)$. Denote by j_b the restriction of f to $A(b)$. Let a be an arbitrary element from A. According to the action of C on D we have

$$f(a(b)) = f(a(1)^b) = f^b(a(1)) = f(a(1)),$$

that is, $j_b(a(b)) = j_1(a(1))$. Hence, there is an element j in $J_0(A)$ such that $j_b(a(b)) = j(a)$ for any $a \in A$ and for any $b \in B$. Therefore, for any $u = a_1(b_1)a_2(b_2)\cdots a_k(b_k)$ the relation

$$f(a_1(b_1)a_2(b_2)\cdots a_k(b_k)) = \sum_{i=i}^{k} j(a_i)$$

holds. Let $b, b_1, b_2 \in C;\ d, d_1, d_2 \in D$ and $u = b_1 d_1,\ v = b_2 d_2$. Then

$$\begin{aligned} &|f(uv) + f(uv^{-1}) - 2f(u)| \\ &\quad = |f(b_1 b_2 d_1^{b_2} d_2) + f(b_1 b_2^{-1} d_1^{b_2^{-1}} d_2^{-1 b_2^{-1}}) - 2f(b_1 d_1)| \le c. \end{aligned} \tag{4.1}$$

Further we have

$$\begin{aligned} &|f(db) + f(db^{-1}) - 2f(d)| \le c, \\ &|f(db) + f(db) - 2f(d)| \le c, \\ &|2f(db) - 2f(d)| \le c \end{aligned}$$

or

$$|f(db) - f(d)| \le \frac{c}{2}.$$

The latter is equivalent to

$$|f(bd^b) - f(d)| \le \frac{c}{2}. \tag{4.2}$$

Taking into account $f(d^b) = f(d)$ we get

$$|f(bd) - f(d)| \le \frac{c}{2}. \tag{4.3}$$

Putting $b_1 = b_2$ in (4.1) we obtain

$$|f(d_1^{b_2} d_2) + f(d_1^{b_2} d_2^{-1 b_2}) - 2f(b_2 d_1)| \le c. \tag{4.4}$$

Now taking into account (4.3) and the relation $f(d^b) = f(d)$ we obtain from (4.4)

$$\begin{aligned} 2c &\ge |f(d_1^{b_2} d_2) + f(d_1 d_2^{-1}) - 2f(d_1) \\ &= |f(d_1^{b_2} d_2) - f(d_1 d_2) + f(d_1 d_2) + f(d_1 d_2^{-1}) - 2f(d_1)|. \end{aligned}$$

Hence

$$\begin{aligned} &|f(d_1^b d_2) - f(d_1 d_2)| \\ &\quad = |f(d_1^b d_2) - f(d_1 d_2) + [f(d_1 d_2) + f(d_1 d_2^{-1}) - 2f(d_1)] \\ &\qquad - [f(d_1 d_2) + f(d_1 d_2^{-1}) - 2f(d_1)]| \\ &\quad \le |f(d_1^b d_2) - f(d_1 d_2) + [f(d_1 d_2) + f(d_1 d_2^{-1}) - 2f(d_1)]| \\ &\qquad + |[f(d_1 d_2) + f(d_1 d_2^{-1}) - 2f(d_1)]| \\ &\quad \le 2c + c = 3c \end{aligned} \tag{4.5}$$

for any $d_1, d_2 \in D$ and any $b \in C$.

Let $b \neq 1$. Then from (4.5) we obtain $|f(a_1(b)a_2(1)) - f(a_1 a_2)| \le 3c$ for any $a_1, a_2 \in A$, that is, $|j(a_1(b)) + j(a_2(1)) - j(a_1 a_2)| \le 3c$ for any $a_1, a_2 \in A$, and $|j(a_1) + j(a_2) -$

$j(a_1a_2)| \le 3c$ for any $a_1, a_2 \in A$. Hence, $j \in PX(A)$. But $PX(A) \cap J_0(A) = X(A)$ and we see that $j \in X(A)$.

Let $\psi = f\big|_D$. Then ψ is an element of $X(D)$ invariant relative to C. Let us extend ψ to G as follows: $\psi_1(bd) = \psi(d)$. It is easy to see that $\psi_1 \in KX(G)$. From Theorem 2.16 it follows that $\psi_1 \in KJ(G)$. From Lemma 2.8 we see that $\hat{\psi}_1 \in PJ(G)$. Let us verify that $\hat{\psi}_1(xy) = \hat{\psi}_1(yx)$ for all $x, y \in G$.

Indeed, by Lemma 2.8 there is $q > 0$ such that

$$|\hat{\psi}_1(x) - \psi_1(x)| \le q, \quad \forall x \in G. \tag{4.6}$$

From the relation $\psi_1 \in KX(G)$ we see that for some $\delta > 0$,

$$|\psi_1(xy) - \psi_1(x) - \psi_1(y)| \le \delta, \quad \forall x, y \in G. \tag{4.7}$$

This implies that for $x, y, z \in G$ the following relation holds:

$$|\psi_1(xyz) - \psi_1(x) - \psi_1(y) - \psi_1(z)| \le 2\delta, \tag{4.8}$$

From (4.6) and (4.8) we have

$$|\hat{\psi}_1((xy)^{n+1}) - \psi_1(x) - \psi_1((yx)^n) - \psi_1(y)| \le q + 2\delta, \ \ \forall x, y \in G. \tag{4.9}$$

Now applying (4.7) we obtain

$$|\hat{\psi}_1((xy)^{n+1}) - \psi_1((yx)^{n+1})| \le q + 3\delta, \quad \forall x, y \in G. \tag{4.10}$$

Similarly, we get

$$|\hat{\psi}_1((yx)^{n+1}) - \psi_1(x) - \psi_1((xy)^n) - \psi_1(y)| \le q + 2\delta, \ \ \forall x, y \in G. \tag{4.11}$$

Now again using (4.6), (4.7) and (4.8) we obtain

$$|\hat{\psi}_1((xy)^{n+1}) - \psi_1((yx)^n) - \psi_1(yx)| \le q + 3\delta, \quad \forall x, y \in G.$$

From the equality

$$\hat{\psi}_1((yx)^{n+1}) = \hat{\psi}_1((yx)^n) + \hat{\psi}_1(yx)$$

and (4.6) we get

$$|\hat{\psi}_1((yx)^{n+1}) - \psi_1((yx)^n) - \psi_1(yx)| \le 2q. \tag{4.12}$$

From (4.11) and (4.12) we obtain that for $p = 3q + 3\delta$ the following relations hold:

$$|\hat{\psi}_1((xy)^{n+1}) - \hat{\psi}_1((yx)^{n+1})| \le p \quad \forall x, y \in G \quad \text{and} \quad \forall n \in \mathbb{N}.$$

$$(n+1)|\hat{\psi}_1(xy) - \hat{\psi}_1(yx)| \le p.$$

This implies that

$$|\hat{\psi}_1(xy) - \hat{\psi}_1(yx)| \le \frac{p}{n+1} \quad \forall x, y \in G \quad \text{and} \quad \forall n \in \mathbb{N}.$$

The latter is possible only if $\hat{\psi}_1(xy) \equiv \hat{\psi}_1(yx)$.

Now by Theorem 2.16 we get that $\hat{\psi}_1 \in PX(G)$ such that $\hat{\psi}_1\big|_D = \psi$.

It is clear that $g = f - \hat{\psi}_1 \in PJ(G)$ and $g\big|_{C\cup D} \equiv 0$. Let us verify that $g \equiv 0$ on G. Indeed, for any $bd \in G$ we have $2g(bd) = g((bd)^2) = g(b^2 d^b d) = g(d^b d) = 0$. Hence $g(bd) = 0$. So, we see that $f = \hat{\psi}_1$ and $f \in PX(G)$.

Now let us verify that $f \in X(G)$. To do it we verify that $2f$ is a character of the group G. Indeed,

$$\begin{aligned}
&2f(b_1d_1b_2d_2) - 2f(b_1d_1) - 2f(b_2d_2)\\
&\quad = f(d_1^{b_2b_1b_2} d_2^{b_1b_2} d_1^{b_1} d_2) - f(d_1^{b_1} d_1) - f(d_2^{b_2} d_2)\\
&\quad = f(d_1^{b_1}) + f(d_2^{b_1b_2}) + f(d_1^{b_1}) + f(d_2) - f(d_1)\\
&\qquad - f(d_1) - f(d_2) - f(d_2)\\
&\quad \equiv 0.
\end{aligned}$$

So, $2f \in X(G)$ and we obtain that $f \in X(G)$. Hence, $f \in J_0(G)$ and the equation (2.1) is stable on G. This completes the proof. □

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